On the global well-posedness of discrete Boltzmann systems with chemical reaction

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Abstract

We study an initial-boundary value problem in one-space dimension for the discrete Boltzmann equation extended to a diatomic gas undergoing both elastic multiple collisions and chemical reactions. By integration of conservation equations, we prove a global existence result in the half-space for small initial data $N_0 \in \mathcal{B}_+^1 \cap L^1$.

1 Introduction

In recent years, increasing attention has been devoted to Boltzmann type models for gases with inelastic molecular interactions, due to the presence of chemically reacting mixtures in several applications as combustion engines, chemical reactors and plasma physics. Let us quote, for example, papers [6], [7], [16] in the context of the full Boltzmann Equation (BE), and papers [8], [11], [13], [15] in the context of the Discrete Boltzmann Equation (DBE). This discrete model consists of a system of hyperbolic semi-linear partial differential equations describing the time-space evolution of a gas whose particles can attain a finite number of preselected velocities only ([5]). For what concerns the initial-boundary value problem within the DBE, a rather vast bibliography is available in the context of inert gases (see for instance [4], [9], [10]). However, only few results are available in the literature concerning reacting gases. In this context, in [17], an existence result is proven for a particular model concerning a chemically active gas ([11]).

As pointed out in [1], in spite of the simple mathematical structure of the DBE, the developments of the last decades have shown that the results on existence and uniqueness

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of solutions are, generally, weaker than the corresponding ones for the BE. Nevertheless, when dealing with fluid dynamical applications and chemistry and physics of moderately dense gases, models of DBE provide an accurate description of the flow pattern, quite suitable for numerical implementations. This justifies the renewed interest in models of DBE extended to chemically reacting gases, several years after the pioneering papers of Broadwell [2] and Cabannes [3] that constituted a breakthrough in the study of the propagation of shock waves within one component inert gases.

In fact, due to the simple structure of the discrete collision operator and the increasing computer capacities, several numerical simulations have been implemented with the aim of studying rather complex processes. In particular, in [14], the steady detonation problem has been solved by means of the kinetic model of [13] for gases undergoing autocatalytic reversible reactions. In this model, the particles can experience binary and triple elastic collisions and chemical reactions. The associated initial-value problem has been sucessfully studied from a numerical point of view in [12], although no existence theorem is available.

In the present paper, we prove the global wellposedness of the initial boundary value problem in one space dimension for the discrete model of [13].

The inclusion of multiple collisions with chemical reactions represents an additional difficulty, since cubic terms appear in the collision operator and the conservation laws differ from the ones of classical inert gases.

We continue this introduction by presenting the model.

Kinetic equations

$$
\begin{cases}
\dfrac{\partial}{\partial t} N_i^A(x,t) + v_i \dfrac{\partial}{\partial x} N_i^A(x,t) = F_i^A(N(x,t)) & (i \in \{1;2;3\}), \\[2ex]
\dfrac{\partial}{\partial t} N_i^{A_2}(x,t) + \dfrac{v_i}{2} \dfrac{\partial}{\partial x} N_i^{A_2}(x,t) = F_i^{A_2}(N(x,t)) & (i \in \{1;2;3\}), \\[2ex]
\dfrac{\partial}{\partial t} N_i^{A^*}(x,t) + v_i \dfrac{\partial}{\partial x} N_i^{A^*}(x,t) = F_i^{A^*}(N(x,t)) & (i \in \{1;3\}),
\end{cases}
\tag{1}
$$

where $(x,t) \in [0;+\infty[\times [0;+\infty[$, and the other terms have the following meaning:

- $N = \left(N_1^A, N_2^A, N_3^A, N_1^{A^*}, N_3^{A^*}, N_1^{A_2}, N_2^{A_2}, N_3^{A_2}\right)$ is the vector number density associated to particles of a diatomic gas constituted by atoms A, accelerated atoms A^* and molecules A_2, subjected to autocatalytic chemical reaction of type

$$
A_2 + M \rightleftharpoons A + A^* + M , \quad M = A, A^*, A_2
\tag{2}
$$
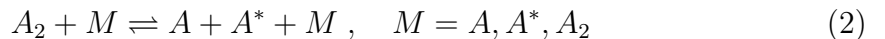
  M being the catalyst of the reaction;

- $(v_1, v_2, v_3) = (c, 0, -c)$, where c is a positive constant defined in terms of the atom mass m and molecular bond energy ϵ of the diatomic molecule A_2, through the relation $c = \sqrt{(4\epsilon)/(5m)}$; for sake of simplicity, we will assume $m = 1$;

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- the collision terms F_i^M are defined by polynomials of the form

$$F_i^M(N) = \left(P_{i,M}^{(1)}(N) + P_{i,M}^{(2)}(N) \right) - N_i^M \left(Q_{i,M}^{(1)}(N) + Q_{i,M}^{(2)}(N) \right),$$

which describe the elastic scattering and chemical mechanism among particles; the polynomials $P_{i,M}^{(k)}$, $Q_{i,M}^{(k)}$ are homogeneous with positive coefficients, $P_{i,M}^{(1)}$ and $Q_{i,M}^{(1)}$ of degree 2, $P_{i,M}^{(2)}$ and $Q_{i,M}^{(2)}$ of degree 3 and 1, respectively. In particular, $P_{i,M}^{(1)}$ and $P_{i,M}^{(2)}$ represent the rate of creation of particles of species M with velocity along i-direction due to inert or reactive collisions of binary and triple type, respectively. On the other hand, the terms $N_i^M Q_{i,M}^{(1)}$ and $N_i^M Q_{i,M}^{(2)}$ represent the rate of disappearance of particles of species M with velocity along i-direction due to inert or reactive collisions of triple and binary type, respectively.

Let us observe that inert collisions involve the same number of incoming and outgoing particles, *i.e.* two or three particles in correspondence of binary or triple collisions, respectively. Conversely, collisions with chemical reaction (2) lead to the formation of two atoms A, A^* and one M-particle when a molecule A_2 dissociates in the presence of the catalist M. Therefore, a triple reactive source term corresponds to a binary reactive sink term, and *vice-versa*. The detailed evaluation of the terms $F_i^M(N)$ is provided in paper [13].

The particular form of $F_i^M(N)$ considered in the present paper is appropriate for all those elementary reactions of catalytic type ocurring, for example, in the brain chain of the Hydrogen–Oxygen and Methane combustion mechanisms.

The other case of elementary bimolecular reaction, including only binary encounters and then only quadratic collision terms, has been considered in paper [17], but the existence result is much weaker than the one of the present paper.

Global chemical reactions involving a greater number of gas particles can be considered but, in general, they decompose in a brain chain of elementary reactions of bimolecular and autocatalytic type. In this case, the collision terms obviously become rather cumbersome, but still polynomials, in the unknown number densities, whose degree is equal to the maximum number of the colliding particles. The technique exploited here can also be applied, provided that the reactions are two–way.

Conservation equations

We assume that (1) possesses the following independent conservation laws (see [13] for details), corresponding, respectively, to the conservation of partial number densities of species $A - A_2$, $A^* - A_2$ and the total momentum of the system in the x-direction.

$$\frac{\partial}{\partial t} \left(\sum_{i=1}^{3} (N_i^A + N_i^{A_2}) \right) + \frac{\partial}{\partial x} \left(\sum_{i=1}^{3} v_i (N_i^A + \frac{1}{2} N_i^{A_2}) \right) = 0 \tag{3}$$

$$\frac{\partial}{\partial t}\left(\sum_{i=1}^{3} N_i^{A_2} + \sum_{i=1,3} N_i^{A^*}\right) + \frac{\partial}{\partial x}\left(\sum_{i=1}^{3} \frac{1}{2} v_i N_i^{A_2} + \sum_{i=1,3} v_i N_i^{A^*}\right) = 0 \qquad (4)$$

$$\frac{\partial}{\partial t}\left(\sum_{i=1}^{3} v_i(N_i^A + N_i^{A_2}) + \sum_{i=1,3} v_i N_i^{A^*}\right)$$
$$+ \frac{\partial}{\partial x}\left(\sum_{i=1}^{3} v_i^2(N_i^A + \frac{1}{2} N_i^{A_2}) + \sum_{i=1,3} v_i^2 N_i^{A^*}\right) = 0. \qquad (5)$$

Initial-boundary value problem

We consider the initial-boundary value problem associated to Eqs. (1) with initial data

$$N_i^M(x,0) = N_{i_0}^M(x). \qquad (6)$$

where N_{i_0} are prescribed functions of $x \geq 0$.

We complete the I.B.V.P. (1)-(6) with the following boundary conditions:

$$\begin{pmatrix} N_1^A(0,t) \\ N_1^{A^*}(0,t) \\ N_1^{A_2}(0,t) \end{pmatrix} = \begin{pmatrix} \beta_A^A & \beta_{A^*}^A & \beta_{A_2}^A \\ \beta_A^{A^*} & \beta_{A^*}^{A^*} & \beta_{A_2}^{A^*} \\ \beta_A^{A_2} & \beta_{A^*}^{A_2} & \beta_{A_2}^{A_2} \end{pmatrix} \begin{pmatrix} N_3^A(0,t) \\ N_3^{A^*}(0,t) \\ N_3^{A_2}(0,t) \end{pmatrix}, \quad \forall t > 0, \qquad (7)$$

with

$$\sum_{M'} \beta_{M'}^M \leq 1, \quad \beta_M^A + \beta_M^{A^*} + \frac{1}{2}\beta_M^{A_2} \leq \delta_M, \qquad (8)$$

where $\delta_M = 1$ if $M = A, A^*$ and $\delta_{A_2} = \frac{1}{2}$.

Conditions (7) mean that the contribution to the incoming flux of each fixed species A, A^*, A_2 on the boundary at $x = 0$ is given by a fraction of outgoing molecules of the same species that have been reflected back from the boundary, and a fraction of molecules of other species that have been reacted.

Moreover, condition (8) assures that the macroscopic flow of the gas is not inward to the boundary, since

$$\begin{aligned} \text{Netflux} &= \sum_{i=1,3} v_i \left[N_i^A(0,t) + N_i^{A^*}(0,t) + \frac{1}{2} N_i^{A_2}(0,t) \right] \\ &= \sum_M \left(\beta_M^A + \beta_M^{A^*} + \frac{1}{2} \beta_M^{A_2} - \delta_M \right) N_3^M(0,t) \\ &\leq 0 \end{aligned}$$

Compatibility conditions

Finally, we suppose that the initial data satisfies the following compatibility conditions:

$$
\begin{pmatrix} N_{1_0}^A(0) \\ N_{1_0}^{A^*}(0) \\ N_{1_0}^{A_2}(0) \end{pmatrix} = \begin{pmatrix} \beta_A^A & \beta_{A^*}^A & \beta_{A_2}^A \\ \beta_A^{A^*} & \beta_{A^*}^{A^*} & \beta_{A_2}^{A^*} \\ \beta_A^{A_2} & \beta_{A^*}^{A_2} & \beta_{A_2}^{A_2} \end{pmatrix} \begin{pmatrix} N_{3_0}^A(0) \\ N_{3_0}^{A^*}(0) \\ N_{3_0}^{A_2}(0) \end{pmatrix}
\tag{9}
$$

and

$$
\begin{pmatrix} \dfrac{d}{dx} N_{1_0}^A(0) \\[2mm] \dfrac{d}{dx} N_{1_0}^{A^*}(0) \\[2mm] \dfrac{d}{dx} N_{1_0}^{A_2}(0) \end{pmatrix} = \begin{pmatrix} \beta_A^A & \beta_{A^*}^A & \beta_{A_2}^A \\ \beta_A^{A^*} & \beta_{A^*}^{A^*} & \beta_{A_2}^{A^*} \\ \beta_A^{A_2} & \beta_{A^*}^{A_2} & \beta_{A_2}^{A_2} \end{pmatrix} \begin{pmatrix} \dfrac{d}{dx} N_{3_0}^A(0) \\[2mm] \dfrac{d}{dx} N_{3_0}^{A^*}(0) \\[2mm] \dfrac{d}{dx} N_{3_0}^A(0) \end{pmatrix}.
\tag{10}
$$

We end this introduction with a few notations:

For $N(x,t) = \left(N_1^A(x,t), \ldots, N_3^{A_2}(x,t) \right)$ and $X \subset \mathbb{R}^+ \times \mathbb{R}^+$, we put

$$
L^\infty(X) = \left\{ N / \|N\|_{X,\infty} = \max_{i,M} \operatorname*{ess\,sup}_{(x,t)\in X} |N_i^M(x,t)| < +\infty \right\}
$$

Also, we define the space of bounded continuously differentiable functions with bounded derivatives:

$$
\mathcal{B}^1(X) = \left\{ f \in C^1(X) \;/\; f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial t} \in L^\infty(X) \right\}
$$

and we denote by $\mathcal{B}_+^1(X)$ its positive cone, composed of all positive functions of $\mathcal{B}^1(X)$.

As usual, for $N_0(x) = \left(N_{1_0}^A(x), \ldots, N_{3_0}^{A_2}(x) \right)$, we introduce the space

$$
L^1(\mathbb{R}^+) = \left\{ N_0 / \|N_0\|_1 = \sum_{i,M} \int_0^{+\infty} |N_{i_0}^M(x)| d\mu(x) < +\infty \right\}.
$$

Finally, for $T > 0$ and $N \in L^\infty([0;+\infty[\times[0;T])$, we introduce the upper bound

$$
E(T) = \max_{i,M} \sup_{(x,t)\in\mathbb{R}^+\times[0;T]} |N_i^M(x,t)| \;,\;\; E_0 = E(0),
$$

and for $N_0 \in L^1(\mathbb{R}^+)$, we put

$$
m_0 = \int_0^{+\infty} \left(\sum_{i=1}^3 (N_{i_0}^A + 2N_{i_0}^{A_2}) + \sum_{i=1,3} N_{i_0}^{A^*} \right)(x)\, d\mu(x)
$$

corresponding to the initial mass of the system.

2 Local solutions

Theorem 2.1
Let $N_o \in \mathcal{B}^1_+([0; +\infty[)$ satisfying the compatibility conditions (9),(10). Then there exists a unique solution

$$N \in \mathcal{B}^1_+([0; +\infty[\times[0; T_o])$$

for the mixed-problem (1),(6),(7),where the life-span $T_o > 0$ depends exclusively on

$$E_o = \max_{i,M} \sup_{x \in \mathbb{R}^+} |N^M_{i_o}(x)|.$$

Proof of Theorem 2.1:
The proof of this theorem follows the lines of Theorem 3.1 in [9], where the existence of local solutions for a dicrete Boltzmann model with binary collisions is treated.
For self-completeness, we sketch here the above mentioned proof:

For convenience, we put

$$v^M_i = v_i \text{ if } M = A, A^* \quad \text{and} \quad v^{A_2}_i = \frac{v_i}{2}.$$

For some parameter $\lambda > 0$, we put (1) in the form

$$\frac{\partial}{\partial t} N^M_i(x,t) + v^M_i \frac{\partial}{\partial x} N^M_i(x,t) + \lambda N^M_i(x,t) = F^{M,\lambda}_i(N(x,t)), \tag{11}$$

where

$$F^{M,\lambda}_i(N(x,t)) = F^M_i(N(x,t)) + \lambda N^M_i(x,t).$$

Let $(x,t) \in [0; +\infty[\times[0; +\infty[$.

If $x - v^M_i t \geq 0$, integrating (11) along the v^M_i-characteristic through (x,t) leads to

$$N^M_i(x,t) = e^{-\lambda t} N_{i_o}(x - v^M_i t) + \int_0^t e^{-\lambda(t-\tau)} F^{M,\lambda}_i(N)(x - v^M_i(t-\tau), \tau)d\tau. \tag{12}$$

If $x - v^M_i t < 0$, one gets

$$N^M_i(x,t) = e^{-\lambda(t-t^M_i)} N^M_i(0, t^M_i) + \int_{t^M_i}^t e^{-\lambda(t-\tau)} F^{M,\lambda}_i(N)(x - v^M_i(t-\tau), \tau)d\tau, \tag{13}$$

where $t^M_i = t - \dfrac{x}{v^M_i}$ is the intersection of the v^M_i-characteristic with the boundary $x = 0$.

Noticing that in this case $v^M_i > 0$ $(i = 1)$, condition (7) yields

$$N^M_i(x,t) = e^{-\lambda(t-t^M_i)} \sum_{M'} \beta^M_{M'} N^{M'}_3(0, t^M_i) + \int_{t^M_i}^t e^{-\lambda(t-\tau)} F^{M,\lambda}_i(N)(x - v^M_i(t-\tau), \tau)d\tau. \tag{14}$$

Finally, integrating along the $v_3^{M'}$-characteristic through $(0, t_i^M)$,

$$
\begin{aligned}
N_i^M(x,t) = \ & e^{-\lambda t} \sum_{M'} \beta_{M'}^M e^{-\lambda t_i^M} N_{3_0}^{M'}(-v_3^{M'} t_i^M) \\
& + e^{-\lambda t} \sum_{M'} \beta_{M'}^M \int_0^{t_i^M} e^{-\lambda(t_i^M - \tau)} F_3^{M',\lambda}(N)(x - v_3^{M'}(t-\tau), \tau) d\tau \\
& + \int_{t_i^M}^t e^{-\lambda(t-\tau)} F_i^{M,\lambda}(N)(x - v_i^M(t-\tau), \tau) d\tau.
\end{aligned}
$$

Now, for $T > 0$, we put $\Omega_T = [0; +\infty[\times [0;T]$, $\Omega_{M,i,T}^+ = \{(x,t) \in \Omega_T \; ; \; x \geq v_i^M t\}$ and $\Omega_{M,i,T}^- = \{(x,t) \in \Omega_T \; ; \; x < v_i^M t\}$.

We introduce the Banach space

$$
X^1(T) = \{N = (N_i^M)_{i,M} \in C^o(\Omega_T) \cap L^\infty(\Omega_T) \; ; \; \frac{\partial N_i^M}{\partial x} \in C^o(\Omega_T) \cap L^\infty(\Omega_T)\}
$$

equipped with the norm

$$
\|N\|_{1,T} = \|N\|_{\infty,\Omega_T} + \|\frac{\partial N}{\partial x}\|_{\infty,\Omega_T} \text{ where } \|N\|_{\infty,\Omega_T} = \max_{i,M} \sup_{(x,t)\in\Omega_T} |N_{i,M}(x,t)|.
$$

For $E, G > 0$ we consider the closed convex subset $S(T, E, G)$ of $X^1(T)$ formed by all functions $N \in X^1(T)$ such that for all i, M,

$$
N_i^M(x,0) = N_{i_o}^M(x) \; , \; x \in [0; +\infty[
$$

and

$$
0 \leq N_i^M(x,t) \leq E \; , \; |\frac{\partial N_i^M}{\partial x}(x,t)| \leq G \; , \; (x,t) \in \Omega_T.
$$

Using the compatibility conditions (9),(10) and the fact that all the nonlinear terms involved in (1) are Lipschitz (see [9] for details) one can prove that the map

$$
\psi_\lambda : N \rightarrow \psi_\lambda(N)
$$

given by

$$
\begin{aligned}
\psi_\lambda(N)_i^M(x,t) = \ & e^{-\lambda t} N_{i_o}(x - v_i^M t) \\
& + \int_0^t e^{-\lambda(t-\tau)} F_i^{M,\lambda}(N)(x - v_i^M(t-\tau), \tau) d\tau \quad \text{if } (x,t) \in \Omega_{M,i,T}^+ ,
\end{aligned} \tag{15}
$$

and

$$\psi_\lambda(N)_i^M(x,t) = e^{-\lambda t} \sum_{M'} \beta_{M'}^M e^{-\lambda t_i^M} N_{3_o}^{M'}(-v_3^{M'} t_i^M)$$

$$+ e^{-\lambda t} \sum_{M'} \beta_{M'}^M \int_0^{t_i^M} e^{-\lambda(t_i-\tau)} F_3^{M',\lambda}(N)(x - v_3^{M'}(t-\tau),\tau) d\tau \qquad (16)$$

$$+ \int_{t_i^M}^t e^{-\lambda(t-\tau)} F_i^{M,\lambda}(N)(x - v_i^M(t-\tau)),\tau) d\tau \qquad \text{if } (x,t) \in \Omega_{M,i,T}^-$$

is a contraction of $S(T,E,G)$ for adequate values of λ, T, E, and G, where λ, T and E depend exclusively on E_o.

Then, the unique fix-point N of ψ_λ is a local solution of (1), with

$$N \in C^0(\Omega_T) \cap L^\infty(\Omega_T) \qquad \text{and} \qquad \frac{\partial N}{\partial x} \in C^0(\Omega_T) \cap L^\infty(\Omega_T).$$

It is a technicality to prove that N is continuously differentiable with respect to t and that N solves Eq.(1). The reader can refer to [9] (proof of Theorem 3.1).

3 Global solutions

We consider the new set of variables

$$\begin{aligned}
M_1 &= N_1^A + N_1^{A^*} \\
M_2 &= N_1^{A_2} \\
M_3 &= N_2^A + N_2^{A_2} \\
M_4 &= N_3^{A_2} \\
M_5 &= N_3^A + N_3^{A^*}.
\end{aligned}$$

This new set satisfies the system

$$\frac{\partial}{\partial t} M_i(x,t) + w_i \frac{\partial}{\partial x} M_i(x,t) = R_i(N(x,t)), \quad i \in \{1;\dots;5\}, \qquad (17)$$

where $(w_1, w_2, w_3, w_4, w_5) = (c, \frac{c}{2}, 0, -\frac{c}{2}, -c)$ and

$$\begin{aligned}
R_1 &= F_1^A + F_1^{A^*} \\
R_2 &= F_1^{A_2} \\
R_3 &= F_2^A + F_2^{A_2} \\
R_4 &= F_3^{A_2} \\
R_5 &= F_3^A + F_3^{A^*}.
\end{aligned}$$

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We make the usual following assumption: for every $j \in \{1, \ldots, 5\}$, R_j can be put in the form

$$R_j(N) = N_i^M R_j^{(1)} - R_j^{(2)}, \tag{18}$$

where $v_i^M \neq w_j$ and $R^{(k)}$, $k = 1, 2$ are polynomials with positive cefficients. Note that this condition holds in the systems studied in [12].

Under this assumption, we prove in this section the following theorem:

Theorem 3.1
Let $N_o \in \mathcal{B}_+^1([0; +\infty[) \cap L^1([0; +\infty[)$ satisfying the compatibility conditions (9)-(10). Then there exists $\epsilon, \epsilon' > 0$ such that if

$$m_o = \int_0^{+\infty} \left(\sum_{i=1}^3 (N_{i_o}^A + 2N_{i_o}^{A_2}) + \sum_{i=1,3} N_{i_o}^{A^*} \right)(x)dx < \epsilon \quad and \quad E_o = \max_{i,M} \sup_{x \in \mathbb{R}^+} |N_{i_o}^M(x)| < \epsilon',$$

the mixed problem (1)-(6)-(7) has a unique solution

$$N \in \mathcal{B}_+^1([0; +\infty[\times [0; +\infty[).$$

Proof:

Let $N_o \in \mathcal{B}_+^1([0; +\infty[) \cap L^1(]0; +\infty[)$ satisfying (9)-(10) and $N \in \mathcal{B}_+^1([0; +\infty[\times [0; T_o])$ the solution to (1)-(6)-(7) given by Theorem 2.1.

In order to establish Theorem 3.1, we prove the following key lemma:

Lemma 3.2 *For all $t \in [0; T_o]$,*

$$E(t) \leq CE_o + C'm_o(E(t) + E(t)^2) \tag{19}$$

where the constants C,C' are independent of T_o and N_o.

Proof of Lemma 3.2:

Following the ideas in [4] and [10], the proof of this lemma is based on the integration of the conservation laws (3),(4),(5) along closed curves formed by the characteristic lines of (1).

In what follows we will denote indifferently by C and C' some positive constants.

Let $X = (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+$. We begin by considering the case $ct - x > 0$.

In this case the c-characteristic through X meets the boundary $x = 0$ at a time $t_c = t - \dfrac{x}{c} > 0$.

Let $j \in \{1; \ldots; 5\}$.

We first assume that $x - w_j t < 0$.
In this case $w_j \in \{\dfrac{c}{2}, c\}$ (i.e. $j \in \{1, 2\}$) and the characteristic corresponding to the speed w_j meets the boundary $x = 0$ at the time $t_j = t - \dfrac{x}{w_j}$. Integrating (17) along this line,

$$M_j(x, t) = M_j(0, t_j) + \int_{t_j}^{t} R_j(N(x - w_j(t - \tau), \tau))d\tau. \tag{20}$$

Also, by (7),

$$M_j(0, t_j) \le \sum_{M, M'} \beta_M^{M'} N_3^M(0, t_j) \le C(M_4(0, t_j) + M_5(0, t_j))$$

Hence

$$
\begin{aligned}
M_j(x, t) \ \le\ & C(M_4(0, t_j) + M_5(0, t_j)) + \int_{t_j}^{t} R_j(N(x - w_j(t - \tau), \tau))d\tau \quad\quad (21) \\[2mm]
\le\ & C(M_4(0, t_j) + M_5(0, t_j)) + C'(E(t) + E(t)^2) \int_{t_j}^{t} \sum_{v_i^M \ne w_j} N_i^M(x - w_j(t - \tau), \tau)d\tau,
\end{aligned}
$$

where we have used condition (18).
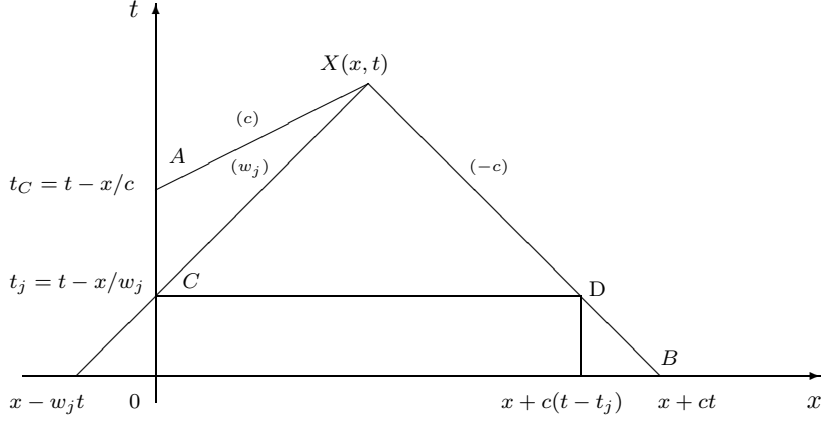We now estimate $I = \displaystyle\int_{t_j}^{t} \sum_{v_i^M \ne w_j} N_i^M(x - w_j(t - \tau), \tau)d\tau$.

Let us consider the differential form

$$w = -\left(\sum_{i=1}^{3}(N_i^A + 2N_i^{A_2}) + \sum_{i=1,3} N_i^{A^*} \right) dx + \left(\sum_{i=1}^{3} v_i(N_i^A + N_i^{A_2}) + \sum_{i=1,3} v_i N_i^{A^*} \right) dt.$$

By making use of the conservation laws (3),(4) we easily check that w is an exact form, hence by putting $A = (0, t_c)$ and $B = (x + ct, 0)$,

$$\int_{[OA]} w + \int_{[AX]} w + \int_{[XB]} w + \int_{[BO]} w = 0. \tag{22}$$

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Furthermore,

$$\int_{[OA]} w = \int_0^{t_c} \left(\sum_{i=1}^3 v_i(N_i^A + N_i^{A_2}) + \sum_{i=1,3} v_i N_i^{A^*} \right) (0,\tau)d\tau \leq 0, \qquad (23)$$

by the boundary condition (7).
Also,

$$\int_{[AX]} w = \int_{t_c}^t \left(\sum_{i=1}^3 ((v_i - c)N_i^A + (v_i - 2c)N_i^{A_2}) + \sum_{i=1,3} (v_i - c)N_i^{A^*} \right) (x - c(t-\tau),\tau)d\tau,$$
$$(24)$$

$$\int_{[XB]} w = -\int_0^t \left(\sum_{i=1}^3 ((v_i + c)N_i^A + (v_i + 2c)N_i^{A_2}) + \sum_{i=1,3} (v_i + c)N_i^{A^*} \right) (x + c(t-\tau),\tau)d\tau$$
$$(25)$$

and

$$\int_{[BO]} w = \int_0^{x+ct} \left(\sum_{i=1}^3 (N_i^A + 2N_i^{A_2}) + \sum_{i=1,3} N_i^{A^*} \right) (x + ct - \xi, 0)d\xi \leq m_o. \qquad (26)$$

By (22),

$$\int_0^t \left(\sum_{i=1}^3 ((v_i + c)N_i^A + (v_i + 2c)N_i^{A_2}) + \sum_{i=1,3} (v_i + c)N_i^{A^*} \right) (x + c(t-\tau),\tau)d\tau +$$

$$+ \int_{t_c}^t \left(\sum_{i=1}^3 ((c - v_i)N_i^A + (2c - v_i)N_i^{A_2}) + \sum_{i=1,3} (c - v_i)N_i^{A^*} \right) (x - c(t-\tau),\tau)d\tau \leq m_o,$$
$$(27)$$

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and
$$\int_0^t \left(\sum_{i=1,2} N_i^A + \sum_{i=1}^3 N_i^{A_2} + N_1^{A^*} \right) (x + c(t-\tau), \tau) d\tau \leq C m_o. \tag{28}$$

We now consider $C = (0, t_j)$ and $D = (x + c(t - t_j), t_j)$.

As before,
$$\int_{[OC]} w + \int_{[CD]} w + \int_{[DB]} w + \int_{[BO]} w = 0. \tag{29}$$

An elementary computation leads to
$$\int_{[OC]} w = \int_0^{t_j} \left(\sum_{i=1}^3 v_i (N_i^A + N_i^{A_2}) + \sum_{i=1,3} v_i N_i^{A^*} \right) (0, \tau) d\tau \leq 0, \tag{30}$$

$$\int_{[CD]} w = -\int_0^{x+c(t-t_j)} \left(\sum_{i=1}^3 (N_i^A + 2N_i^{A_2}) + \sum_{i=1,3} N_i^{A^*} \right) (\xi, t_j) d\xi, \tag{31}$$

$$\int_{[DB]} w = -\int_0^{t_j} \left(\sum_{i=1}^3 ((v_i + c)N_i^A + (v_i + 2c)N_i^{A_2}) + \sum_{i=1,3} (v_i + c)N_i^{A^*} \right) (x + c(t-\tau), \tau) d\tau$$
$$\leq 0. \tag{32}$$

By (29),
$$\int_0^{x+c(t-t_j)} \left(\sum_{i=1}^3 (N_i^A + 2N_i^{A_2}) + \sum_{i=1,3} N_i^{A^*} \right) (\xi, t_j) d\xi \leq m_o. \tag{33}$$

Next, we consider the differential form w' defined by
$$w' = -\left(\sum_{i=1}^3 ((w_j - v_i)N_i^A + (2w_j - v_i)N_i^{A_2}) + \sum_{i=1,3} (w_j - v_i)N_i^{A^*} \right) dx$$
$$+ \left(\sum_{i=1}^3 v_i((w_j - v_i)N_i^A + (w_j - \frac{1}{2}v_i)N_i^{A_2}) + \sum_{i=1,3} v_i(w_j - v_i)N_i^{A^*} \right) dt. \tag{34}$$

By (5), w' is an exact form.

$$\int_{[XC]} w' =$$
$$\int_{t_j}^t \left(\sum_{i=1}^3 ((w_j - v_i)^2 N_i^A + 2(w_j - \frac{1}{2}v_i)^2 N_i^{A_2}) + \sum_{i=1,3} (w_j - v_i)^2 N_i^{A^*} \right) (x - w_j(t-\tau), \tau) d\tau, \tag{35}$$

$$\int_{[CD]} w' = - \int_0^{x+c(t-t_j)} \left(\sum_{i=1}^3 ((w_j - v_i)N_i^A + (2w_j - v_i)N_i^{A_2}) + \sum_{i=1,3} (w_j - v_i)N_i^{A^*} \right)(\xi, t_j)d\tau, \tag{36}$$

and

$$\int_{[DX]} w' =$$

$$\int_{t_j}^t c \left(\sum_{i=1}^3 ((w_j - v_i)N_i^A + (2w_j - v_i)N_i^{A_2}) + \sum_{i=1,3} (w_j - v_i)N_i^{A^*} \right)(x + c(t-\tau), \tau)d\tau +$$

$$+ \int_{t_j}^t \left(\sum_{i=1}^3 v_i((w_j - v_i)N_i^A + (w_j - 2v_i)N_i^{A_2}) + \sum_{i=1,3} v_i(w_j - v_i)N_i^{A^*} \right)(x + c(t-\tau), \tau)d\tau$$

$$= \int_{t_j}^t \left(\sum_{i=1}^3 \left(((v_i + c)(w_j - v_i)N_i^A + 2(w_j - \frac{v_i}{2})(c + \frac{v_i}{2})N_i^{A_2}) \right) \right)((x + c(t-\tau), \tau))d\tau +$$

$$+ \sum_{i=1,3} \left((v_i + c)(w_j - v_i)N_i^{A^*} \right)((x + c(t-\tau), \tau))d\tau. \tag{37}$$
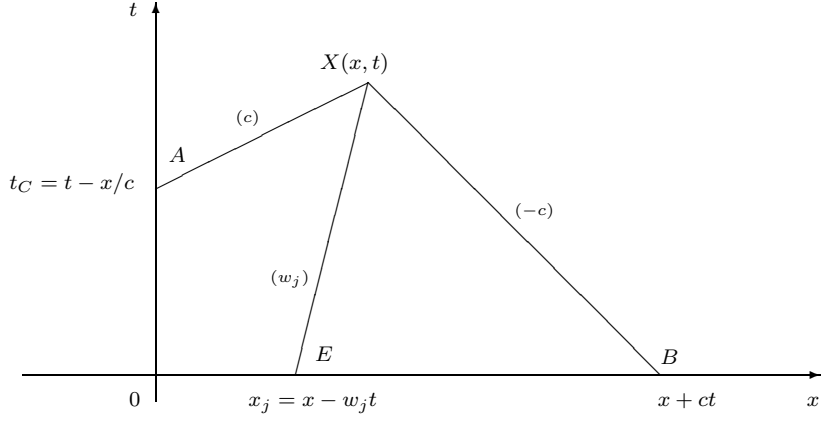
Hence,

$$\begin{aligned}
\int_{[XC]} w' &= - \int_{[CD]} w' - \int_{[DX]} w' \\
&\leq \left| \int_{[CD]} w' \right| + \left| \int_{[DX]} w' \right| \\
&\leq C \int_0^{x+c(t-t_j)} \left(\sum_{i=1}^3 (N_i^A + 2N_i^{A_2}) + \sum_{i=1,3} N_i^{A^*} \right)(\xi, t_j)d\xi \\
&\quad + C \int_0^t \left(\sum_{i=1,2} N_i^A + \sum_{i=1}^3 N_i^{A_2} + N_1^{A^*} \right)(x + c(t-\tau), \tau)d\tau \\
&\leq Cm_o
\end{aligned}$$

by (28) and (33).
Finally, by (35),

$$I = \int_{t_j}^t \sum_{v_i^M \neq w_j} N_i^M(x - w_j(t-\tau), \tau)d\tau \leq Cm_o. \tag{38}$$

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Next, we consider the case where $x_j = x - w_j t \geq 0$.

In this case the characteristic corresponding to a speed w_j meets $t = 0$ at $x_j = x - w_j t$. We put $E = (x_j, 0)$.

Here, for $j \in \{1; \dots 5\}$,

$$M_j(x,t) = M_j(x_j, 0) + \int_0^t R_j(N(x - w_j(t - \tau), \tau))d\tau, \qquad (39)$$

and, as before,

$$M_j(x,t) \leq CE_o + C'(E(t) + E(t)^2) \int_0^t \sum_{v_i^M \neq w_j} N_i^M(x - w_j(t - \tau), \tau)d\tau \qquad (40)$$

In order to estimate $J = \int_0^t \sum_{v_i^M \neq w_j} N_i^M(x - w_j(t - \tau), \tau)d\tau$, we integrate w' on the triangle EXB:

$$\int_{[XE]} w' =$$
$$\int_0^t \left(\sum_{i=1}^3 ((w_j - v_i)^2 N_i^A + 2(w_j - \frac{v_i^2}{2}) N_i^{A_2}) + \sum_{i=1,3} (w_j - v_i)^2 N_i^{A^*} \right)(x - w_j(t - \tau), \tau)d\tau, \qquad (41)$$

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and

$$\left| \int_{[EB]} w' \right| + \left| \int_{[BX]} w' \right| \leq$$

$$\int_{x_j}^x \left(\sum_{i=1}^3 ((w_j - v_i)N_i^A + (2w_j - v_i)N_i^{A_2}) + \sum_{i=1,3}(w_j - v_i)N_i^{A^*} \right)(\xi, 0)d\tau +$$

$$+ \int_0^t \left(\sum_{i=1}^3 \left(((v_i + c)(w_j - v_i)N_i^A + 2(w_j - \frac{v_i}{2})(c + \frac{v_i}{2})N_i^{A_2}) \right) \right)((x + c(t - \tau), \tau))d\tau +$$

$$+ \sum_{i=1,3} \left((v_i + c)(w_j - v_i)N_i^{A^*} \right)((x + c(t - \tau), \tau))d\tau,$$

hence

$$\left| \int_{[EB]} w' \right| + \left| \int_{[BX]} w' \right| \leq Cm_o +$$

$$+ C' \int_0^t \left(\sum_{i=1,2} N_i^A + \sum_{i=1}^3 N_i^{A_2} + N_1^{A^*} \right)(x + c(t - \tau), \tau)d\tau \leq Cm_o$$

by (28).

Finally,

$$J = \int_0^t \sum_{v_i^M \neq w_j} N_i^M(x - w_j(t - \tau), \tau)d\tau \leq C \int_{[XE]} w' \leq Cm_o. \tag{42}$$

By combining this estimate with (40), we obtain, in the case where $x - w_j t \geq 0$,

$$\forall j \ , \ M_j(x, t) \leq CE_o + Cm_o(E(t) + E(t)^2). \tag{43}$$

If $x - w_j t < 0$, by (21) and (38),

$$M_j(x, t) \leq C(M_4(0, t_j) + M_5(0, t_j)) + Cm_o(E(t) + E(t)^2). \tag{44}$$

For $j \in \{4, 5\}$, integration along the characteristic of speed w_j leads to

$$
\begin{aligned}
M_j(0, t_j) &= M_j(w_j t_j, 0) + \int_0^{t_j} R_j(N(-w_j(\tau - t_j), \tau))d\tau \\
&\leq CE_o + (E(t) + E(t)^2) \int_0^{t_j} \sum_{v_i^M \neq w_j} N_i^M(-w_j(t - \tau), \tau)d\tau \\
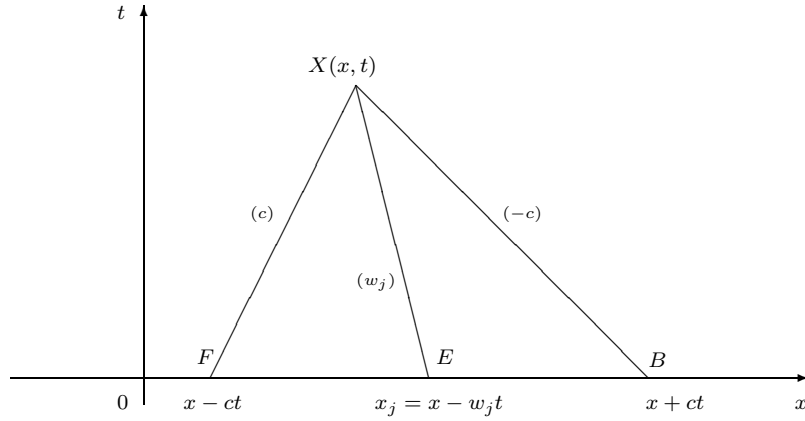&\leq CE_o + C'm_o(E(t) + E(t)^2)
\end{aligned}
$$

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by (42).

Replacing in (44) leads to

$$M_j(x,t) \leq CE_o + Cm_o(E(t) + E(t)^2)$$

also in the case $x - w_j t < 0$.

Finally, if the characteristic corresponding to a speed c does not meet the boundary $x = 0$, we put $x_c = x - ct$ and $F = (x_c, 0)$.

Integrating w on the triangle XBF and w' on the triangle XFE leads by analogous computations to (43) which concludes the proof of Lemma 3.2. ∎

We now prove the following lemma:

Lemma 3.3 *There exists $\epsilon > 0, \epsilon' > 0$ such that if $m_o < \epsilon$ and $E_o < \epsilon'$ then there exists a constant $K = K(N_o) > 0$ depending exclusively on the initial data N_o such that*

$$\forall 0 \leq t \leq T_o \ , \ E(t) \leq K(N_o).$$

One can easily show by standard arguments that Theorem 3.1 can be obtained from Lemma 3.3.

Proof of Lemma 3.3

We define the polynomial

$$P : Y \in \mathbb{R} \rightarrow C'm_o Y^2 + (C'm_o - 1)Y + CE_o.$$

Let $m_o < \frac{1}{C'} := \epsilon$ and $\epsilon' > 0$ small enough such that if $E_o < \epsilon'$,

$$(C'm_o - 1)^2 - 4CC'm_o E_o > 0.$$

Then P possesses the two roots

$$
\begin{aligned}
Y_1(E_o, m_o) &= \frac{1 - C'm_o - \sqrt{(C'm_o - 1)^2 - 4CC'm_o E_o}}{C'm_o} > 0, \\
Y_2(E_o, m_o) &= \frac{1 - C'm_o + \sqrt{(C'm_o - 1)^2 - 4CC'm_o E_o}}{C'm_o} > 0.
\end{aligned}
$$

Since for all $t \in [0; T_o]$, $P(E(t)) \geq 0$ and $t \to E(t)$ is continuous,

$$\forall 0 \leq t \leq T_o \ , \ E(t) \leq Y_1 \ \ \text{or} \ \ \forall 0 \leq t \leq T_o \ , \ E(t) \geq Y_2.$$

Clearly, $Y_2 \leq E_o$ cannot hold for E_o small enough, since

$$\lim_{E_o \to 0} Y_2(E_o, m_o) = 2\frac{1 - C'm_o}{C'm_o} > 0.$$

Hence Lemma 3.3 holds with $K(N_o) = Y_1(E_o, m_o)$. ∎

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