

# Equilibrium Fluctuations for Totally Asymmetric Particle Systems

exclusion and zero-range processes

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Ao meu pai  
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# Chapter 1

## Introduction

The contents of this monograph were chosen in order to present a framework to derive the equilibrium fluctuations for two kinds of interacting particle systems of asymmetric jump rates. The idea of how to present this result was chosen in order that someone who does not know the theory of interacting particle systems, understands the fundamental results and techniques and feel comfortable with the ideas behind the theory of *hydrodynamic limit*. In this monograph we analyze two interacting particle systems: the totally asymmetric simple exclusion process and the totally asymmetric zero-range process.

The idea is to start from the definition of the processes, going through the hydrodynamic limit result, with the objective of establishing the equilibrium fluctuations for these processes. Both processes are put in context since as we will see below, they share mostly the same properties. In spite of considering only asymmetric systems, the hydrodynamic limit will be derived for the symmetric versions of these processes, using the Entropy method introduced by Guo, Papanicolau and Varadhan in [14] and by the Relative Entropy method introduced by Yau in [33]. The equilibrium fluctuations for these symmetric processes were presented in [25] and [18].

We opt for presenting the hydrodynamic limit for these processes, since for the asymmetric systems the proof is more involved and tricky, and we want to keep the presentation of this result the simplest as possible. For

a proof on the hydrodynamic limit for the corresponding asymmetric processes, we refer the interested reader to [26]. There the hydrodynamic limit was derived by applying the Entropy method, using the fact that the hydrodynamic equation has a unique entropy solution. The equilibrium fluctuations will be derived for both asymmetric processes by showing that the limit density fluctuation field is the unique weak solution of the corresponding stochastic partial differential equation. We start by reviewing some of the latest results obtained for these processes.

The exclusion process on  $\mathbb{Z}^d$  has been extensively studied in the literature. In this process, particles evolve on  $\mathbb{Z}^d$  according to interacting random walks with an exclusion rule which prevents to have more than one particle per site. The dynamics can be informally described as follows. Fix a probability  $p(\cdot)$  on  $\mathbb{Z}^d$ . Each particle, independently from the others, waits a mean one exponential time, at the end of which being at the site  $x$  it jumps to  $x + y$  at rate  $p(y)$ . If the site is occupied the jump is suppressed in order to respect the exclusion rule. In both cases, the particle waits a new exponential time.

The space state of the process is  $\{0, 1\}^{\mathbb{Z}^d}$  and we denote the configurations by the Greek letter  $\eta$ , so that  $\eta(x) = 0$  if the site  $x$  is vacant and  $\eta(x) = 1$  otherwise. The case in which  $p(y) = 0 \forall |y| > 1$  is referred as the simple exclusion process. In the asymmetric simple exclusion process the probability  $p$  is such that  $p(1) = p$ ,  $p(-1) = 1 - p = q$  with  $p \neq 1/2$  while in the symmetric simple exclusion process  $p = 1/2$ . For the special case  $p = 1$ , jumps occur only to the right neighboring site and the process is known as the totally asymmetric simple exclusion process.

For  $0 \leq \alpha \leq 1$ , denote by  $\nu_\alpha$  the Bernoulli product measure on  $\{0, 1\}^{\mathbb{Z}^d}$  with density  $\alpha$ . It is known that  $\nu_\alpha$  is an invariant measure for the exclusion process and that all invariant and translation invariant measures are convex combinations of  $\nu_\alpha$ , if  $p(\cdot)$  is such that  $p_t(x, y) + p_t(y, x) > 0$ ,  $\forall x, y \in \mathbb{Z}^d$  and  $\sum_x p(x, y) = 1$ ,  $\forall y \in \mathbb{Z}^d$ , see [20]. We will consider the one-dimensional case, i.e. the process evolving in  $\mathbb{Z}$  and a remark is made when the results are valid for the  $d$ -dimensional case.

The hydrodynamic limit for the asymmetric simple exclusion process was

shown by Rezakhanlou in [26]. There it was shown, that for the asymmetric simple exclusion process, taken on the hyperbolic scaling  $tn$ , the macroscopic particle density profile evolves according to the inviscid Burgers equation, namely:

$$\partial_t \rho(t, u) + \nabla(\rho(t, u)(1 - \rho(t, u))) = 0.$$

This result is a Law of Large Numbers for the empirical measure, for the process starting from a general set of initial measures associated to a profile  $\rho_0$ .

To establish the Central Limit Theorem for the empirical measure we need to consider the density fluctuation field as defined below. We show that, in this time scale, the time evolution of the limit density fluctuation field is deterministic, in the sense that at any given time  $t$ , the density field is a translation of the initial one. We notice that, this result was previously obtained in [8]. In order to observe fluctuations from the dynamics one has to change to the diffusive scaling  $tn^2$ .

The translation or velocity of the system is given by  $(1 - 2\alpha)$  and for  $\alpha = 1/2$ , the limit density field does not evolve in time, and one is forced to go beyond the hydrodynamic scaling  $tn$ . We can consider the density fluctuation field in a longer time scale, where we subtract the velocity of the system and any value of  $\alpha$  can be considered in this setting.

It is conjectured that until the time scale  $tn^{3/2}$  the density fluctuation field does not evolve in time, see Chapter 5 of [32] and references therein. The result we present here is a contribution in this direction, since the result can be accomplished up to the time scale  $n^{4/3}$ . The main difficulty in proving the Central Limit Theorem for the empirical measure is showing the Boltzmann-Gibbs Principle, which can be proved for this longer time scale  $tn^{4/3}$  using a multi-scale argument. As a consequence of this translation behavior, one obtains the dependence on the initial configuration of the current through a time dependent bond and the position of the tagged particle in the longer time scale. For simplicity we present the results here for totally asymmetric jumps, i.e.  $p = 1$ , but the same results hold for more general asymmetric rates as the asymmetric simple exclusion process. For this case we refer the reader to [11].

We also mention that the study of the transition time scales for the asymmetric exclusion process was studied in the two dimensional setting in [34] and in dimension three in [5]. In the one-dimensional case, the problem of the flux was studied in [3] and for the density in [11].

Now, assume that the origin is occupied at time 0. Tag this particle and denote by  $X_t$  its position at time  $t$ . Applying an invariance principle due to Newman and Wright [22], Kipnis in [17] proved a Central Limit Theorem for the position of the tagged particle in the one-dimensional asymmetric simple exclusion process, provided the initial configuration is distributed according to  $\nu_\alpha^*$ , the Bernoulli product measure conditioned to have a particle at the origin. Transforming the exclusion process into a series of queues, namely a constant rate asymmetric zero-range process, the position of the tagged particle in the exclusion process becomes the current through the bond  $[-1, 0]$  in the zero-range representation. Kipnis in [17], was able to apply Newman and Wright results to the zero-range process and derive the Law of Large Numbers and the Central Limit Theorem for the position of the tagged particle.

Few years later, Ferrari and Fontes [9] proved that the position at time  $t$  of the tagged particle,  $X_t$ , can be approximated by a Poisson Process. More precisely, they proved that for all  $t \geq 0$ , if the initial distribution is  $\nu_\alpha^*$  and  $p > q$ ,

$$X_t = N_t - B_t + B_0,$$

where  $N_t$  is a Poisson Process with rate  $(p - q)(1 - \alpha)$  and  $B_t$  is a stationary process with bounded exponential moments. As a corollary they obtained the weak convergence of

$$\frac{X_{t\epsilon^{-1}} - (p - q)(1 - \alpha)t\epsilon^{-1}}{\sqrt{(p - q)(1 - \alpha)t\epsilon^{-1}}}$$

to a Brownian motion. The argument is divided in two steps. The convergence of the finite-dimensional distributions [7] is consequence of the fact that in the scale  $t^{\frac{1}{2}}$ , the position  $X_t$  can be read from the initial configuration:  $X_t$  is given by the initial number of empty sites in the interval  $[0, (p - q)\alpha t]$  divided by  $\alpha$ . Tightness follows from the sharp approximation



of  $X_{t\epsilon^{-1}}$  by the Poisson process and the weak convergence of the Poisson process to Brownian motion. Using the approximation of  $X_t$  by a Poisson process and Kipnis results for the tagged particle, the same authors prove equilibrium density fluctuations for the asymmetric simple exclusion process in [8]. The density fluctuations for the totally asymmetric simple exclusion process (i.e. the case  $p = 1$ ) have also been obtained by Rezakhanlou in [27] in a more general setting than for the process starting from an equilibrium state.

Recently, Jara and Landim in [15] showed that the asymptotic behavior of the tagged particle in the one-dimensional nearest neighbor exclusion process, can be recovered from a joint asymptotic behavior of the empirical measure and the current through a fixed bond. From this observation they proved a non-equilibrium Central Limit Theorem for the position of the tagged particle in the symmetric simple exclusion process, under the diffusive scaling  $n^2$ .

Here, besides presenting this general method used in [15], to reprove Ferrari and Fontes result on the convergence of the re-scaled position of the tagged particle to a Brownian motion in the hyperbolic time scale, we present an extension of their result by showing that in a longer time scale the position of the tagged particle still depends on the initial configuration of the system.

The advantage of the approach presented here, is that it relates the Central Limit Theorem for the position of the tagged particle to the Central Limit Theorem for the empirical measure, a problem which is relatively well understood, see [18].

As mentioned above, here we will also consider another process known in the literature as the zero-range process. We will consider the process evolving in  $\mathbb{Z}$  and a remark is made when the results are valid for the  $d$ -dimensional case. In this process, if particles are present at a site  $x$ , then after a mean one exponential time, one of them jumps to  $x + 1$  at rate  $p(1)$ , or to  $x - 1$  at rate  $p(-1)$ , independently from the number of particles at other sites. This is a Markov process  $\xi$  with space state  $\mathbb{N}^{\mathbb{Z}}$ , where the configurations are denoted by  $\xi$ , so that for a site  $x$ ,  $\xi(x)$  represents the

number of particles at that site.

For each density  $\rho$  of particles, there exists an invariant measure denoted by  $\mu_\rho$ , which is translation invariant and is such that  $E_{\mu_\rho}[\eta(0)] = \rho$ , that is the Geometric product measure of parameter  $\frac{1}{1+\rho}$ . If  $p(\cdot)$  is symmetric, namely  $p(1) = p(-1) = 1/2$  the process is known as the symmetric zero-range process and if  $p(1) = 1 - p(-1) \neq 1/2$ , then the process is known as the asymmetric zero-range process. In the special case  $p(1) = 1$  the process is called the constant rate totally asymmetric zero-range process and this is the case that we consider here. The results presented here also hold for more general asymmetric rates, but for simplicity we present the results for the totally asymmetric zero-range process.

Since the work of Rezakhanlou in [26], it is known that for the totally asymmetric zero-range process the macroscopic particle density profile in the Euler scaling of time  $n$ , evolves according to the hyperbolic conservation law

$$\partial_t \rho(t, u) + \nabla \phi(\rho(t, u)) = 0,$$

where  $\phi(\rho) = \frac{\rho}{1+\rho}$ . Since  $\phi$  is differentiable, last equation can also be written as  $\partial_t \rho(t, u) + \phi'(\rho(t, u)) \nabla \rho(t, u) = 0$  and characteristics of partial differential equations of this type are straight lines with slope  $\phi'(\rho)$ . This result is a Law of Large Numbers for the empirical measure related to this process starting from a general set of initial measures associated to a profile  $\rho_0$ , see [26] for details. If one wants to go further and show a Central Limit Theorem for the empirical measure starting from the equilibrium state  $\mu_\rho$ , one has to consider the density fluctuation field as defined below.

It is not difficult to show that under the hydrodynamic time scale  $n$ , the limit density fluctuation field at time  $t$  is just a translation of the initial density field, which is a Gaussian white noise. The translation or velocity of the system is given by  $\phi'(\rho) = \frac{1}{(1+\rho)^2}$  which is the characteristic speed. As we have mentioned above, this same phenomenon happens for the asymmetric simple exclusion process on the hyperbolic time scale, starting from the Bernoulli product measure  $\nu_\alpha$ , which is an invariant state for that process and the velocity of that system is given by  $1 - 2\alpha$ .

If we consider the particle system moving in a reference frame with this

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constant velocity, then the limit density fluctuation field does not evolve in time and one is forced to consider the process evolving on a longer time scale. As above, this result can be accomplished for the constant rate totally asymmetric zero-range process speed up to the time scale  $n^{4/3}$ , i.e. in this case the limit field at time  $t$  still coincides with the initial field.

In the approach to this problem, the main difficulty in proving the Central Limit Theorem for the empirical measure is showing that the Boltzmann-Gibbs Principle holds for this process, which can be handled by generalizing the multi-scale argument of the asymmetric simple exclusion process. This Principle says, roughly speaking, that non-conserved quantities fluctuate in a faster scale than conserved ones, so when averaging in time a local field, what survives in the limit is its projection over the conserved quantities. To prove last result for the totally asymmetric zero-range process there are some extra computations due to the large space state, which we can overcome by using the equivalence of ensembles and a Taylor expansion of the instantaneous flux, in order to avoid the correlation terms. In fact, as happens for the asymmetric simple exclusion process, this result should be valid until the time scale  $n^{3/2}$  and on this time scale a phase transition should occur.

Since up to the time scale  $n^{4/3}$ , the macroscopic behavior of the system does only depend on the initial state, this implies that the flux or current of particles across a characteristic vanishes on this longer time scale. If one wants to observe non-trivial fluctuations of this current the process should be speeded up on a longer time scale. In fact, it was recently proved by [2] that the variance of the current across a characteristic is of order  $t^{2/3}$  and this translates by saying that this result should hold until the time scale  $n^{3/2}$ . Indeed, this result should hold for more general systems than the totally asymmetric zero-range process or the asymmetric simple exclusion process, but for the case of one-dimensional systems with one conserved quantity and hydrodynamic equation of hyperbolic type, whose flux is a concave function. This is a step towards showing the universality behavior of the scaling exponent for these systems.

We notice here that all the results presented below for the totally asym-

metric zero-range process, also hold for a more general zero-range process, namely one could take a zero-range dynamics in which the jump rate from  $x$  to  $x + 1$  is given by  $g(\eta(x))$ , with  $g(\cdot)$  nondecreasing and satisfying conditions of Definition 3.1 of Chapter 2 of [18] and also for partial asymmetric jumps, in the sense that a particle jumps from  $x$  to  $x + 1$  at rate  $pg(\eta(x))$  and from  $x$  to  $x - 1$  at rate  $qg(\eta(x))$ , where  $p + q = 1$ ,  $p \neq 1/2$  and with  $g$  as general as above.

Finally, we mention that the results presented here should hold for a more general class of processes of asymmetric jump rates and with hydrodynamic equation of hyperbolic type. These processes belong to a universal class of processes that share the same scaling exponent and distributional properties, see [32].

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