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Perturbations and scattering of spherically symmetric d–dimensional α'–corrected black holes in string theory

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Abstract. We compute the tensorial gravitational perturbations of general spherically symmetric black holes in d dimensions with string–theoretical α' corrections. We then study the scattering of minimally coupled massless scalar fields by such black holes. We obtain a general formula for the low frequency absorption cross section for every black hole of this kind, which we apply to known black hole solutions. In each case we compare the results for the absorption cross section with the black hole entropy, obtained through Wald's formula.

1. Introduction

The low frequency limit of the absorption cross section for minimally coupled scalar fields is equal to the area of the black hole horizon, a result which can also be extended to higher spin fields [1]. Equivalently, one can say that the low frequency cross section equals four times the Bekenstein–Hawking black hole entropy: $\sigma = 4GS$.

This relation was only established classically. It is important to check if and how such relation is maintained in the presence of higher derivative terms, namely string α' corrections. This question leads us to study α' corrections to the absorption cross section. But such study is interesting and important by its own, since gravitational wave astronomy is becoming an experimental reality which could allow for the detection and measurement of (small) string effects.

The first work to discuss the effects of leading α' corrections quadratic in the Riemann tensor in the absorption cross section of spherically symmetric black holes for generic d dimensions was article [2], but just dealing with a particular black hole solution. Here we perform such study for any d dimensional spherically symmetric black hole with such corrections.

This work is organized as follows. In section 2 we review the formalism for gravitational perturbations of black holes in d spacetime dimensions, focusing specifically on the tensorial perturbations. In section 3 we present the α'–corrected action and field equations we are about to address and their generic spherically symmetric solution. We obtain the tensorial perturbations to this solution and the respective tensor potential and master equation. In section 4 we solve the master equation in different regions of spacetime, using different approximations: close to the horizon, at asymptotic infinity and in the intermediate region. We present solutions, in closed form, of the master equation for these three regions. After matching these three different solutions, we are able to obtain a general formula for the α'–corrected low frequency absorption cross section. All we have been describing is performed for a generic spherically symmetric metric; in section 5, we apply our result to known metrics with α' corrections. We also compute the α'–corrected entropy, to be compared to the cross section. We end by discussing our results.

2. General setup of the perturbation theory

2.1. Perturbations on a $(d-2)$–sphere

We will study the behavior, under gravitational perturbations, of string–corrected black hole solutions in a generic spacetime dimension d. For such analysis we use the framework developed by Ishibashi and Kodama [3, 4, 5] for black holes. This framework applies to generic spacetimes of the form $\mathcal{M}^d = \mathcal{N}^{d-n} \times \mathcal{K}^n$, with coordinates $\{x^\mu\} = \left\{y^a, \theta^i\right\}$. In here $\mathcal{K}^n$ is a manifold with constant sectional curvature K. The metric in the total space $\mathcal{M}^d$ is then written as

$$g = g_{ab}(y)\ dy^a\ dy^b + r^2(y)\ \gamma_{ij}(\theta)\ d\theta^i\ d\theta^j. \tag{1}$$

For our purposes, we take $n = d-2$ and the manifold $\mathcal{K}^n$, describing the geometry of the black hole event horizon, will be a $(d-2)$–sphere (thus, with $K=1$). Also, $\mathcal{N}^{d-n}$ coordinates will be $\{y^a\} = \{t, r\}$, with $\left\{r, \theta^i\right\}$ being the usual spherical coordinates so that $r(y) = r$ and $\gamma_{ij}(\theta)\ d\theta^i d\theta^j = d\Omega^2_{d-2}$.

Defining generic perturbations to the metric as $h_{\mu\nu} = \delta g_{\mu\nu}$, $h^{\mu\nu} = -\delta g^{\mu\nu}$, we get for the variation of the Levi-Civita connection

$$\delta\Gamma^\rho_{\mu\nu} = \frac{1}{2}\left(\nabla_\mu h_\nu{}^\rho + \nabla_\nu h_\mu{}^\rho - \nabla^\rho h_{\mu\nu}\right) \tag{2}$$

From this variation and the Palatini equation

$$\delta\mathcal{R}^\rho{}_{\sigma\mu\nu} = \nabla_\mu\ \delta\Gamma^\rho_{\nu\sigma} - \nabla_\nu\ \delta\Gamma^\rho_{\mu\sigma}, \tag{3}$$

one can easily derive the variation of the Riemann tensor:

$$\delta\mathcal{R}_{\rho\sigma\mu\nu} = \frac{1}{2}\Big(\mathcal{R}_{\mu\nu\rho}{}^\lambda h_{\lambda\sigma} - \mathcal{R}_{\mu\nu\sigma}{}^\lambda h_{\lambda\rho} - \nabla_\mu\nabla_\rho h_{\nu\sigma} + \nabla_\mu\nabla_\sigma h_{\nu\rho} - \nabla_\nu\nabla_\sigma h_{\mu\rho} + \nabla_\nu\nabla_\rho h_{\mu\sigma}\Big). \tag{4}$$

General tensors, of rank at most equal to two, can be uniquely decomposed in tensor, vector and scalar components, according to their tensorial behavior on the $(d-2)$–sphere, the geometry of the black hole event horizon [4]. In particular, this is also true for the perturbations to the metric, but one should note that metric perturbations of tensor type only exist for dimensions $d > 4$, unlike perturbations of vector and scalar type, which also exist for $d = 4$. This is because the 2–sphere does not admit any tensor harmonics [6]. Tensor perturbations are therefore intrinsically higher–dimensional.

2.2. Tensorial perturbations of a spherically symmetric static metric

In this work we will only consider tensor type gravitational perturbations to the metric field, for α'–corrected $\mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma}$ black holes in string theory. One should consider perturbations to all the fields present in the low–energy effective action (in our case, the metric and the dilaton), but, as we will show later, one can consistently set tensor type perturbations to the dilaton field to zero. These metric perturbations were studied in [4], where it is shown that they can be written as

$$h_{ij} = 2r^2(y^a)\ H_T(y^a)\ \mathcal{T}_{ij}(\theta^i), \quad h_{ia} = 0, \quad h_{ab} = 0, \tag{5}$$

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with $\mathcal{T}_{ij}$ satisfying

$$\left(\gamma^{kl} D_k D_l + k_T\right) \mathcal{T}_{ij} = 0, \quad D^i \mathcal{T}_{ij} = 0, \quad g^{ij} \mathcal{T}_{ij} = 0. \tag{6}$$

Here, D_i is the covariant derivative on the $(d-2)$–sphere, associated to the metric γ_{ij}. Thus, the tensor harmonics $\mathcal{T}_{ij}$ are the eigentensors of the $(d-2)$–laplacian D^2, whose eigenvalues are given by $k_T + 2 = \ell(\ell + d - 3)$, with $\ell = 2, 3, 4, \ldots$. It should be further noticed that the expansion coefficient H_T is gauge–invariant by itself. This is rather important: when dealing with linear perturbations to a system with gauge invariance one might always worry that final results could be an artifact of the particular gauge one chooses to work with. Of course the simplest way out of this is to work with gauge–invariant variables, and this is precisely implemented in the Ishibashi–Kodama framework [3, 4, 5]. As it was noticed in [2], the Ishibashi–Kodama gauge–invariant variables are also valid for higher derivative theories as long as diffeomorphisms keep implementing gauge transformations. This is because up to now we have only chosen the background metric we wish to perturb: so far, no choice of equations of motion has been done.

Now we consider a static, spherically symmetric background metric. Such a metric is clearly of the type (1), and is given by

$$ds^2 = -f(r)\ dt^2 + g^{-1}(r)\ dr^2 + r^2 d\Omega^2_{d-2}. \tag{7}$$

One first needs to obtain the variation of the Riemann tensor under generic perturbations of the metric. If one collects the expressions for $h_{\mu\nu}$ given in (5), their covariant derivatives, and further the components of the Riemann tensor, and replaces them on the Palatini equation (4), one obtains

$$\begin{aligned}
\delta\mathcal{R}_{ijkl} &= \Big((3g-1)H_T + rg\partial_r H_T\Big)\Big(g_{il}\mathcal{T}_{jk} - g_{ik}\mathcal{T}_{jl} - g_{jl}\mathcal{T}_{ik} + g_{jk}\mathcal{T}_{il}\Big) + \\
&+ r^2 H_T\Big(D_iD_l\mathcal{T}_{jk} - D_iD_k\mathcal{T}_{jl} - D_jD_l\mathcal{T}_{ik} + D_jD_k\mathcal{T}_{il}\Big), && (8)\\
\delta\mathcal{R}_{itjt} &= \left(-r^2\partial_t^2 H_T + \frac{1}{2}r^2 f f'\partial_r H_T + r f f' H_T\right)\mathcal{T}_{ij}\,, && (9)\\
\delta\mathcal{R}_{itjr} &= \left(-r^2\partial_t\partial_r H_T - r\partial_t H_T + \frac{1}{2}r^2\frac{f'}{f}\partial_t H_T\right)\mathcal{T}_{ij}\,, && (10)\\
\delta\mathcal{R}_{irjr} &= \left(-r\frac{g'}{g}H_T - \frac{1}{2}r^2\frac{g'}{g}\partial_r H_T - 2r\partial_r H_T - r^2\partial_r^2 H_T\right)\mathcal{T}_{ij}\,, && (11)\\
\delta\mathcal{R}_{abcd} &= 0, && (12)
\end{aligned}$$

and further

$$\begin{aligned}
\delta\mathcal{R}_{ij} &= \frac{r^2}{f}\left(\partial_t^2 H_T\right)\mathcal{T}_{ij} - r^2 g\left(\partial_r^2 H_T\right)\mathcal{T}_{ij} - \frac{1}{2}r^2\left(f'+g'\right)\left(\partial_r H_T\right)\mathcal{T}_{ij} - r\left(f'+g'\right)H_T\mathcal{T}_{ij} + \\
&+ (2-d)\,rg\left(\partial_r H_T\right)\mathcal{T}_{ij} + (2d-4)\,H_T\mathcal{T}_{ij} + (6-2d)\,gH_T\mathcal{T}_{ij} + k_T H_T\mathcal{T}_{ij}\,, && (13)\\
\delta\mathcal{R}_{ia} &= 0, \quad \delta\mathcal{R}_{ab} = 0, \quad \delta\mathcal{R} = 0. && (14)
\end{aligned}$$

These are the equations we will need in order to perturb the α'–corrected field equations.

3. Gravitational perturbations to the α'–corrected field equations

The d–dimensional effective action with α' corrections we will be dealing with is given, in the Einstein frame, by

$$\frac{1}{16\pi G}\int\sqrt{-g}\left(\mathcal{R} - \frac{4}{d-2}\left(\partial^\mu\phi\right)\partial_\mu\phi + \mathrm{e}^{\frac{4}{d-2}\phi}\frac{\lambda}{2}\ \mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma}\right)\mathrm{d}^d x. \tag{15}$$

Here $\lambda = \frac{\alpha'}{2}, \frac{\alpha'}{4}$ and 0, for bosonic, heterotic and type II strings, respectively. We are only considering gravitational terms: we can consistently settle all fermions and gauge fields to zero for the moment. That is not the case of the dilaton, as it can be seen from the resulting field equations:

$$\nabla^2\phi - \frac{\lambda}{4}\,\mathrm{e}^{\frac{4}{2-d}\phi}\left(\mathcal{R}_{\rho\sigma\lambda\tau}\mathcal{R}^{\rho\sigma\lambda\tau}\right) = 0, \tag{16}$$

$$\mathcal{R}_{\mu\nu} + \lambda\,\mathrm{e}^{\frac{4}{2-d}\phi}\left(\mathcal{R}_{\mu\rho\sigma\tau}\mathcal{R}_{\nu}{}^{\rho\sigma\tau} - \frac{1}{2(d-2)}g_{\mu\nu}\mathcal{R}_{\rho\sigma\lambda\tau}\mathcal{R}^{\rho\sigma\lambda\tau}\right) = 0. \tag{17}$$

From (16) one sees that the correction term $\mathcal{R}_{\rho\sigma\lambda\tau}\mathcal{R}^{\rho\sigma\lambda\tau}$ acts as a source for the dilaton and, therefore, one cannot set the dilaton to zero without setting this term to zero too. Still, as it was shown in [13] and we will review later, for a spherically symmetric metric like (7), at order $\lambda = 0$ the dilaton is a constant (which can be always set to 0). The dilaton only gets nonconstant terms at order λ; this is why we could neglect terms which are quadratic in ϕ while deriving these field equations, since we are only working perturbatively to first order in λ.

In the present context, any black hole solution is built perturbatively in λ, and a solution will only be valid in regions where $r^2 \gg \lambda$, *i.e.*, any perturbative solution is only valid for black holes whose event horizon is much bigger than the string length.

We want to study scattering processes associated to solutions to the field equations above and, therefore, they are the ones which we will perturb.

By perturbing (16) and (17) one gets

$$\delta\nabla^2\phi - \frac{\lambda}{4}\,\mathrm{e}^{\frac{4}{2-d}\phi}\,\delta\left(\mathcal{R}_{\rho\sigma\lambda\tau}\mathcal{R}^{\rho\sigma\lambda\tau}\right) + \frac{\lambda}{d-2}\,\mathrm{e}^{\frac{4}{2-d}\phi}\,\mathcal{R}_{\rho\sigma\lambda\tau}\mathcal{R}^{\rho\sigma\lambda\tau}\,\delta\phi = 0, \tag{18}$$

$$\begin{aligned}\delta\mathcal{R}_{ij} &+ \lambda\,\mathrm{e}^{\frac{4}{2-d}\phi}\left[\delta\left(\mathcal{R}_{i\rho\sigma\tau}\mathcal{R}_{j}{}^{\rho\sigma\tau}\right) - \frac{1}{2(d-2)}\mathcal{R}_{\rho\sigma\lambda\tau}\mathcal{R}^{\rho\sigma\lambda\tau}\,h_{ij} - \right.\\ &- \left.\frac{1}{2(d-2)}\,g_{ij}\,\delta\left(\mathcal{R}_{\rho\sigma\lambda\tau}\mathcal{R}^{\rho\sigma\lambda\tau}\right)\right] + \frac{4}{d-2}\,\mathcal{R}_{ij}\,\delta\phi = 0.\end{aligned} \tag{19}$$

Using the explicit form of the Riemann tensor together with the variations (5) and (8–14), one can compute the terms in (18) and (19). From (18), the information one obtains is that one can consistently set $\delta\phi = 0$, as expected for a tensorial perturbation of a scalar field.

Collecting the several expressions, the result for (19) finally becomes

$$\begin{aligned}&\left(1 - 2\lambda\frac{f'}{r}\right)\frac{r^2}{f}\partial_t^2 H_T - \left(1 - 2\lambda\frac{g'}{r}\right)r^2 g\,\partial_r^2 H_T - \\ &- \left[(d-2)rg + \frac{1}{2}r^2\left(f' + g'\right) + 4\lambda(d-4)\frac{g\,(1-g)}{r} - 4\lambda g g' - \lambda r\left(f'^2 + g'^2\right)\right]\partial_r H_T + \\ &+ \left[\left(\ell\,(\ell + d - 3) - 2\right)\left(1 + \frac{4\lambda}{r^2}\,(1-g)\right) + 2(d-2) - 2(d-3)g - r\left(f' + g'\right) + \right.\\ &+ \left.\lambda\left(8\frac{1-g}{r^2} + 2\,(d-3)\,\frac{(1-g)^2}{r^2} - \frac{r^2}{d-2}\left[f'' + \frac{1}{2}\left(\frac{f'g'}{g} - \frac{f'^2}{f}\right)\right]^2\right)\right]H_T = 0.\end{aligned} \tag{20}$$

This is a second order partial differential equation for the perturbation function H_T. If we now divide (20) by $\left(1 - 2\lambda\frac{f'}{r}\right)\frac{r^2}{f}$, we obtain an equation of the form

$$\partial_t^2 H_T - F^2(r)\,\partial_r^2 H_T + P(r)\,\partial_r H_T + Q(r)\,H_T = 0. \tag{21}$$

For our purposes, we would like to re–write the above equation (21) in a more tractable form, as a Schrödinger–like master equation. In order to achieve so, we follow a procedure similar to the one in [9], defining a gauge–invariant "master variable" for the gravitational perturbation as

$$\Phi = k(r) H_T, \quad k(r) = \frac{1}{\sqrt{F}} \exp\left(- \int \frac{P}{2F^2}\, dr \right), \tag{22}$$

and replacing $\partial/\partial r$ by $\partial/\partial r_*$, r_* being the tortoise coordinate defined in this case by $dr_* = \frac{dr}{F(r)}$. It is then easy to see that an equation like (21) may be written as a master equation:

$$\frac{\partial^2 \Phi}{\partial r_*^2} - \frac{\partial^2 \Phi}{\partial t^2} = \left(Q + \frac{F'^2}{4} - \frac{FF''}{2} - \frac{P'}{2} + \frac{P^2}{4F^2} + \frac{PF'}{F} \right) \Phi \equiv V_{\mathsf{T}} \left[f(r), g(r) \right] \Phi, \tag{23}$$

with

$$\begin{aligned} V_{\mathsf{T}}[f(r), g(r)] &= \frac{1}{16 r^2 f g} \left[(16 \ell (\ell + d - 3) f^2 g + r^2 f^2 f'^2 + 3 r^2 g^2 f'^2 - 2 r^2 f (f + g) f' g' \right. \\ &- 4 r^2 f g (g - f) f'' + 16 r f g^2 f' + 4 r (d - 6) f^2 g f' \\ &+ \left. 4 (d - 2) r f^2 g g' + 4 (d - 4)(d - 2) f^2 g^2 \right] \\ &+ \frac{\lambda}{8 r^4 f g} \left[32 \ell (\ell + d - 3) f^2 (1 - g) g + 16 \ell (d + \ell - 3) f^2 g f' r \right. \\ &+ 3 r^3 g^2 f'^2 \left(f' - g' \right) - r^3 f^2 f'^2 \left(f' - g' \right) - 2 r^3 f g f' \left(f' - g' \right) g' \\ &+ 2 r^3 f g^2 \left(-3 f' f'' + 2 g' f'' + f' g'' \right) - 4 r^3 f^2 g f' \left(f'' - g'' \right) \\ &- 2 r^3 f^2 g g' \left(f'' - g'' \right) - 4 r^3 f^2 g^2 \left(f^{(3)} - g^{(3)} \right) \\ &+ 18 r^2 f g^2 f'^2 - 12 r^2 f^2 g f'^2 - 10 r^2 f^2 g g'^2 - 2 r^2 f g^2 f' g' \\ &+ 2 r^2 (4d - 13) f^2 g f' g' + 8 r^2 f^2 g^2 f'' + 8 (d - 5) r^2 f^2 g^2 g'' \\ &+ 4 r (d - 4)^2 f^2 g^2 (f' + g') + 8 r f^2 g^2 (g' - f') \\ &+ \left. 8 (d - 4) r f^2 g (f' + g' - 4 g g') + 16 (d - 5)(d - 4) f^2 g^2 (1 - g) \right]. \end{aligned} \tag{24}$$

Equation (24) gives the generic expression for the potential for tensor–type gravitational perturbations of any kind of static, spherically symmetric $\mathcal{R}^2$ string–corrected black hole in d–dimensions of the form (7). We are now ready to start studying scattering processes in the background of such a black hole.

4. Scattering by spherically symmetric α'–corrected black holes

The master equation (21) (or equivalently (20)), describing gravitational tensorial perturbations, is also the equation allowing for a study of scattering of tensor–type gravitational waves by the corresponding black hole solution (in our case, $d-$dimensional spherically symmetric black holes with string $\mathcal{R}^2$ corrections): the perturbation function can then be seen as a wave function. The master equation (21) also describes a minimally coupled massless scalar field on the the background of such black holes. In this case the integer $\ell = 0$ can be arbitrary: it does not need to be greater than one, as in the case of the tensorial perturbations.

Knowing the master equation, we are able to compute the absorption cross section, a quantity which is directly related to the greybody factors. A classical result in Einstein gravity is that, for any spherically symmetric black hole in arbitrary dimension, the absorption cross section of minimally coupled massless scalar fields is equal to the area of the black hole horizon [1], or equivalently $\sigma = 4S$, S being the Bekenstein–Hawking entropy. In order to extend such study to an effective theory with string $\mathcal{R}^2$ corrections, we shall use the technique of matching solutions,

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which was first developed for Einstein gravity in $d = 4$ in [10], and later extended to arbitrary d dimensions in [11]. That was also the technique which was used in [2], where for the first time black hole scattering with $\mathcal{R}^2$ α' corrections was studied. In that paper, a formula for the absorption cross–section was derived for a particular d–dimensional solution [7]. We are looking for a general formula for the absorption cross section, applicable to a general solution like (7). The idea of this technique is to separately solve the master equation above in different regions of the parameter r, where in each region we take a different approximation in order to simplify the equation.

We will be considering scattering at low frequencies, $R_H\omega \ll 1$. The low frequency requirement is necessary in order to use the technique of matching solutions: it is precisely when the wavelength of the scattered field is very large, compared to the radius of the black hole, that one can actually match solutions near the event horizon to solutions at asymptotic infinity [10, 11]. At low frequencies, only the mode with lowest angular momentum contributes to the cross section [1]. This is the context we will be considering; from now on, we will always take $\ell = 0$.

We assume that the solutions to the master equation are of the form $\Phi(r_*, t) = e^{i\omega t}\phi(r_*)$, such that $\frac{\partial \Phi}{\partial t} = i\omega\Phi$ (the same being valid for $H_T(r,t)$). This way the master equation looks like Schrödinger equation.

We consider a generic metric of the form of (7). We make the general assumption that the functions $f(r), g(r)$ have the form

$$f(r) = f_0(r)\left(1 + \frac{\lambda}{R_H^2} f_c(r)\right),\ g(r) = f_0(r)\left(1 + \frac{\lambda}{R_H^2} g_c(r)\right). \tag{25}$$

The function $f_0(r)$ is a solution to the classical Einstein equations, while the functions $f_c(r), g_c(r)$ encode the α' higher–derivative corrections.

The spherically symmetric solution to the vacuum Einstein equation in d dimensions is the Tangherlini solution, with

$$f_0(r) =: f_0^T(r) = 1 - \left(\frac{R_H}{r}\right)^{d-3}, \tag{26}$$

R_H being the horizon radius. For later convenience and application to more general black holes, we will allow for a multiplicative factor $c(r)$, which would encode string effects:

$$f_0(r) = c(r)\left(1 - \left(\frac{R_H}{r}\right)^{d-3}\right). \tag{27}$$

This will be the form of the function $f_0(r)$ we will be considering.

4.1. Scattering close to the event horizon

We start by solving the master equation near the black hole event horizon. In this region, the functions $f(r), g(r)$ from (25) have the form

$$f(r) \simeq f_0'(R_H)\left(1 + \frac{\lambda}{R_H^2} f_c(R_H)\right)(r - R_H),\ g(r) \simeq f_0'(R_H)\left(1 + \frac{\lambda}{R_H^2} g_c(R_H)\right)(r - R_H). \tag{28}$$

This means at the precise location of the horizon, the potential (24) vanishes; and as long as $\frac{r-R_H}{R_H} \ll (R_H\omega)^2$ one will have $V_\mathsf{T}(r) \ll \omega^2$ and in this near–horizon region one may neglect the potential $V_\mathsf{T}(r)$ in the master equation. One thus obtains, very close to the event horizon,

$$\left(\frac{d^2}{dr_*^2} + \omega^2\right)\Big(k(r)H_T(r)\Big) = 0. \tag{29}$$

In this same region, with f_0 given by (27), $k(r)$ may be taken as a constant. One also has $f_0(r) \simeq f_0'(R_H)\,(r-R_H)\,,\ f_0'(R_H) = \frac{(d-3)c(R_H)}{R_H}$, and

$$r_*(r) = \frac{R_H}{(d-3)c(R_H)}\left(1 - \frac{\lambda}{R_H^2}\frac{f_c(R_H)+g_c(R_H)}{2}\right)\log\left(\frac{r-R_H}{R_H}\right) + \mathcal{O}\left(r-R_H\right). \tag{30}$$

The solutions to (29) are plane waves. As we are interested in studying the absorption cross section, we shall consider the general solution for a purely incoming plane wave:

$$H_T(r_*) = A_{\text{near}} e^{i\omega r_*}. \tag{31}$$

Using (30) in (31), one finally obtains in this region

$$H_T(r) \simeq A_{\text{near}}\left(1 + i\frac{R_H\omega}{(d-3)c(R_H)}\left(1 - \frac{\lambda}{R_H^2}\frac{f_c(R_H)+g_c(R_H)}{2}\right)\log\left(\frac{r-R_H}{R_H}\right)\right). \tag{32}$$

4.2. Scattering at asymptotic infinity
We now analyze the solution to the master equation close to infinity.

In this article we consider asymptotically flat black holes which, at infinity, behave like flat Minkowski spacetime. This is equivalent to saying that, in the metric (7), functions $f(r), g(r)$ tend to the constant value 1 in the limit of very large r, and their derivatives tend to 0 in the same limit. From (24) we see that, asymptotically, the potential $V_{\mathsf{T}}(r)$ behaves at most as $1/r^2$, and therefore it vanishes in the limit $r \to \infty$.

In this limit, with vanishing potential, the master equation reduces to a simple free–field equation whose solutions are either incoming or outgoing plane–waves in the tortoise coordinate. One can also solve the master equation in the original radial coordinate in terms of Bessel functions, obtaining [1, 10, 11]

$$H_T(r) = (r\omega)^{(3-d)/2}\left[A\, J_{(d-3)/2}\,(r\omega) + B\, N_{(d-3)/2}\,(r\omega)\right].$$

At low–frequencies, with $r\omega \ll 1$, such solution becomes

$$H_T(r) \simeq A_{\text{asymp}}\,\frac{1}{2^{\frac{d-3}{2}}\Gamma\left(\frac{d-1}{2}\right)} + B_{\text{asymp}}\,\frac{2^{\frac{d-3}{2}}\Gamma\left(\frac{d-3}{2}\right)}{\pi\,(r\omega)^{d-3}} + \mathcal{O}\left(r\omega\right). \tag{33}$$

In order to compute the absorption cross–section, we will need to relate the coefficients A_{asymp} and B_{asymp} to A_{near}, obtained in (32). This can be done by the technique of matching near–horizon to asymptotic solutions, and requires us to solve the master equation in an intermediate region, between the event horizon and asymptotic infinity [10, 11]. This is what we will do in the following.

4.3. Scattering in the intermediate region
We now consider the intermediate region: far from the horizon, but not asymptotic infinity. We keep working in the low–frequency regime, but this time there are no restrictions to the magnitude of the potential, which may be large (but it is always regular, as one can see from (24)).

We want to solve the master equation or, equivalently, equation (21). Since we work perturbatively in λ, we define the expansion

$$H_T(r) = H_0(r) + \lambda H_1(r), k(r) = k_0(r) + \lambda k_1(r)$$

and the previous assumptions (25) for f, g. Taking the $\lambda = 0$ terms for f, g and from (24), we get from (23) the following equation for $H_0(r)$, written in the r coordinate (where if $\lambda = 0$ $\frac{d}{dr_*} = f_0 \frac{d}{dr}$):

$$\left[-f_0(r)\frac{d}{dr}\left(f_0(r)\frac{d}{dr}\right) + f_0(r)\left(\frac{(d-2)(d-4)f_0(r)}{4r^2} + \frac{(d-2)f_0'(r)}{2r}\right)\right]\left(r^{\frac{d-2}{2}}H_0(r)\right) = 0, \quad (34)$$

whose most general solution is[1]

$$H_0(r) = A^0_{\text{inter}} + B^0_{\text{inter}} \int \frac{d\,r}{r^{d-2} f_0(r)}. \quad (35)$$

In order to solve for $H_1(r)$, we take for F, P, Q similar expansions as we did for H_T, k : $F = F_0 + \lambda F_1, P = P_0 + \lambda P_1, Q = Q_0 + \lambda Q_1$. We then expand every term of (21). To zero order in λ we obtain

$$H_0'' - \frac{P_0}{F_0^2} H_0' - \frac{Q_0}{F_0^2} H_0 = 0, \quad (36)$$

which is completely equivalent to (34), with solution (35).

The terms of first order in λ are $-F_0^2 H_1'' - F_1^2 H_0'' + P_0 H_1' + P_1 H_0' + Q_0 H_1 + Q_1 H_0$, which may be rewritten as

$$H_1'' - \frac{P_0}{F_0^2} H_1' - \frac{Q_0}{F_0^2} H_1 = R(r), \; R(r) = -\left(\frac{F_1}{F_0}\right)^2 H_0'' + \frac{P_1}{F_0^2} H_0' + \frac{Q_1}{F_0^2} H_0 \quad (37)$$

This is a second–order linear nonhomogeneous differential equation for H_1. The homogeneous part is exactly the same as the differential equation (36) for H_0, with general solution (35), replacing $H_0(r), A^0_{\text{inter}}, B^0_{\text{inter}}$ by $H_1(r), A^1_{\text{inter}}, B^1_{\text{inter}}$.

According to the method of variation of constants, a particular solution to the nonhomogeneous equation (37) is given by

$$H_1^{\text{part}}(r) = v_1(r) + v_2(r) \int \frac{d\,r}{r^{d-2} f_0(r)}. \quad (38)$$

To obtain the most general solution to (37) one just needs to add to $H_1^{\text{part}}(r)$ the most general solution (35) to the homogeneous equation (36), including the contributions H_0, H_1 as $H = H_0 + \lambda H_1$:

$$H(r) = (A_{\text{inter}} + \lambda v_1(r)) + (B_{\text{inter}} + \lambda v_2(r)) \int \frac{d\,r}{r^{d-2} f_0(r)} = A_{\text{inter}} + B_{\text{inter}} \int \frac{d\,r}{r^{d-2} f_0(r)} + \lambda H_1^{\text{part}}(r). \quad (39)$$

To summarize: we were able to solve the master equation in the intermediate region. This is a linear nonhomogeneous equation; for its general solution, we should add to the solution to the homogeneous equation a particular solution $H_1^{\text{part}}(r)$, which we found by the method of variation of constants. We checked that this particular solution $H_1^{\text{part}}(r)$ vanishes at infinity and at the black hole horizon; close to these regions, we can ignore $H_1^{\text{part}}(r)$ and simply consider the solution to the homogeneous equation. This will be a key feature for the matching process.

[1] The integrals in this subsection are all meant to be indefinite.

4.4. Calculation of the absorption cross section

We are now ready to start the matching process, using f_0 given by (27).

If we evaluate (39) near the horizon, we obtain

$$H_T(r) \simeq A_{\text{inter}} + \frac{B_{\text{inter}}}{(d-3)R_H^{d-3}c\,(R_H)} \log\left(\frac{r-R_H}{R_H}\right) + \cdots . \tag{40}$$

Evaluating (39) at asymptotic infinity, one may match the coefficients above to the ones in (33), yielding

$$\begin{aligned} A_{\text{asymp}} &= 2^{\frac{d-3}{2}}\Gamma\left(\frac{d-1}{2}\right)A_{\text{inter}} = 2^{\frac{d-3}{2}}\Gamma\left(\frac{d-1}{2}\right)A_{\text{near}}, \\ B_{\text{asymp}} &= -\frac{\pi\omega^{d-3}}{2^{\frac{d-3}{2}}(d-3)\Gamma\left(\frac{d-3}{2}\right)}B_{\text{inter}} = -\frac{i\pi\,(R_H\omega)^{d-2}}{2^{\frac{d-1}{2}}\Gamma\left(\frac{d-1}{2}\right)}\left(1-\frac{\lambda}{R_H^2}\frac{f_c(R_H)+g_c(R_H)}{2}\right)A_{\text{near}} \end{aligned} \tag{41}$$

Computing the low frequency absorption cross section is now a simple exercise in scattering theory [10, 11]. Near the black hole event horizon, from (31), the incoming flux per unit area is

$$J_{\text{near}} = \frac{1}{2i}\left(H_T^\dagger(r_*)\frac{dH_T}{dr_*} - H_T(r_*)\frac{dH_T^\dagger}{dr_*}\right) = \omega\left|A_{\text{near}}\right|^2 . \tag{42}$$

The outgoing flux per unit area at asymptotic infinity, where r_* and r coincide, is, from (33),

$$J_{\text{asymp}} = \frac{1}{2i}\left(H_T^\dagger(r)\frac{dH_T}{dr} - H_T(r)\frac{dH_T^\dagger}{dr}\right) = \frac{2}{\pi}r^{2-d}\omega^{3-d}\left|A_{\text{asymp}}B_{\text{asymp}}\right| . \tag{43}$$

In order to compute the cross section, this same flux per unit area at asymptotic infinity must be integrated over a sphere of (large) radius r, and the result should be divided by the incoming flux per unit area:

$$\sigma = \frac{\int r^{d-2}J_{\text{asymp}}d\Omega_{d-2}}{J_{\text{near}}} = \frac{2}{\pi}\omega^{2-d}\frac{\left|A_{\text{asymp}}B_{\text{asymp}}\right|}{\left|A_{\text{near}}\right|^2}\Omega_{d-2}. \tag{44}$$

Replacing the results from (41), the final result is

$$\sigma = A_H\left(1-\frac{\lambda}{R_H^2}\;\frac{f_c(R_H)+g_c(R_H)}{2}\right), \tag{45}$$

where $A_H = R_H^{d-2}\Omega_{d-2}$ is the horizon area.

4.5. The α' corrections to the temperature

The temperature T of a black hole given by a metric of the form (7) is given by $T = \lim_{r\to R_H}\frac{\sqrt{g}}{2\pi}\frac{d\sqrt{f}}{dr}$. In the case f, g are given by (25), this temperature comes as

$$T = \frac{f_0'(R_H)}{4\pi}\left(1+\frac{\lambda}{R_H^2}\;\frac{f_c(R_H)+g_c(R_H)}{2}\right). \tag{46}$$

We see that the α' correction to the temperature is the same we obtained to the absorption cross section in (45), but with opposite sign: when one of these quantities increases, the other one decreases by the same magnitude. This means the product σT does not get α' corrections to first order.

5. Application to concrete string-corrected black hole solutions

We now apply our result to the computation of the absorption cross section for a few specific black hole solutions in string theory. Although our result can of course be applied to concrete solutions in specific d dimensions, we prefer to consider in this article only solutions in which d remains arbitrary.

5.1. The Callan–Myers–Perry black hole

The Callan–Myers–Perry solution was the first d–dimensional black hole solution with $\mathcal{R}^2$ corrections to be obtained (in [7]). It is a simple generalization of the Tangherlini solution of the form (25), with $f_0 = f_0^T$ given by (26) and

$$f_c(r) = g_c(r) = f_c^{CMP}(r) := -\frac{(d-3)(d-4)}{2} \left(\frac{R_H}{r}\right)^{d-3} \frac{1 - \left(\frac{R_H}{r}\right)^{d-1}}{1 - \left(\frac{R_H}{r}\right)^{d-3}}, \tag{47}$$

from which we obtain, using (45), the absorption cross section [2]

$$\sigma^{CMP} = A_H \left(1 + \frac{(d-1)(d-4)}{2} \frac{\lambda}{R_H^2}\right). \tag{48}$$

5.2. The string-corrected dilatonic d–dimensional black hole

The Callan–Myers–Perry solution expresses the effect of the string $\mathcal{R}^2$ corrections, but it does not consider any other string effects, namely the fact that string theories live in d_S spacetime dimensons ($d_S = 10$ or 26 on heterotic or bosonic strings, respectively), and have to be compactified to d dimensions on a $d_S - d$–dimensional manifold. When passing from the string to the Einstein frame, the volume of the compactification manifold becomes spatially varying. In the simple case when such manifold is a flat torus, that volume depends only on the $d-$dimensional part of the dilaton ϕ and, after solving the α'–corrected field equation (19) the metrics of the compactification manifold and of the d–dimensional spacetime decouple.

The explicit solution was worked out in [13]. The general solution for the dilaton, in the background of the spherically symmetric Tangherlini black hole (26), is necessarily of order λ : $\phi(r) := \frac{\lambda}{R_H^2}\varphi(r)$, with $\varphi(r)$ given by

$$\begin{aligned}\varphi(r) \;&=\; \frac{(d-2)^2}{4}\left[\ln\left(1-\left(\frac{R_H}{r}\right)^{d-3}\right) - \frac{d-3}{2}\left(\frac{R_H}{r}\right)^2 - \frac{d-3}{d-1}\left(\frac{R_H}{r}\right)^{d-1}\right.\\ &+\; \left. B\left(\left(\frac{R_H}{r}\right)^{d-3}; \frac{2}{d-3}, 0\right)\right],\end{aligned} \tag{49}$$

$B(x;\, a, b) = \int_0^x t^{a-1}\,(1-t)^{b-1}\, dt$ being the incomplete Euler beta function.

The d–dimensional part of the metric is of the form (7), with f, g given by (25), $f_0 = f_0^T$ given by (26) and

$$g_c(r) = f_c^{CMP}(r),\ f_c(r) = f_c^{CMP}(r) + 4\frac{d_s - d}{(d_s-2)^2}\left(\varphi - r\varphi'\right). \tag{50}$$

Evaluating these functions at the horizon allows us to obtain, again from (45) [2],

$$\begin{aligned}\sigma &= A_H\Bigg(1+\Bigg(\frac{(d-1)(d-4)}{2}\\ &+ \frac{d_s-d}{(d_s-2)^2}\frac{(d-2)^2}{4(d-1)}\left(3d^2-6d-1+2(d-1)\left(\psi^{(0)}\left(\frac{2}{d-3}\right)+\gamma\right)\right)\Bigg)\ \frac{\lambda}{R_H^2}\Bigg). \qquad (51)\end{aligned}$$

We have numerically evaluated the λ–correction for the cross section: it is always positive, for every relevant value of d.

5.3. The double–charged black hole

In article [14] one can find black holes in any dimension formed by a fundamental string compactified on an internal circle with any momentum n and winding w, both at leading order and with leading α' corrections. One starts with the Callan–Myers–Perry solution in the string frame [7], which is of the form (7), with f, g replaced by f_S^{CMP}, g_S^{CMP}, given by

$$\begin{aligned}f_S^{CMP}(r) &= f_0^T\left(1+2\frac{\lambda}{R_H^2}\mu(r)\right),\\ g_S^{CMP}(r) &= f_0^T\left(1-2\frac{\lambda}{R_H^2}\epsilon(r)\right),\\ \epsilon(r) &= \frac{d-3}{4}\frac{\left(\frac{R_H}{r}\right)^{d-3}}{1-\left(\frac{R_H}{r}\right)^{d-3}}\left[\frac{(d-2)(d-3)}{2}-\frac{2(2d-3)}{d-1}+(d-2)\left(\psi^{(0)}\left(\frac{2}{d-3}\right)+\gamma\right)\right.\\ &+ \left. d\left(\frac{R_H}{r}\right)^{d-1}+\frac{4}{d-2}\varphi(r)\right],\\ \mu(r) &= -\epsilon(r)+\frac{2}{d-2}(\varphi(r)-r\varphi'(r)). \qquad (52)\end{aligned}$$

f_0^T is given by (26) and $\varphi(r)$ is given by (49).

This metric is lifted to an additional dimension by adding an extra coordinate, taken to be compact (this means to produce a uniform black string). One then performs a boost along this extra direction, with parameter α_w, and T–dualizes around it (to change string momentum into winding), obtaining a $(d+1)$–dimensional black string winding around a circle. Finally one boosts one other time along this extra direction, with parameter α_p, in order to add back momentum charge. One finally obtains a spherically symmetric black hole in d dimensions with two electrical charges.

The whole process is worked out in detail in [14]; the final metric, in the Einstein frame, is of the form (7), with f, g given by (25), but this time with f_0 given by

$$f_0^I = \frac{f_0^T}{\sqrt{\Delta(\alpha_n)\Delta(\alpha_w)}},\ \Delta(x) := 1+\left(\frac{R_H}{r}\right)^{d-3}\sinh^2 x,$$

[2] The digamma function is given by $\psi(z) = \Gamma'(z)/\Gamma(z)$, $\Gamma(z)$ being the usual Γ function. For positive n, one defines $\psi^{(n)}(z) = d^n\,\psi(z)/d\,z^n$. This definition can be extended for other values of n by fractional calculus analytic continuation. These are meromorphic functions of z with no branch cut discontinuities.

γ is Euler's constant, defined by $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^n \frac{1}{k} - \ln n\right)$, with numerical value $\gamma \approx 0.577216$.

f_0^T being given by (26). This is clearly a metric of the type (27), with $c(r) = \frac{1}{\sqrt{\Delta(\alpha_n)\Delta(\alpha_w)}}$. f_c, g_c are given by

$$\begin{aligned} f_c^I(r) &= \frac{1}{2\Delta(\alpha_n)\Delta(\alpha_w)}\Big(2\left(2 - f_0^T\right)\left(\Delta(\alpha_n)\sinh^2(\alpha_w) + \Delta(\alpha_w)\sinh^2(\alpha_n)\right)\mu(r) \\ &+ 4\left(1 - \left(\frac{R_H}{r}\right)^{2(d-3)}\sinh^2(\alpha_w)\sinh^2(\alpha_n)\right)\mu(r) \\ &+ (d-3)^2 f_0^T\left(\frac{R_H}{r}\right)^{2(d-2)}\sinh^2(\alpha_w)\sinh^2(\alpha_n) - 4\Delta(\alpha_n)\Delta(\alpha_w)\varphi(r)\Big), \\ g_c^I(r) &= \frac{1}{2\Delta(\alpha_n)\Delta(\alpha_w)}\Big(2\left(\Delta(\alpha_n)\sinh^2(\alpha_w) + \Delta(\alpha_w)\sinh^2(\alpha_n)\right)\mu(r) f_0^T \\ &+ (d-3)^2 f_0^T\left(\frac{R_H}{r}\right)^{2(d-2)}\sinh^2(\alpha_w)\sinh^2(\alpha_n) + 4\Delta(\alpha_n)\Delta(\alpha_w)\left(\varphi(r) - \epsilon(r)\right)\Big) \end{aligned} \tag{53}$$

After determining the respective limits when $r \to R_H$, (45) allows us to obtain

$$\sigma = A_H\left(1 + \frac{\lambda}{R_H^2}\frac{3d^3 - 16d^2 + 19d - 2 + 2(d-2)(d-1)\left(\psi^{(0)}\left(\frac{2}{d-3}\right) + \gamma\right)}{4(d-1)}\right). \tag{54}$$

We have again numerically evaluated the λ–correction for the cross section: like in the previous case, it is always positive, for every relevant value of d.

5.4. *Comparison with the entropy*

As we have seen, in classical Einstein gravity the low frequency limit of the absorption cross section of minimally coupled massless fields, for any spherically symmetric black hole in arbitrary d dimensions, equals the area of the black hole horizon [1]. In terms of a physical quantity, the Bekenstein–Hawking entropy, this statement may be written as $\sigma|_{\alpha'=0} = 4G\, S|_{\alpha'=0}$.

It is an interesting physical question to figure out if such relation is preserved in the presence of α' corrections, i.e. to verify if the corrections to the cross sections we have been obtaining and to the black hole entropy are the same. The α'–corrected entropy can be obtained through Wald's formula

$$S = -2\pi\int_H \frac{\partial\mathcal{L}}{\partial R^{\mu\nu\rho\sigma}}\varepsilon^{\mu\nu}\varepsilon^{\rho\sigma}\sqrt{h}\,d\,\Omega_{d-2}, \tag{55}$$

$\mathcal{L}$ being the α'–corrected lagrangian (15) and H the black hole horizon, with area $A_H = R_H^{d-2}\Omega_{d-2}$ and metric h_{ij} induced by the spacetime metric $g_{\mu\nu}$. This entropy has been computed for a generic metric of the form (7) with first order α' corrections in [13, 12], the result being

$$S = \frac{1}{4G}\int_H\left(1 + \frac{\lambda}{R_H^2}(d-3)(d-2)\right)\sqrt{h}\,d\,\Omega_{d-2} = \frac{A_H}{4G}\left(1 + (d-3)(d-2)\frac{\lambda}{R_H^2}\right). \tag{56}$$

From the cases we have studied, we conclude that the relation $\sigma = 4GS$ is not verified, at least for a generic black hole solution, in the presence of α' corrections.

6. Conclusions

We have obtained a general formula for the low frequency absorption cross section for spherically symmetric d–dimensional black holes with leading α' corrections in string theory, which we applied to three different known black hole solutions. We have compared the values of the α' corrections to the cross section with those for the black hole entropy, having obtained different results. There are examples where such agreement has been found for supersymmetric black holes in $d = 4$ and 5 (for a discussion see [12]); the cases we have studied are all nonsupersymmetric and in generic d. It is important to figure out in general when such agreement is verified or not: does it depend on the spacetime dimension or on the amount of supersymmetry preserved by the black hole solution? These are topics which currently keep being researched.

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