ISSUES ON CHILDREN'S IDEAS OF FRACTIONS WHEN QUOTIENT INTERPRETATION IS USED

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This paper focuses on children's understanding of fractions when quotient interpretation is used to introduce them this concept. An intervention program was conducted with a 7-years-old classroom, from a public primary school, in Fafe, Portugal. This intervention program comprised seven sessions in which children learned the representation of fractions and were challenged to solve some problems of ordering and equivalence of fractions. These sessions were organised following the official curricular content (starting by the equal sharing problems) but also according to the children's rhythms and demands. Children's performance and their arguments solving the tasks of ordering and equivalence of fractions are presented here. Issues on their learning process are characterized and discussed.

FRAMEWORK

Fractions is one of the most complex concept that children have to learn during the elementary school, but also a necessary one. Literature already provided information about students' difficulties (see Behr et al., 1984; Hart, 1981; Kerslake, 1986) with fractions. More recently, literature has been discussing the issues related to the effects of the interpretations for fractions on children's understanding of this concept (see Mamede, Nunes & Bryant, 2006; Mamede & Nunes, 2008; Nunes, Bryant, Pretzlik, Wade, Evans & Bell, 2004) and on the children's schemes of action (Nunes, 2008; Nunes & Bryant, 2008).

Distinct interpretations of fractions seem to affect differently children's understanding of the ideas of fraction. At the primary school, the children are supposed to understand at least fractions in quotient, part-whole and operator interpretations. But in these interpretations the meaning of the numerator and denominator differ. In part-whole interpretation, the denominator designates the number of parts into which a whole has been cut and the numerator designates the number of parts taken. So, 2/4 in a part-whole situation means that a whole – for example – a chocolate was divided into four equal parts, and two were taken. In quotient interpretation, the denominator designates the number of recipients and the numerator designates the number of items being shared. In a quotient situation, 2/4 means that 2 items – for example, two chocolates – were shared among four people. Furthermore, it should be noted that in quotient situation a fraction can have two meanings: it represents the division and also the amount that each recipient receives, regardless of how the chocolates were cut. For example, the fraction 2/4 can represent two chocolates shared among four children and also can represent the part that each child receives, even if each of the chocolates was only cut in half each

(Mack, 2001; Nunes, Bryant, Pretzlik, Evans, Wade & Bell, 2004). In operator situations, the denominator indicates the number of equal groups into which a set was divided and the numerator is the number of groups taken (Nunes et al., 2004). In an operator interpretation, if a boy is given 2/4 of 12 marbles, means that the 12 marbles are organized into 4 groups (of 3 marbles each) and the boy receives 6 marbles – that is 2 groups of the 4 into which the 12 marbles were organized. Thus number meanings differ across these interpretations. These differences affect children's understanding of fractions when building on their informal knowledge.

Mamede, Nunes and Bryant (2006) conducted a survey on 80 first-grade children, aged 6 and 7 to compare their understanding of ordering and equivalence of fraction presented to them in quotient and part-whole interpretations. These children had received no school instruction about fractions. The results show that children's performance on problems presented in quotient interpretation was much better than in part-whole interpretation. In quotient interpretation the rates of success were 55% for 6-year-olds children and 71% for 7-year-olds children, for ordering problems; and 35% for 6-year-olds and 77% for 7-year-olds children, for equivalence problems. In part-whole interpretation the rates of success were 24% for 6-year-olds children and 20% for 7-year-olds children, for ordering problems; and 9% for 6-year-olds and 10% for 7-year-olds children, for equivalence problems. The children's resolutions were also analysed giving evidence that strategies based on correspondence combined with partitioning were popular among the group of children who solved the problems in quotient interpretation whereas partitioning was the strategy adopt by those who worked in part-whole situations.

More recently Mamede (2008) conducted an intervention program with 37 first graders (ages 6-7) to introduce fractions in distinct interpretations. The children were addressed randomly to work in part-whole, quotient and operator interpretations of fractions. Again the children had received no instruction about fractions. The results showed that those who were introduced to quantities represented by fractions in quotient interpretation could succeed in ordering, equivalence and labeling tasks; those who were introduced to fractions using part-whole and operator situations were able to succeed only on the labeling of fractions, but not on the ordering and equivalence tasks.

Thus, the type of interpretation used to work with fraction in the school interferes with students understanding of fractions. This idea is also supported by Nunes et al. (2004) who describe the results of a survey conducted with 130 students in Year 4 and 5 (8- and 9-year-olds) to analyse the pupil's ability to compare equivalent fractions presented in Quotient and Part-whole situations. In quotient situation item the pupils were asked to compare the fractions 1/4 and 2/8; in part-whole situation they were asked to compare 2/4 and 4/8. Results show that the rates of correct responses were 46% for the part-whole item and 77% for the quotient item. Thus, in spite of considering different fractions in each situation, these results suggest that children perform differently in these two situations.

Research has been giving evidence that quotient situations are more suitable for children to build on their informal knowledge for fractions. The informal ideas about fractional quantities appear much earlier than the formal learning of fractions in school. Research developed with younger children shows that in a division situation, there are some children as young as 6-year-olds who can understand the inverse relation between the divisor and the quotient, when the dividend is the same (Correa, Nunes & Bryant, 1998) when discrete quantities are involved, and when continuous quantities are involved (Empson, 1999; Kornilaki & Nunes, 2005). This understanding of the inverse relation between the divisor and the quotient can be seen as a precursor of understanding of the logic of fractions: the greater the divisor (which would be represented by the denominator in a quotient interpretation), the smaller the quantity.

Streefland (1991, 1997) recommends the use of quotient situations to introduce fractions to children because these situations rely on the idea of fair sharing, which can provide the model for fractions and the part-whole concept related to equivalence and operational relations. The author not only recommends but also provides evidence of success in the use of the quotient interpretation to introduce fractions to children, describing a theory for teaching fractions based on the realistic approach that uses this type of interpretation to introduce fractions to children (see Streefland, 1991). Starting from problems using situations taken from daily life focused on division situations, Streefland produced good improvements on children's understanding of fractions, helping them to perceive the meaning of numerator and denominator as connected to each other, forming a correct mental object for the concept of fraction.

Traditionally, in many European countries, including Portugal, and the U.S. (see Behr, Harel, Post & Lesh, 1992; Behr, Lesh, Post & Silver, 1983; Kerslake, 1986; Mack, 1990; DEB, 1998) children are introduced to fractions at school using the part-whole interpretation and then this work with fractions is extended to include operator situations. In Portugal, in the primary school levels (1^{st} to 4^{th}-grades) students are introduced to fractions representation using the part-whole interpretation, and in some cases students have their first contact with fractions on the 5^{th} grade. Portugal is experiencing a new curriculum for the elementary school levels. This new curriculum refers that fractions should be introduced to children in an informal way, in the second grade, relying in partitioning and equal sharing; and explored in the third and fourth grades in the quotient, part-whole, operator and measure interpretations should be explored. Nevertheless that document gives no other indication for teachers to introduce and explore fractions in the classroom. Literature already provided evidence of success when children are introduced to fractions in quotient interpretation (see Streefland, 1991, 1997; Mamede, 2008).

However, for many Portuguese primary school teachers the concept of fraction only makes sense when the part-whole interpretation is involved. Knowing that quotient interpretation of fractions can help children to build on their informal knowledge with understanding, how can teachers explore this interpretation in the classrooms? This paper tries to give evidence of a well succeeded experience conducted in the classroom in which fractions are introduced to children using quotient interpretation.

The study reported here describes children's understanding of fractions when they experience partitioning and equal sharing activities, and then received instruction on fractions using the quotient interpretation. The teaching experiment follows the Portuguese official curriculum for the 2nd grade mathematics, but goes further anticipating children's first contact with fractions to this level. Previous related studies give evidence of success of children's understanding of quantities represented by fractions when quotient interpretation is used but they do not follow the Portuguese curriculum in the classroom.

The part of the study reported here focuses on children's understanding of fractions when they are introduced to them using quotient situations, after a contact with partitioning and equal shared activities. It tries to address two questions: (1) How do children understand ordering of fractions when introduced to this concept using the quotient interpretation? (2) How do children understand the equivalence of fraction in this interpretation?

METHODS

An intervention study was conducted using qualitative methods to describe children's performances and characterize the processes involved in their learning to represent and compare fractions. Children's answers, as well as their arguments and solving strategies were analysed to reach an insight on their ideas of fraction.

Participants

The participants were a class of 8 students from a public primary school from Fafe, in the north of Portugal. The children were all 7 years-old. The teacher of the class is one of the researchers. These children had received no instruction about fractions.

Design

The intervention comprised 7 sessions, of approximately 90 minutes each, in which children were introduce to fractions using quotient situations. In the first two sessions children were challenged to solve problems involving equal sharing; they were also introduced to the symbolic representation of fractions, in which the quotient situation or interpretation was used. The remaining sessions were designed to explore ordering and equivalence of fractions in quotient situations.

There were 6 task of ordering of fractions and 4 of equivalence of fractions. The fractions used in these tasks were all less than 1. In the ordering tasks children were asked to solve a problem such as: “Two girls are going to share fairly a chocolate bar, and there is nothing left; four boys are going to share fairly a chocolate bar and there is nothing left. These chocolate bars are equal. Do you think that each girl is going to eat more chocolate than each boy, each boy is going to eat more chocolate than each girl, each girl and each boy are eating the same amount of chocolate? Can you write the number that represents the amount of chocolate that each child eats?”. They were also asked to compare fraction given only symbolically. Analogous tasks were presented to them involving equivalence of fractions, in a problem such as: “Two girls are going to share fairly a chocolate bar, and there is nothing left; four boys are going to share fairly two chocolate bars and there is nothing left. These chocolate bars are equal. Do you think that each girl is going to eat more chocolate than each boy, each boy is going to eat more chocolate than each girl, each girl and each boy are eating the same amount of chocolate? Can you write the number that represents the amount of chocolate that each child eats?”. In some sessions the ordering and equivalence problems were presented with no pictorial support.

Procedure

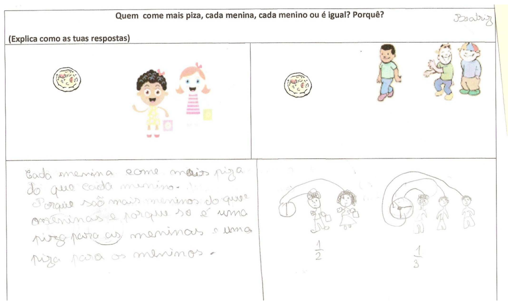
In all sessions the tasks were presented to the children with the support of PowerPoint slides. Each child had a worksheet with the same information presented by the teacher, in which they could draw as they wish; and manipulative aid was provided as coloured paper with squared, rectangular and circular shapes were available. In each session the tasks were presented by the teacher to the class orally to ensure children’s understanding of the problem, as they usually do in the math class. Then children were asked to solve the problems presented to them and justify their results. Then the teacher challenged them to write down their arguments and verify their solutions.

Data collection was carried out with the use of video and audio records, students’ worksheets and field notes token by the researcher.

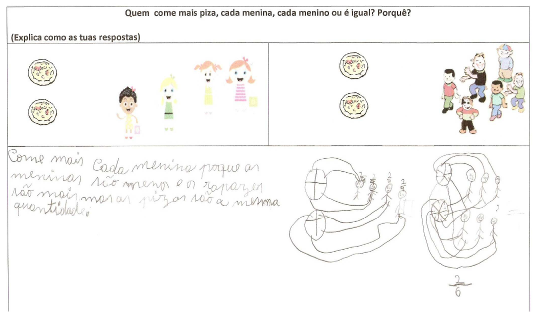
Results

In this section it is presented the results concerning children’s performance and arguments solving ordering and equivalence problems of fractions. The children solved the tasks individually and wrote their answers on their worksheet; then they wrote their justification and only after that they were challenged to verify their solution.

When the children were asked to compare 1/2 and 1/3, 7 of the 8 children succeeded; these 7 children gave a correct answer and then wrote down the explanation. Majority of children wrote down the explanation and then drew the pictures in their worksheets to verify their reasoning. Children’s performance was even better when they were asked to compare 2/4 and 2/6. Figure 1 gives examples of children’s performance on these ordering problems.



‘Each girl eats more pizza than each boy. Because there are more boys than girls and there is one pizza for the girls and one pizza for the boys.’



‘Each girl ate 1/2. Because there are only two girls. And the boys ate 1/3.’

Figure 1: Children’s resolutions comparing 1/2 and 1/3, and 2/4 and 2/6.

The children’s arguments were improving along the sessions. By the fourth session it was possible to hear valid explanations such as “... each girl eats more pizza because there are fewer than boys and there is equal pizzas” referring to the equal number of pizzas; or “Each girl eats more pizza because each girl eats a bigger piece and the boy eats a smaller piece than the girls”; or “Each girl eats more because there are two pizzas for four girls and each boy eats less because there are two pizzas for six boys”. The children’s arguments were improving along the sessions.

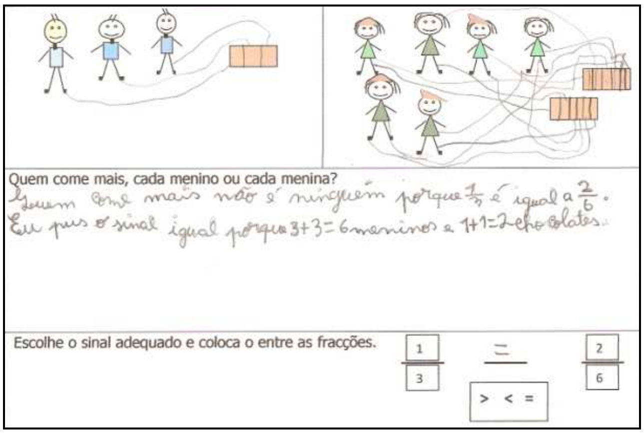
In another episode involving an ordering task, the children were able to recognise and generalize the inverse relation between the divisor and the quotient, when the dividend is the same, as it shows the following transcription of children’s discussion.

Tutor:	So, if you wish to get the biggest amount of paper which fraction would you choose?
M:	One half... it´s more than the others.
Tutor:	How do you know that is more?
M:	One paper for two children is more than one paper for three...
J:	And more than for four and more than three...
Tutor:	If it is so, what sign did you write J?
J:	The ‘bigger’ sign...
Tutor:	Why did you put the ‘bigger’ sign?
J:	It’s always 1 paper and there is always more children. They were 2, then 3, then 4....
R:	The smallest is one-fifth... I circled that one!

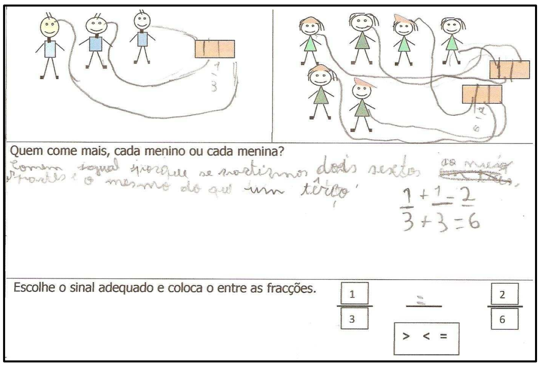
Transcription 1: Children’s explanation of the inverse relation between divisor and quotient for the same dividend.

The children were also able to solve problems of equivalence of fractions (7 out of 8 succeeded in all these tasks). Some episodes in which children revealed difficulties will be presented in the conference.

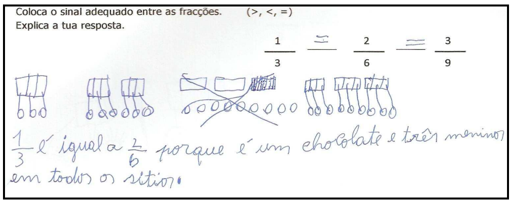
Figure 2 shows children's resolutions on two equivalence tasks. In these tasks some children found more difficult to explain their arguments in a written mode, in spite of solving the tasks correctly. These examples as well as children's difficulties will be presented in the conference. However, their oral justifications improved after drawing their schemes to verify the solutions, as their pictures were giving them some support in this task.



'No one eats more because 1/3 is equal to 2/6. I put the equal sign because 3+3=6 boys and 1+1=2 chocolates.'



'Eat the same because if we cut two-sixths the pieces are the same of one-third.'



'1/3 is the same of 2/6 because there is one chocolate and three children in all places.'

Figure 2: Children's resolutions of distinct equivalence tasks.

In the majority of the tasks presented, children seemed to rely on the use of correspondence to reach the solution. When solving equivalence tasks the use of correspondence is even more evident as in many cases this idea is also supported by their justifications (see Figure 2).

When asked to compare 1/3 and 2/6 almost all succeeded (7 out of 8 children). Children's resolutions relied mainly on partitioning and correspondence. But a few children seemed to reveal some type of proportional reasoning when presenting their justifications. This is suggested when they try to explain that fact with numbers and expression familiar to them, such as '1+1=2' and '3+3=6' trying to express the double of quantities involved (see Figure 2).

DISCUSSION AND CONCLUSIONS

The results of this study allow us to establish some remarks. First, this experience gives evidence that children can understand fractions when introduced to them in quotient situations, in agreement with Streefland (1991, 1997), Mamede, Nunes and Bryant (2006), Mamede and Nunes (2008) who previously studied this issues. In the sessions of this study, the ordering problems seemed to help the children to easily understand the inverse relation between the divisor and the quotient, when the dividend is the same. This relation is essential to understand the meaning of fractions. Second, these children learned easily fractions labels. They were introduced to the representation of fractions in the beginning of the intervention, and soon they master the symbolic representation of fractions, when quotient interpretation was used. In this type of interpretation, the magnitudes involved in the fractions refer to two variables of different nature (Nunes *et al.*, 2004), - numerator refers to the number of items to share, denominator refers to the number of recipients - and this may facilitate children's learning of fractions labels. Third, because in quotient interpretation the numerator and the denominator relate to variables that are different in nature (Nunes *et al.*, 2004), children easily relied on the use of correspondence to solve many of the tasks. This finding was also documented previously by Mamede, Nunes and Bryant (2006) when observing 6-7-year-olds children's strategies solving ordering and equivalence problems, when interviewed individually. Nunes (2008) argues that in a division situation, there are two types of action schemes: partitioning, which involves dividing the whole into equal parts; and correspondence which involves two quantities (a quantity to be shared and a number of recipients of the shares). The development of these action schemes defers. Children of 5 to 6-year-olds can establish correspondence to produce equal shares (see Kornilaki & Nunes, 2005; Mamede, Nunes & Bryant, 2006; Nunes, 2008), but they find more difficult to accomplish partitioning of continuous quantities. These schemes of action (Nunes, 2008) are fundamental for the learning of the mathematical concepts. Fourth, the equivalence problems presented in quotient interpretation gave the children an opportunity to promote their proportional reasoning. When solving the equivalence problems many children establish a proportional relation between the numbers of items to share and the number of recipients in order to reach the solution; some of them could express that relation in a written way, others by drawings. Proportional reasoning was also a strategy identified by Mamede (2007) and Nunes *et al.* (2004) when analysing students' strategies solving equivalence problems presented to them in quotient interpretation of fractions. To conclude, this short intervention program allowed the teacher to understand children's possibilities of success with fractions when they are introduced to the children using quotient situations. We hope that this experiment can contribute to promote a change in the classroom practices, following the Portuguese official curriculum, giving the primary teachers an example of a well succeeded experience. As correspondence seems to have an important role on children's reasoning on fractions problems, it seems to be relevant to give young children the opportunity to develop sharing experiences based on correspondence (one-to-one and one-to-many) since kindergarten.

More research is needed in order to explore other ways of introducing fractions to children in the classroom, using quotient interpretation of fractions. In this experiment, the children learned the fraction representation in the beginning sessions. For further research in this area it would be interesting to develop a longitudinal research to analyse the influence of interventions based on quotient interpretation of fractions on young children's understanding of other interpretations of fractions.

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