Some pseudovariety joins involving locally trivial semigroups

José Carlos Costa*

December, 2001

ABSTRACT

This paper is concerned with the computation of pseudovariety joins involving the pseudovariety $\mathcal{L}\mathbf{I}$ of locally trivial semigroups. We compute, in particular, the join of $\mathcal{L}\mathbf{I}$ with any subpseudovariety of $\mathbf{CR} \circledcirc \mathbf{N}$, the Mal'cev product of the pseudovariety of completely regular semigroups and the pseudovariety of nilpotent semigroups. Similar studies are conducted for the pseudovarieties $\mathbf{K}$, $\mathbf{D}$ and $\mathbf{N}$, where $\mathbf{K}$ (resp. $\mathbf{D}$) is the pseudovariety of all semigroups S such that $eS = e$ (resp. $Se = e$) for each idempotent e of S.

1 Introduction

As one recalls, the join $\mathbf{V} \vee \mathbf{W}$ of two pseudovarieties $\mathbf{V}$ and $\mathbf{W}$ is the least pseudovariety containing both $\mathbf{V}$ and $\mathbf{W}$. In spite of its simple definition, the determination of pseudovarieties of the form $\mathbf{V} \vee \mathbf{W}$ is in general very difficult. For instance, Albert, Baldinger and Rhodes [1] have exhibited, quite surprisingly, two decidable pseudovarieties whose join is undecidable. At that time it was already known that the join of two finitely based pseudovarieties can be non finitely based [23]. Thus, the calculation of joins seams to be a problem that depends greatly on the specific pseudovarieties involved and so general results are not easily obtained.

Before the development of the theory of implicit operations by Almeida, relatively few results of this type were known. With the use of the implicit operations this situation has changed significantly. The first significant success was obtained by Almeida [3], who showed that $\mathbf{G} \vee \mathbf{Com} = \mathbf{ZE}$ where $\mathbf{G}$, $\mathbf{Com}$ and $\mathbf{ZE}$ are, respectively, the pseudovarieties of groups, of commutative semigroups and of semigroups in which idempotents are central (i.e., commute with every element). Another remarkable example is the calculation of $\mathbf{R} \vee \mathbf{L}$, the join of the pseudovarieties of $\mathcal{R}$-trivial and $\mathcal{L}$-trivial semigroups, by Almeida and Azevedo [7]. More recently, several other calculations and answers to decision problems were performed by Almeida, Azevedo, Trotter, Volkov, Weil, Zeitoun and the author [9, 10, 11, 13, 14, 15, 17, 24, 25, 26, 27].

This article solves, in particular, the problem of the calculation of the join $\mathcal{L}\mathbf{I} \vee \mathbf{V}$ where $\mathbf{V}$ is any subpseudovariety of $\mathbf{CR} \circledcirc \mathbf{N}$. This result is extremely natural: we show that if $\mathbf{V}$ is defined by a set Σ of pseudoidentities and if a and b are symbols

*Research supported by INVOTAN, grant 4/A/94/PO and by Centro de Matemática da Universidade do Minho. The author is member of the project *AGC* contract Praxis/2/2.1/MAT/63/94.

not appearing in Σ, then $\mathcal{L}\mathbf{I} \vee \mathbf{V}$ is the subpseudovariety of $\mathbf{CR} \textcircled{m} \mathbf{N}$ defined by the pseudoidentities of the form $a^\omega x b^\omega = a^\omega y b^\omega$ where $x = y$ is an element of Σ. Therefore, the property of finite basis is preserved. This supports a conjecture that states that the join of $\mathcal{L}\mathbf{I}$ with an arbitrary finitely based pseudovariety is finitely based. Our result is quite general since $\mathbf{CR} \textcircled{m} \mathbf{N}$ contains, for instance, the pseudovariety $\mathbf{CR}$ of completely regular semigroups. It is based on a strong property of the semigroups S in $\mathbf{CR} \textcircled{m} \mathbf{N}$, which is the fact that the ideal generated by the idempotents of S is a subsemigroup, and so we do not expect that it can be easily extended. However, we are also able to compute the joins $\mathcal{L}\mathbf{I} \vee \mathbf{V}$ where $\mathbf{V}$ is a pseudovariety verifying a certain property of cancellation. This last result is valid, for instance, when $\mathbf{V}$ is one of $\mathbf{R}$ and $\mathbf{ZE}$. All results obtained for $\mathcal{L}\mathbf{I}$ have similar analogues for $\mathbf{K}$ and $\mathbf{D}$. We then derive some results for $\mathbf{N}$.

The results obtained generalize a number of results already known, namely the computation involving the pseudovarieties: **Com** [2]; **J** of $\mathcal{J}$-trivial semigroups [13]; **B** of idempotent semigroups [27]; **Sl** of semilattices [6, 27]; subpseudovarieties of **CR** (for joins with **N**) [21].

2 Preliminaries

We assume the reader is familiar with the basic notions of finite semigroup theory. We will briefly recall some notions and notation concerning pseudovarieties and implicit operations. For more details and proofs, the reader may wish to consult the books of Eilenberg [18] and Pin [19] for an introduction to the theories of semigroups and pseudovarieties, and the book of Almeida [6] for a comprehensive treatment of the theory of implicit operations.

2.1 Pseudovarieties

We fix a finite alphabet $A_n = \{a_1, a_2, \ldots, a_n\}$ and we set $A = \bigcup_{n\in\mathbb{N}} A_n$. The free semigroup (resp. monoid) on A_n is denoted by A_n^+ (resp. A_n^*). The *content* $c(u)$ of a word $u \in A_n^+$ is the set of all letters occurring in u and the length of u is denoted by $|u|$.

A *pseudovariety* of semigroups is a class of finite semigroups closed under taking subsemigroups, homomorphic images and finite direct products. The pseudovariety of all finite semigroups is denoted by $\mathbf{S}$ and $\mathbf{I}$ denotes the trivial pseudovariety.

As usual, we denote by $E(S)$ the set of idempotents of a semigroup S. We define an operator on the lattice of pseudovarieties as follows. For a pseudovariety $\mathbf{V}$, we set

$$\mathcal{L}\mathbf{V} = \{S \in \mathbf{S} \mid eSe \in \mathbf{V} \text{ for all } e \in E(S)\}.$$

This operator admits two asymmetrical versions. For a pseudovariety $\mathbf{V}$, we define

$$\mathcal{L}_l\mathbf{V} = \{S \in \mathbf{S} \mid eS \in \mathbf{V} \text{ for all } e \in E(S)\}$$
$$\mathcal{L}_r\mathbf{V} = \{S \in \mathbf{S} \mid Se \in \mathbf{V} \text{ for all } e \in E(S)\}.$$

Particularly important in this article are the pseudovarieties $\mathcal{L}\mathbf{I}$, $\mathbf{K} = \mathcal{L}_l\mathbf{I}$, $\mathbf{D} = \mathcal{L}_r\mathbf{I}$ and $\mathbf{N} = \mathbf{K} \cap \mathbf{D}$. We recall that $\mathcal{L}\mathbf{I} = \mathbf{K} \vee \mathbf{D}$.

Another important operator is defined as follows, for a pseudovariety $\mathbf{V}$,

$$\mathcal{D}\mathbf{V} = \{S \in \mathbf{S} \mid \text{ the regular } \mathcal{D}\text{-classes of } S \text{ are semigroups of } \mathbf{V}\}.$$

An important example of a pseudovariety of this type is $\mathcal{D}\mathbf{S}$, the pseudovariety of semigroups whose regular $\mathcal{D}$-classes are semigroups. We will also encounter the pseudovarieties $\mathcal{D}\mathbf{RH}$, where, for a pseudovariety $\mathbf{H}$ of groups, $\mathbf{RH}$ denotes the pseudovariety of right groups all of whose subgroups lie in $\mathbf{H}$. Notice that $\mathbf{R} = \mathcal{D}\mathbf{RI}$.

2.2 Implicit operations and pseudoidentities

Let $\mathbf{V}$ be a pseudovariety. An n-ary implicit operation on $\mathbf{V}$ is a collection (x_S) indexed by the semigroups $S \in \mathbf{V}$, where each x_S is a mapping from S^n into S and such that if $\varphi : S \to T$ is a morphism with $S, T \in \mathbf{V}$, then the following diagram commutes.

$$\begin{array}{ccc} S^n & \xrightarrow{x_S} & S \\ {\scriptstyle\varphi^n}\downarrow & & \downarrow{\scriptstyle\varphi} \\ T^n & \xrightarrow{x_T} & T \end{array}$$

The set of n-ary implicit operations on $\mathbf{V}$ is denoted by $\hat{F}_n(\mathbf{V})$. The most simple, yet fundamental, examples of implicit operations are the *explicit operations*: for a word $u = a_{i_1} \cdots a_{i_k} \in A_n^+$, the collection $(u_S)_{S \in \mathbf{V}}$ defined by $u_S(s_1, \ldots, s_n) = s_{i_1} \cdots s_{i_k}$ is an implicit operation, denoted simply by u. The set of n-ary explicit operations on $\mathbf{V}$ is denoted by $F_n(\mathbf{V})$.

We define a semigroup structure on $\hat{F}_n(\mathbf{V})$ by letting $(x_S) \cdot (y_S) = (x_S \cdot y_S)$. Moreover, it is known that $\hat{F}_n(\mathbf{V})$ — endowed with the initial topology for the evaluation morphisms

$$\begin{array}{rccc} e_T : & \hat{F}_n(\mathbf{V}) & \to & T^{T^n} \\ & (x_S)_{S \in \mathbf{V}} & \mapsto & x_T \end{array}$$

where T runs through $\mathbf{V}$ and where each finite semigroup T^{T^n} is endowed with the discrete topology, — is a compact totally disconnected semigroup in which $F_n(\mathbf{V})$ is a dense subsemigroup.

Given an implicit operation $x \in \hat{F}_n(\mathbf{V})$, it is easy to verify that the sequence $(x^{k!})_{k \in \mathbb{N}}$ converges in $\hat{F}_n(\mathbf{V})$. Its limit, denoted by x^ω, is the only idempotent in the topological closure of the subsemigroup generated by x.

If $\mathbf{W}$ is a subpseudovariety of $\mathbf{V}$, the mapping

$$\begin{array}{rccc} \pi : & \hat{F}_n(\mathbf{V}) & \to & \hat{F}_n(\mathbf{W}) \\ & (x_S)_{S \in \mathbf{V}} & \mapsto & (x_S)_{S \in \mathbf{W}} \end{array}$$

is a continuous onto morphism, called the *canonical projection of* $\hat{F}_n(\mathbf{V})$ *onto* $\hat{F}_n(\mathbf{W})$. The image $\pi(x)$ of an element $x \in \hat{F}_n(\mathbf{V})$ is called the *restriction of* x *to* $\mathbf{W}$. In particular, when $\mathbf{W}$ is the pseudovariety $\mathbf{Sl}$,— recall that $\hat{F}_n(\mathbf{Sl})$ is the semigroup 2^{A_n} of non empty subsets of A_n under union— the canonical projection $c : \hat{F}_n(\mathbf{V}) \to \hat{F}_n(\mathbf{Sl})$ is called the *content morphism* on $\mathbf{V}$. This projection c extends to the elements of $\hat{F}_n(\mathbf{V})$ the notion of content for words of A_n^+.

Every morphism $\varphi : A_n^+ \to \hat{F}_m(\mathbf{V})$ can be uniquely extended to a continuous morphism $\bar{\varphi} : \hat{F}_n(\mathbf{V}) \to \hat{F}_m(\mathbf{V})$. Let $y_1, \ldots, y_n$ be m-ary implicit operations on $\mathbf{V}$, and let $\varphi : A_n^+ \to \hat{F}_m(\mathbf{V})$ be the morphism mapping a_i $(i = 1, \ldots, n)$ to y_i. The image $\bar{\varphi}(x)$, of an n-ary implicit operation x on $\mathbf{V}$, is an m-ary implicit operation on $\mathbf{V}$ which is said to be obtained by substituting y_i for a_i in x.

A *pseudoidentity* is a pair (x, y) of elements of $\hat{F}_n(\mathbf{S})$, for some n, and is usually denoted $x = y$. We say that a finite semigroup S *satisfies* a pseudoidentity $x = y$, written $S \models x = y$, if $x_S = y_S$. We say that a class $\mathcal{C}$ of finite semigroups satisfies a set Σ of pseudoidentities, written $\mathcal{C} \models \Sigma$, if each element of $\mathcal{C}$ satisfies each element of Σ. The class of all finite semigroups which satisfy Σ is said to be *defined by* Σ and is denoted $[\![\Sigma]\!]$. For instance, the following are definitions by pseudoidentities of some important pseudovarieties in this paper:

$$
\begin{array}{rclrcl}
\mathbf{B} & = & [\![a = a^2]\!], & \mathbf{Com} & = & [\![ab = ba]\!] \\
\mathbf{CR} & = & [\![a = a^{\omega+1}]\!], & \mathbf{D}_1 & = & [\![ab = b]\!] \\
\mathcal{D}\mathbf{RG} & = & [\![(ab)^\omega (ba)^\omega = (ab)^\omega]\!], & \mathcal{D}\mathbf{S} & = & [\![((ab)^\omega (ba)^\omega (ab)^\omega)^\omega = (ab)^\omega]\!] \\
\mathcal{E}\mathbf{Com} & = & [\![a^\omega b^\omega = b^\omega a^\omega]\!], & \mathbf{G} & = & [\![a^\omega = 1]\!] \\
\mathbf{J} & = & [\![(ab)^\omega = (ba)^\omega, a^{\omega+1} = a^\omega]\!], & \mathbf{K}_1 & = & [\![ab = a]\!] \\
\mathbf{R} & = & [\![(ab)^\omega a = (ab)^\omega]\!], & \mathbf{RG} & = & [\![ab^\omega = a]\!] \\
\mathbf{Sl} & = & [\![a = a^2, ab = ba]\!], & \mathbf{ZE} & = & [\![a^\omega b = ba^\omega]\!].
\end{array}
$$

Let Σ be a set of pseudoidentities defining a pseudovariety $\mathbf{V}$. Then $\mathcal{L}\mathbf{V}$ (resp. $\mathcal{L}_l\mathbf{V}$, $\mathcal{L}_r\mathbf{V}$) is defined by the set of all pseudoidentities which are obtained from Σ by substituting each variable b by $a^\omega b a^\omega$ (resp. $a^\omega b$, ba^ω) where a is a variable that does not occur in Σ. For instance, we have that

$$
\begin{array}{rclrcl}
\mathcal{L}_l\mathbf{CR} & = & [\![a^\omega b = (a^\omega b)^{\omega+1}]\!], & \mathcal{L}_r\mathbf{CR} & = & [\![ba^\omega = (ba^\omega)^{\omega+1}]\!] \\
\mathcal{L}\mathbf{G} & = & [\![(a^\omega b a^\omega)^\omega = a^\omega]\!], & \mathcal{L}\mathbf{I} & = & [\![a^\omega b a^\omega = a^\omega]\!] \\
\mathcal{L}_l\mathbf{I} & = & [\![a^\omega b = a^\omega]\!], & \mathcal{L}_r\mathbf{I} & = & [\![ba^\omega = a^\omega]\!].
\end{array}
$$

The following fundamental theorem, due to Reiterman [22], shows that all pseudovarieties are defined by pseudoidentities.

Theorem 2.1 *Let* $\mathbf{V}$ *be a class of finite semigroups. Then* $\mathbf{V}$ *is a pseudovariety if and only if there exists a set* Σ *of pseudoidentities such that* $\mathbf{V} = [\![\Sigma]\!]$. □

If $\mathbf{V} = [\![\Sigma]\!]$, the set Σ is said to be a *basis* (of pseudoidentities) of $\mathbf{V}$.

2.3 Some important pseudovarieties

Let $\mathbf{V}$ and $\mathbf{W}$ be pseudovarieties. It is clear that a pseudoidentity is satisfied by $\mathbf{V} \vee \mathbf{W}$ if and only if it is satisfied by both $\mathbf{V}$ and $\mathbf{W}$. Therefore, in order to compute $\mathbf{V} \vee \mathbf{W}$ it is important to understand, the best one can, the implicit operations on $\mathbf{V}$ and $\mathbf{W}$. Since we are interested in joins involving the pseudovarieties $\mathcal{L}\mathbf{I}$, $\mathbf{K}$, $\mathbf{D}$ and $\mathbf{N}$, it is important to remember some well-known facts concerning these pseudovarieties.

We begin by selecting from the considerations of [6, pp. 88-91] on these pseudovarieties the following, simple but fundamental, observation.

Lemma 2.2 *Let* $\mathbf{V}$ *be a pseudovariety of semigroups containing* $\mathbf{N}$, *and let* $(x_k)_{k\in\mathbb{N}}$ *be a sequence of* n*-ary explicit operations on* $\mathbf{V}$ *converging to an implicit operation* $x \in \hat{F}_n(\mathbf{V})$.

(1) *The pseudovariety* $\mathbf{V}$ *does not satisfy any non-trivial identity, that is,* $F_n(\mathbf{V}) = A_n^+$ *for every* n*. Furthermore, if* $\mathbf{V}$ *satisfies a pseudoidentity* $y = u$*, with* u *explicit, then* y *and* u *are equal.*

(2) *The sequence* $(|x_k|)_{k\in\mathbb{N}}$ *tends to* $+\infty$ *if and only if* x *is not explicit.*

(3) *If* $\mathbf{V}$ *contains* $\mathbf{K}$ *(resp.* $\mathbf{D}$*) and* x *is not explicit, then, for every* $i \in \mathbb{N}$*,* x_k *and* x_l *have the same prefix (resp. suffix) of length* i*, for any sufficiently large* k *and* l*.* □

As a consequence, we derive the following useful result about the pseudovarieties $\mathcal{L}\mathbf{I}$, $\mathbf{K}$ and $\mathbf{D}$.

Corollary 2.3 *If* $\mathcal{L}\mathbf{I}$ *(resp.* $\mathbf{K}$*,* $\mathbf{D}$*) satisfies a pseudoidentity* $x = y$*, with* x *and* y *not explicit, then there exist* $r, s, u, v \in \hat{F}_n(\mathbf{S})$*, with* $r, s \notin A_n^+$*, such that* $x = rus$ *and* $y = rvs$ *(resp.* $x = ru$ *and* $y = rv$*,* $x = us$ *and* $y = vs$*).*

Proof. Consider sequences $(x_k)_k$ and $(y_k)_k$ in A_n^+ converging, respectively, to x and y in $\hat{F}_n(\mathbf{S})$. By Lemma 2.2, taking subsequences if necessary, we may assume that, for every $k \in \mathbb{N}$,

$$x_k = r_k u_k s_k \qquad \text{and} \qquad y_k = r_k v_k s_k$$

for some $r_k, u_k, v_k, s_k \in A_n^+$ with r_k and s_k of length k. Furthermore, by compactness of $\hat{F}_n(\mathbf{S})$, we may assume, taking subsequences if necessary, that the sequences $(r_k)_k$, $(u_k)_k$, $(v_k)_k$ and $(s_k)_k$ converge in $\hat{F}_n(\mathbf{S})$, say to r, u, v and s, respectively. Now the result follows from Lemma 2.2 since neither r nor s can be explicit. □

Now, we recall some fundamental properties of the subpseudovarieties of $\mathcal{D}\mathbf{S}$ (see [8, 6]).

Proposition 2.4 *Let* $\mathbf{V}$ *be a subpseudovariety of* $\mathcal{D}\mathbf{S}$ *containing* $\mathbf{Sl}$ *and let* $x \in \hat{F}_n(\mathbf{S})$*.*

(1) *The restriction of* x *to* $\mathbf{V}$ *is regular if and only if* $\mathbf{V}$ *satisfies* $x^{\omega+1} = x$*.*

(2) x *admits a factorization of the form*

$$x = u_0 x_1 u_1 \cdots x_r u_r$$

where each u_i *is a word and each* x_i *is regular when restricted to* $\mathcal{D}\mathbf{S}$*. Moreover, if* u_i *is empty, then* $c(x_i)$ *and* $c(x_{i+1})$ *are* $\subseteq$*-incomparable and if* u_i *is not empty, its first letter is not in* $c(x_i)$ *and its last letter is not in* $c(x_{i+1})$*.*

(3) *If* y *is another implicit operation on* $\mathbf{S}$ *and* $y = v_0 y_1 v_1 \cdots y_s v_s$ *is a factorization of* y *as in* (2)*, then* $\mathbf{J}$ *satisfies* $x = y$ *if and only if* $r = s$*,* $u_i = v_i$ *and* $c(x_i) = c(y_i)$ *for all* i*.* □

An *identity* is a pseudoidentity whose members are explicit. A pseudovariety is said to be *equational* if it admits a basis of identities. We say that a pseudovariety $\mathbf{V}$ is *locally finite* if $F_n(\mathbf{V})$ is finite for every integer n. Almeida [4] proved the following result.

Proposition 2.5 *Let* $\mathbf{V}$ *be a pseudovariety of semigroups.*

(1) *If* $F_n(\mathbf{V})$ *is finite for some integer* n*, then* $\hat{F}_n(\mathbf{V}) = F_n(\mathbf{V})$*.*

(2) *If* $\mathbf{V}$ *is locally finite, then it is equational.* □

As important examples of locally finite pseudovarieties we have the finitely generated pseudovarieties.

We will need the following result (see [6, Corollary 5.6.2]).

Proposition 2.6 *Let* $\mathbf{V}$ *be a pseudovariety of semigroups and let* $x \in \hat{F}_n(\mathbf{V}) \setminus F_n(\mathbf{V})$. *Then there exist* $x_1, x_2, x_3 \in \hat{F}_n(\mathbf{V})$ *such that* $x = x_1 x_2^\omega x_3$. □

3 Three important operators

Let Σ be a set of pseudoidentities and let a, b, c, d be symbols not appearing in Σ. Define

$$\begin{aligned} \mathbf{U}_\Sigma &= [\![a^\omega bxcd^\omega = a^\omega bycd^\omega \mid x = y \in \Sigma]\!] \\ \mathbf{U}^l_\Sigma &= [\![a^\omega bx = a^\omega by \mid x = y \in \Sigma]\!] \\ \mathbf{U}^r_\Sigma &= [\![xba^\omega = yba^\omega \mid x = y \in \Sigma]\!]. \end{aligned}$$

Now define three operators on the lattice of pseudovarieties as follows. For a pseudovariety $\mathbf{V}$ let

$$\mathcal{U}\mathbf{V} = \mathbf{U}_\Delta, \qquad \mathcal{U}_l\mathbf{V} = \mathbf{U}^l_\Delta \qquad \text{and} \qquad \mathcal{U}_r\mathbf{V} = \mathbf{U}^r_\Delta$$

where Δ is the set of *all* pseudoidentities satisfied by $\mathbf{V}$, that is

$$\Delta = \{x = y \mid \text{there is some } n \in \mathbb{N} \text{ such that } x, y \in \hat{F}_n(\mathbf{S}) \text{ and } \mathbf{V} \models x = y\}.$$

We notice the following immediate properties, for pseudovarieties $\mathbf{V}$ and $\mathbf{W}$:

- $\mathcal{L}\mathbf{I} \vee \mathbf{V} \subseteq \mathcal{U}\mathbf{V}$, $\mathbf{K} \vee \mathbf{V} \subseteq \mathcal{U}_l\mathbf{V}$ and $\mathbf{D} \vee \mathbf{V} \subseteq \mathcal{U}_r\mathbf{V}$;
- if $\mathbf{V} \subseteq \mathbf{W}$, then $\mathcal{U}\mathbf{V} \subseteq \mathcal{U}\mathbf{W}$, $\mathcal{U}_l\mathbf{V} \subseteq \mathcal{U}_l\mathbf{W}$ and $\mathcal{U}_r\mathbf{V} \subseteq \mathcal{U}_r\mathbf{W}$.

In the next result we prove that the values of $\mathcal{U}\mathbf{V}$, $\mathcal{U}_l\mathbf{V}$ and $\mathcal{U}_r\mathbf{V}$ are independent from the basis of $\mathbf{V}$ chosen to define them.

Proposition 3.1 *Let* $\mathbf{V}$ *be a pseudovariety of semigroups and let* Σ *be any basis of pseudoidentities defining* $\mathbf{V}$. *Then,* $\mathcal{U}\mathbf{V} = \mathbf{U}_\Sigma$, $\mathcal{U}_l\mathbf{V} = \mathbf{U}^l_\Sigma$ *and* $\mathcal{U}_r\mathbf{V} = \mathbf{U}^r_\Sigma$.

Proof. The other equalities being similar, we only prove the first one. Let Δ be the set of all pseudoidentities satisfied by $\mathbf{V}$. The inclusion $\mathbf{U}_\Delta \subseteq \mathbf{U}_\Sigma$ is clear since $\Sigma \subseteq \Delta$. To prove the inclusion $\mathbf{U}_\Sigma \subseteq \mathbf{U}_\Delta$, we have to prove that, for all $x = y \in \Delta$,

$$\mathbf{U}_\Sigma \models a^\omega bxcd^\omega = a^\omega bycd^\omega. \tag{1}$$

The proof of this fact is given in three steps.

First step Suppose first that $\mathbf{V}$ is locally finite (and so equational) and that Σ is a basis of *identities* of $\mathbf{V}$. We divide this step in two cases.

First case We begin by proving that (1) holds when x and y are words (which we can suppose distinct). Since $\mathbf{V}$ is an equational pseudovariety with basis Σ and satisfies $x = y$, the completeness of equational logic guarantees that we can obtain the identity $x = y$ from Σ using the following rules of deduction a finite number of times:

$r1)$ $u = v \Rightarrow v = u$;

$r2)$ $u = v$, $v = w \Rightarrow u = w$;

$r3)$ $u = v$, $r, s \in A^* \Rightarrow rus = rvs$;

$r4)$ $u = v$, $c \in A$, $r \in A^+ \Rightarrow u' = v'$ where u' and v' are the words obtained from the words u and v, respectively, by substituting all occurrences of c by r.

That is, there exists a finite sequence of identities

$$u_1 = v_1,\ u_2 = v_2, \ldots, u_k = v_k$$

such that each $u_i = v_i$ is in Σ or is obtained from identities that precede it in the sequence using one of the rules $r1)$ to $r4)$ and $u_k = v_k$ is the identity $x = y$.

To show that $\mathbf{U}_\Sigma$ satisfies $a^\omega bxcd^\omega = a^\omega bycd^\omega$ we prove by induction on i that $\mathbf{U}_\Sigma$ satisfies $a^\omega bu_icd^\omega = a^\omega bv_icd^\omega$ for all identities $u_i = v_i$ in the sequence. The case $i = 1$ follows immediately from the definition of $\mathbf{U}_\Sigma$ since $u_1 = v_1$ is an identity of Σ.

Now let $1 < i \leq k$, suppose that $\mathbf{U}_\Sigma$ satisfies $a^\omega bu_jcd^\omega = a^\omega bv_jcd^\omega$ for all $1 \leq j < i$ and let us show that $\mathbf{U}_\Sigma$ satisfies $a^\omega bu_icd^\omega = a^\omega bv_icd^\omega$. Again, this is clear if $u_i = v_i$ is in Σ. So suppose that $u_i = v_i$ is obtained from the identities $u_j = v_j$ $(1 \leq j < i)$ using one of the rules $r1)$ to $r4)$. That is, $a^\omega bu_icd^\omega = a^\omega bv_icd^\omega$ is obtained from the pseudoidentities $a^\omega bu_jcd^\omega = a^\omega bv_jcd^\omega$ $(1 \leq j < i)$ by the application to the identities $u_j = v_j$ of one of the rules $r1)$ to $r4)$. If the rule used is $r1)$, $r2)$ or $r4)$,— since $\mathbf{U}_\Sigma$ satisfies $a^\omega bu_jcd^\omega = a^\omega bv_jcd^\omega$ $(1 \leq j < i)$ by the induction hypothesis and since the variables a, b, c and d do not occur in $u_j = v_j$,— we easily deduce that the pseudoidentity $a^\omega bu_icd^\omega = a^\omega bv_icd^\omega$ is satisfied by $\mathbf{U}_\Sigma$. There remains the case where the rule used is $r3)$.

Suppose therefore that $u_i = v_i$ is the identity $ru_js = rv_js$ for some $r, s \in A^*$ and $1 \leq j < i$. Then, by the induction hypothesis, $\mathbf{U}_\Sigma$ satisfies $a^\omega bu_jcd^\omega = a^\omega bv_jcd^\omega$ and, hence, satisfies also

$$\begin{aligned} a^\omega bu_icd^\omega &= a^\omega bru_jscd^\omega \\ &= a^\omega brv_jscd^\omega \\ &= a^\omega bv_icd^\omega. \end{aligned}$$

By induction we may now deduce that $\mathbf{U}_\Sigma$ satisfies $a^\omega bu_kcd^\omega = a^\omega bv_kcd^\omega$, that is, that $\mathbf{U}_\Sigma$ satisfies $a^\omega bxcd^\omega = a^\omega bycd^\omega$.

Second case Let us now consider the general case where x and y are arbitrary elements of $\hat{F}_n(\mathbf{S})$. As A_n^+ is dense in $\hat{F}_n(\mathbf{S})$ there exist sequences $(u_k)_k$ and $(v_k)_k$ of words of A_n^+ whose limits are, respectively, x and y. Moreover, the facts that $\mathbf{V}$ satisfies $x = y$ and that $\hat{F}_n(\mathbf{V})$ is finite,— since it coincides with $F_n(\mathbf{V})$ by Proposition 2.5,— permit us to assume that $\mathbf{V}$ satisfies $u_k = v_k$ for all $k \in \mathbb{N}$. As a consequence, as we proved in the first case above, $\mathbf{U}_\Sigma$ satisfies $a^\omega bu_kcd^\omega = a^\omega bv_kcd^\omega$ for all $k \in \mathbb{N}$. We deduce therefore that $\mathbf{U}_\Sigma$ satisfies also $a^\omega bxcd^\omega = a^\omega bycd^\omega$ by passing to the limit.

This concludes the proof of the result when $\mathbf{V}$ is a locally finite pseudovariety and Σ is a set of identities.

Second step Suppose now that $\mathbf{V}$ is an equational pseudovariety (not necessarily locally finite) and that Σ is a set of identities. Let $k \in \mathbb{N}$ and let $\mathbf{S}_k$ be the pseudovariety generated by all the semigroups of cardinality at most k. In particular $\mathbf{S}_k$ is locally finite since it is finitely generated. Let Λ_k be a basis (of identities) of $\mathbf{S}_k$. Now consider the pseudovariety

$$\mathbf{V}_k = \mathbf{V} \cap \mathbf{S}_k.$$

The pseudovariety $\mathbf{V}_k$ is also locally finite (since it is a subpseudovariety of $\mathbf{S}_k$) and admits as basis of identities the set $\Sigma \cup \Lambda_k$.

Let S be a semigroup of $\mathbf{U}_\Sigma$ and let k be its cardinality. In particular S is in $\mathbf{S}_k = [\![\Lambda_k]\!]$. Therefore S lies also in the pseudovariety $\mathbf{U}_{\Lambda_k} \cap \mathbf{U}_\Sigma = \mathbf{U}_{\Lambda_k \cup \Sigma}$. According to the first step this pseudovariety is exactly $\mathcal{U}\mathbf{V}_k$. Now as $\mathbf{V}_k$ is a subpseudovariety of $\mathbf{V}$, we deduce that $\mathcal{U}\mathbf{V}_k \subseteq \mathcal{U}\mathbf{V}$ and so that S lies in $\mathcal{U}\mathbf{V}$.

This proves the inclusion $\mathbf{U}_\Sigma \subseteq \mathbf{U}_\Delta$ which concludes the proof of the Proposition when $\mathbf{V}$ is an equational pseudovariety and Σ is a set of identities.

Third step Let us finally consider the general case where $\mathbf{V}$ and Σ are arbitrary. Let $S \in \mathbf{U}_\Sigma$. Hence, for all pseudoidentities $x = y \in \Sigma$, S satisfies $a^\omega bxcd^\omega = a^\omega bycd^\omega$. For all $x = y \in \Sigma$, consider a sequence of identities $(x_i = y_i)_{i\in\mathbb{N}}$ converging to $x = y$. Then there is an integer k such that S satisfies $a^\omega bx_icd^\omega = a^\omega by_icd^\omega$ for all $i \geq k$. Taking subsequences if necessary, we may assume that, for all $x = y \in \Sigma$, S satisfies all the pseudoidentities $a^\omega bx_icd^\omega = a^\omega by_icd^\omega$. Now consider the following set of identities

$$\Lambda = \{x_i = y_i \mid x = y \in \Sigma,\ i \in \mathbb{N}\}$$

and let $\mathbf{W}$ be the pseudovariety defined by Λ. It is clear that $\mathbf{W} \subseteq \mathbf{V}$ and that $S \in \mathbf{U}_\Lambda$. Furthermore, we know from the second step that $\mathbf{U}_\Lambda = \mathcal{U}\mathbf{W}$. But since $\mathbf{W} \subseteq \mathbf{V}$, we have $\mathcal{U}\mathbf{W} \subseteq \mathcal{U}\mathbf{V}$ which implies that $S \in \mathcal{U}\mathbf{V}$.

This concludes the proof of the inclusion $\mathbf{U}_\Sigma \subseteq \mathcal{U}\mathbf{V}$, from which the result follows. □

4 Some computations

Consider the pseudovariety defined by the pseudoidentity $ab^\omega c = (ab^\omega c)^{\omega+1}$. Notice that this pseudoidentity is obtained from the pseudoidentity $a = a^{\omega+1}$, which defines the pseudovariety $\mathbf{CR}$ of completely regular semigroups, by the substitution of the variable a by $ab^\omega c$. According to Pin and Weil [20] this means that

$$[\![ab^\omega c = (ab^\omega c)^{\omega+1}]\!] = \mathbf{CR} \circledm \mathbf{N}.$$

It is clear that $\mathbf{CR}$ and $\mathcal{L}\mathbf{I}$ are subpseudovarieties of $\mathbf{CR} \circledm \mathbf{N}$, as are $\mathbf{G}$, $\mathbf{B}$, $\mathbf{K}$, $\mathbf{D}$, $\mathbf{N}$, etc. As examples of pseudovarieties that are not contained in $\mathbf{CR} \circledm \mathbf{N}$ we cite $\mathbf{Com}$ and $\mathbf{J}$. We remark that a finite semigroup S lies in $\mathbf{CR} \circledm \mathbf{N}$ if and only if for all $s \in S$ and $e \in E(S)$, if $s \leq_\mathcal{J} e$ then s is a group element. In other words, the ideal of S generated by its idempotents is a completely regular subsemigroup. So, we can say that

S is a nilpotent extension of a completely regular semigroup. In particular, each regular $\mathcal{D}$-class of S is a union of groups which shows that $\mathbf{CR} \circledcirc \mathbf{N} \subseteq \mathcal{D}\mathbf{S}$.

Finally notice that $\mathbf{CR} \circledcirc \mathbf{N}$ can be defined alternatively by

$$\mathbf{CR} \circledcirc \mathbf{N} = [\![a^\omega b = (a^\omega b)^{\omega+1}, ba^\omega = (ba^\omega)^{\omega+1}]\!] = \mathcal{L}_l\mathbf{CR} \cap \mathcal{L}_r\mathbf{CR}.$$

Indeed one can easily verify that $\mathbf{CR} \circledcirc \mathbf{N}$ is contained in $\mathcal{L}_l\mathbf{CR} \cap \mathcal{L}_r\mathbf{CR}$. Conversely, $\mathcal{L}_l\mathbf{CR} \cap \mathcal{L}_r\mathbf{CR}$ satisfies

$$ab^\omega c = (ab^\omega)^\omega ab^\omega c = ((ab^\omega)^\omega ab^\omega c)^{\omega+1} = (ab^\omega c)^{\omega+1}.$$

We can now present the main result of this section which gives, in particular, the characterization of the pseudovarieties of the form $\mathcal{L}\mathbf{I} \vee \mathbf{V}$ where $\mathbf{V}$ is any subpseudovariety of $\mathbf{CR} \circledcirc \mathbf{N}$.

Theorem 4.1 *Let* $\mathbf{V}$ *be a subpseudovariety of, respectively,* $\mathcal{L}_l\mathbf{CR} \cap \mathcal{L}_r\mathbf{CR}$, $\mathcal{L}_r\mathbf{CR}$ *or* $\mathcal{L}_l\mathbf{CR}$. *Then we have, respectively,*

$$\begin{aligned}
\mathcal{L}\mathbf{I} \vee \mathbf{V} &= \mathcal{U}\mathbf{V} \cap \mathcal{L}_l\mathbf{CR} \cap \mathcal{L}_r\mathbf{CR} \\
&= \mathcal{U}\mathbf{V} \cap \mathcal{L}_l(\mathbf{D}_1 \vee \mathbf{V}) \cap \mathcal{L}_r(\mathbf{K}_1 \vee \mathbf{V}) \\
\mathbf{K} \vee \mathbf{V} &= \mathcal{U}_l\mathbf{V} \cap \mathcal{L}_r\mathbf{CR} \\
&= \mathcal{U}_l\mathbf{V} \cap \mathcal{L}_r(\mathbf{K}_1 \vee \mathbf{V}) \\
\mathbf{D} \vee \mathbf{V} &= \mathcal{U}_r\mathbf{V} \cap \mathcal{L}_l\mathbf{CR} \\
&= \mathcal{U}_r\mathbf{V} \cap \mathcal{L}_l(\mathbf{D}_1 \vee \mathbf{V}).
\end{aligned}$$

Proof. The other cases being similar, we show the result only for $\mathbf{K}$. The inclusion $\mathbf{K} \vee \mathbf{V} \subseteq \mathcal{U}_l\mathbf{V} \cap \mathcal{L}_r\mathbf{CR}$ is clear. For the proof of the reverse inclusion, consider a pseudoidentity $x = y$ and suppose that $\mathbf{K} \vee \mathbf{V}$ satisfies $x = y$. By Reiterman's Theorem, it suffices to prove that $\mathcal{U}_l\mathbf{V} \cap \mathcal{L}_r\mathbf{CR}$ satisfies $x = y$.

Since $\mathbf{K}$ satisfies $x = y$, two cases may arise: either x and y are the same word or x and y are both not explicit. In this last case, Corollary 2.3 shows that we can write $x = ru$ and $y = rv$, for some $r, u, v \in \hat{F}_n(\mathbf{S})$ with $r \notin A_n^+$. Furthermore, we have from Proposition 2.6, $r = r_1 r_2^\omega r_3$, for some $r_1, r_2, r_3 \in \hat{F}_n(\mathbf{S})$. Now notice that $\mathcal{L}_r\mathbf{CR}$ satisfies

$$x = ru = r_1 r_2^\omega r_3 u = (r_1 r_2^\omega)^{\omega+1} r_3 u = (r_1 r_2^\omega)^\omega x.$$

Analogously, $\mathcal{L}_r\mathbf{CR}$ satisfies $y = (r_1 r_2^\omega)^\omega y$. Finally, since $\mathbf{V}$ satisfies $x = y$ it is clear, from its definition, that $\mathcal{U}_l\mathbf{V}$ satisfies $(r_1 r_2^\omega)^\omega x = (r_1 r_2^\omega)^\omega y$. Therefore, $\mathcal{U}_l\mathbf{V} \cap \mathcal{L}_r\mathbf{CR}$ satisfies $x = (r_1 r_2^\omega)^\omega x = (r_1 r_2^\omega)^\omega y = y$, which proves the first equality concerning $\mathbf{K}$.

The second equality concerning $\mathbf{K}$ is a consequence of the first one and of the relations $\mathbf{K} \vee \mathbf{V} \subseteq \mathcal{L}_r(\mathbf{K}_1 \vee \mathbf{V}) \subseteq \mathcal{L}_r(\mathcal{L}_r\mathbf{CR}) = \mathcal{L}_r\mathbf{CR}$. □

When we restrict $\mathbf{V}$ to subpseudovarieties of $\mathcal{L}\mathbf{G}$ we can further simplify Theorem 4.1. It suffices to note that $\mathcal{U}\mathcal{L}\mathbf{G} = \mathcal{L}\mathbf{G} \subseteq \mathbf{CR} \circledcirc \mathbf{N}$.

Corollary 4.2 *If* $\mathbf{V}$ *is a subpseudovariety of* $\mathcal{L}\mathbf{G}$, *then* $\mathcal{L}\mathbf{I} \vee \mathbf{V} = \mathcal{U}\mathbf{V}$, $\mathbf{K} \vee \mathbf{V} = \mathcal{U}_l\mathbf{V}$ *and* $\mathbf{D} \vee \mathbf{V} = \mathcal{U}_r\mathbf{V}$. □

On the contrary, as one can easily show, the pseudovariety $\mathcal{U}\mathbf{Sl}$ is not contained in $\mathbf{CR}ⓜ\,\mathbf{N}$ so that $\mathcal{L}\mathbf{I}\vee\mathbf{Sl}$ is strictly contained in $\mathcal{U}\mathbf{Sl}$. Hence $\mathcal{L}\mathbf{I}\vee\mathbf{V}$ is strictly contained in $\mathcal{U}\mathbf{V}$ for any pseudovariety $\mathbf{V}$ in the interval $[\mathbf{Sl}, \mathbf{CR}ⓜ\,\mathbf{N}]$. We recall that a pseudovariety contains $\mathbf{Sl}$ if and only if it is not contained in $\mathcal{L}\mathbf{G}$.

We notice also that in Theorem 4.1, since $\mathcal{L}\mathbf{I}\vee\mathbf{V}$ is contained in $\mathcal{L}_l\mathbf{CR}\cap\mathcal{L}_r\mathbf{CR}$, the variables b and c in the definition of $\mathcal{U}\mathbf{V}$ can be excluded. The same is true for $\mathbf{K}$ (resp. $\mathbf{D}$) provided the pseudovariety $\mathbf{V}$ is contained in $\mathcal{L}_l\mathbf{CR}$ (resp. $\mathcal{L}_r\mathbf{CR}$). That is, we have the following "simplifications".

Corollary 4.3 *Let $\mathbf{V}=[\![\Sigma]\!]$ be a pseudovariety contained in $\mathcal{L}_l\mathbf{CR}\cap\mathcal{L}_r\mathbf{CR}$. Then*

$$\begin{aligned}
\mathcal{L}\mathbf{I}\vee\mathbf{V} &= [\![a^\omega x d^\omega = a^\omega y d^\omega \mid x=y\in\Sigma]\!]\cap\mathbf{U}\\
\mathbf{K}\vee\mathbf{V} &= [\![a^\omega x = a^\omega y \mid x=y\in\Sigma]\!]\cap\mathbf{U}\\
\mathbf{D}\vee\mathbf{V} &= [\![x d^\omega = y d^\omega \mid x=y\in\Sigma]\!]\cap\mathbf{U},
\end{aligned}$$

where $\mathbf{U}$ is alternatively $\mathcal{L}_l\mathbf{CR}\cap\mathcal{L}_r\mathbf{CR}$ or $\mathcal{L}_l(\mathbf{D}_1\vee\mathbf{V})\cap\mathcal{L}_r(\mathbf{K}_1\vee\mathbf{V})$.

Proof. We prove the result for $\mathbf{K}$. Set $\mathbf{W}=[\![a^\omega x = a^\omega y \mid x=y\in\Sigma]\!]$. The inclusion $\mathbf{K}\vee\mathbf{V}\subseteq\mathbf{W}\cap\mathcal{L}_l\mathbf{CR}\cap\mathcal{L}_r\mathbf{CR}$ is clear. For the proof of the reverse inclusion, it suffices from Theorem 4.1 to prove that $\mathbf{W}\cap\mathcal{L}_l\mathbf{CR}\subseteq\mathcal{U}_l\mathbf{V}$. For this, we show that $\mathbf{W}\cap\mathcal{L}_l\mathbf{CR}$ satisfies each pseudoidentity $a^\omega bx = a^\omega by$ with $x=y\in\Sigma$. We deduce successively that $\mathbf{W}\cap\mathcal{L}_l\mathbf{CR}$ satisfies

$$\begin{aligned}
a^\omega bx &= a^\omega b(a^\omega b)^\omega x\\
&= a^\omega b(a^\omega b)^\omega y\\
&= a^\omega by,
\end{aligned}$$

proving the claim. □

Thus, this last corollary permits us to deduce, for instance, the following equalities which were first proved by Zeitoun [27],

$$\begin{aligned}
\mathcal{L}\mathbf{I}\vee\mathbf{B} &= [\![a^\omega bc^\omega = a^\omega b^2 c^\omega, a^\omega b=(a^\omega b)^2, ba^\omega=(ba^\omega)^2]\!]\\
\mathbf{K}\vee\mathbf{B} &= [\![a^\omega b = a^\omega b^2, a^\omega b=(a^\omega b)^2, ba^\omega=(ba^\omega)^2]\!]\\
\mathbf{D}\vee\mathbf{B} &= [\![ba^\omega = b^2 a^\omega, a^\omega b=(a^\omega b)^2, ba^\omega=(ba^\omega)^2]\!].
\end{aligned}$$

But since the pseudoidentity $a^\omega b=(a^\omega b)^2$ is easily derived from $a^\omega b = a^\omega b^2$ (and dually $ba^\omega=(ba^\omega)^2$ is a consequence of $ba^\omega = b^2a^\omega$) we have simply

$$\begin{aligned}
\mathbf{K}\vee\mathbf{B} &= [\![a^\omega b = a^\omega b^2, ba^\omega=(ba^\omega)^2]\!]\\
\mathbf{D}\vee\mathbf{B} &= [\![ba^\omega = b^2 a^\omega, a^\omega b=(a^\omega b)^2]\!].
\end{aligned}$$

Analogously, we deduce also, for instance,

$$\begin{aligned}
\mathcal{L}\mathbf{I}\vee\mathbf{CR} &= [\![a^\omega bc^\omega = a^\omega b^{\omega+1} c^\omega, a^\omega b=(a^\omega b)^{\omega+1}, ba^\omega=(ba^\omega)^{\omega+1}]\!]\\
&= [\![a^\omega bc^\omega = a^\omega b^{\omega+1} c^\omega, ab^\omega c=(ab^\omega c)^{\omega+1}]\!]\\
\mathbf{K}\vee\mathbf{CR} &= [\![a^\omega b = a^\omega b^{\omega+1}, ba^\omega=(ba^\omega)^{\omega+1}]\!]\\
\mathbf{D}\vee\mathbf{CR} &= [\![ba^\omega = b^{\omega+1} a^\omega, a^\omega b=(a^\omega b)^{\omega+1}]\!].
\end{aligned}$$

In the case of the pseudovariety $\mathbf{N}$ of nilpotent semigroups, we have the following result.

Proposition 4.4 *Let* $\mathbf{V}$ *be a pseudovariety such that, for every* $x \in \hat{F}_n(\mathbf{S}) \setminus F_n(\mathbf{S})$, $\mathbf{V}$ *satisfies* $x = \bar{x}^\omega x$ *or* $x = x\bar{x}^\omega$ *for some* $\bar{x} \in \hat{F}_n(\mathbf{S})$. *Then the following equality holds*

$$\mathbf{N} \vee \mathbf{V} = (\mathbf{K} \vee \mathbf{V}) \cap (\mathbf{D} \vee \mathbf{V}).$$

Proof. Let $\mathbf{U} = (\mathbf{K} \vee \mathbf{V}) \cap (\mathbf{D} \vee \mathbf{V})$. That $\mathbf{N} \vee \mathbf{V}$ is included in $\mathbf{U}$ is evident. For the proof of the reverse inclusion, let us suppose that $\mathbf{N} \vee \mathbf{V}$ satisfies a pseudoidentity $x = y$. In particular $\mathbf{N}$ satisfies $x = y$ and so either x and y are identical words or x and y are both not explicit. In this case we may assume, by hypothesis, that $\mathbf{V}$ satisfies $x = \bar{x}^\omega x$ for some $\bar{x} \in \hat{F}_n(\mathbf{S})$. Therefore, since $\mathbf{V}$ satisfies $x = y$ and x and y are not explicit, it is easy to verify that $\mathbf{D} \vee \mathbf{V}$ (and so also $\mathbf{U}$) satisfies $x = \bar{x}^\omega x$ and $y = \bar{x}^\omega y$. Moreover $\mathbf{K} \vee \mathbf{V}$ (and so also $\mathbf{U}$) satisfies $\bar{x}^\omega x = \bar{x}^\omega y$. We deduce, therefore, that $\mathbf{U}$ satisfies $x = y$ which proves the inclusion $\mathbf{U} \subseteq \mathbf{N} \vee \mathbf{V}$. Hence the equality is valid. □

This last result is valid, in particular, when $\mathbf{V}$ is a subpseudovariety of $\mathcal{L}_l\mathbf{CR}$, of $\mathcal{L}_r\mathbf{CR}$ or of $\mathbf{ZE}$. So consider, for instance, the equalities given above for $\mathbf{K} \vee \mathbf{CR}$ and $\mathbf{D} \vee \mathbf{CR}$. We remark that the second pseudoidentity in each of the bases provided for $\mathbf{K} \vee \mathbf{CR}$ and $\mathbf{D} \vee \mathbf{CR}$ becomes superfluous when we take their intersection since it is an immediate consequence of the first pseudoidentity of the other basis. Therefore, Proposition 4.4 permits us to deduce the equality

$$\mathbf{N} \vee \mathbf{CR} = [\![a^\omega b = a^\omega b^{\omega+1}, ba^\omega = b^{\omega+1}a^\omega]\!].$$

More generally, one can show analogously that we have the two following alternative bases of pseudoidentities,— to those given by Reilly and Zhang [21],— for the pseudovarieties of the form $\mathbf{N} \vee \mathbf{V}$, with $\mathbf{V} \subseteq \mathbf{CR}$.

Corollary 4.5 *Let* $\mathbf{V}$ *be a subpseudovariety of* $\mathbf{CR}$ *and let* Σ *be a basis of pseudoidentities defining* $\mathbf{V}$. *Then* $\mathbf{N} \vee \mathbf{V} = [\![a^\omega x = a^\omega y, xa^\omega = ya^\omega \mid x = y \in \Sigma]\!] \cap (\mathbf{N} \vee \mathbf{CR}) = \mathcal{U}_l\mathbf{V} \cap \mathcal{U}_r\mathbf{V}$.

Proof. It is a direct application of the previous results. □

Now we prove a join decomposition which is an easy consequence of Theorem 4.1 and of Proposition 4.4.

Corollary 4.6 *Let* $\mathbf{V}$ *be the pseudovariety* $\mathcal{L}\mathbf{I}$ *(resp.* $\mathbf{K}$, $\mathbf{D}$, $\mathbf{N}$*) and let* $(\mathbf{V}_i)_{i\in I}$ *be a family of pseudovarieties such that* $\bigcap_{i\in I}(\mathbf{V} \vee \mathbf{V}_i)$ *is a subpseudovariety of* $\mathcal{L}_l\mathbf{CR} \cap \mathcal{L}_r\mathbf{CR}$ *(resp.* $\mathcal{L}_r\mathbf{CR}$, $\mathcal{L}_l\mathbf{CR}$, $\mathcal{L}_l\mathbf{CR} \cap \mathcal{L}_r\mathbf{CR}$*). Then*

$$\mathbf{V} \vee (\bigcap_{i\in I} \mathbf{V}_i) = \bigcap_{i\in I}(\mathbf{V} \vee \mathbf{V}_i).$$

Proof. We show the result for $\mathcal{L}\mathbf{I}$. The cases $\mathbf{K}$ and $\mathbf{D}$ are similar and the case $\mathbf{N}$ is an immediate consequence of these results and of Proposition 4.4.

The inclusion $\mathcal{L}\mathbf{I} \vee (\bigcap_{i\in I} \mathbf{V}_i) \subseteq \bigcap_{i\in I}(\mathcal{L}\mathbf{I} \vee \mathbf{V}_i)$ is immediate. Now, for each $i \in I$, let Σ_i be a basis of pseudoidentities for $\mathbf{V}_i$. Then the pseudovariety $\bigcap_{i\in I} \mathbf{V}_i$ is defined by the set $\bigcup_{i\in I} \Sigma_i$. As $\bigcap_{i\in I} \mathbf{V}_i \subseteq \bigcap_{i\in I}(\mathcal{L}\mathbf{I} \vee \mathbf{V}_i) \subseteq \mathcal{L}_l\mathbf{CR} \cap \mathcal{L}_r\mathbf{CR}$ we deduce from Theorem 4.1 that

$$\mathcal{L}\mathbf{I} \vee (\bigcap_{i\in I} \mathbf{V}_i) = [\![a^\omega bxcd^\omega = a^\omega bycd^\omega \mid x = y \in \bigcup_{i\in I} \Sigma_i]\!] \cap \mathcal{L}_l\mathbf{CR} \cap \mathcal{L}_r\mathbf{CR}.$$

To show the inclusion right to left it suffices, therefore, to prove that $\bigcap_{i\in I}(\mathcal{L}\mathbf{I}\vee\mathbf{V}_i)$ satisfies each pseudoidentity $a^\omega bxcd^\omega = a^\omega bycd^\omega$ with $x=y\in\bigcup_{i\in I}\Sigma_i$. But if $x=y\in\bigcup_{i\in I}\Sigma_i$, then $x=y\in\Sigma_j$ for some $j\in I$. So the pseudovariety $\mathcal{L}\mathbf{I}\vee\mathbf{V}_j$ satisfies the pseudoidentity $a^\omega bxcd^\omega = a^\omega bycd^\omega$, whence $\bigcap_{i\in I}(\mathcal{L}\mathbf{I}\vee\mathbf{V}_i)$ satisfies it too. □

5 More computations

Let $\mathbf{V}$ be a pseudovariety of semigroups. Notice that, as in the proof of Theorem 4.1, if $\mathcal{L}\mathbf{I}\vee\mathbf{V}$ satisfies a pseudoidentity $x=y$ with x and y not explicit, then x and y can be written in the form $x=rs^\omega wu^\omega v$ and $y=rs^\omega w'u^\omega v$ for some $r,s,u,v,w,w'\in\hat{F}_n(\mathbf{S})$.

We say that $\mathbf{V}$ *satisfies* $(\mathcal{C})$ if, for every pseudoidentity $x=y$ with x and y not explicit,

$$\mathcal{L}\mathbf{I}\vee\mathbf{V}\models x=y \quad\Rightarrow\quad \begin{array}{l}\text{there exist } r,s,u,v,w,w'\in\hat{F}_n(\mathbf{S}) \text{ such that}\\ \mathcal{U}\mathbf{V}\models x=rs^\omega wu^\omega v,\ y=rs^\omega w'u^\omega v\\ \text{and } \mathbf{V}\models s^\omega wu^\omega=s^\omega w'u^\omega.\end{array}$$

A one-sided version of this property is the following. We say that $\mathbf{V}$ *satisfies* $(\mathcal{C}_l)$ if, for every pseudoidentity $x=y$ with x and y not explicit,

$$\mathbf{K}\vee\mathbf{V}\models x=y \quad\Rightarrow\quad \begin{array}{l}\text{there exist } r,s,w,w'\in\hat{F}_n(\mathbf{S}) \text{ such that}\\ \mathcal{U}_l\mathbf{V}\models x=rs^\omega w,\ y=rs^\omega w' \text{ and } \mathbf{V}\models s^\omega w=s^\omega w'.\end{array}$$

A property $(\mathcal{C}_r)$ is defined symmetrically.

We now present a characterization of the pseudovarieties of the form $\mathcal{L}\mathbf{I}\vee\mathbf{V}$ where $\mathbf{V}$ is any pseudovariety satisfying $(\mathcal{C})$.

Theorem 5.1 *Let* $\mathbf{V}$ *be a pseudovariety satisfying* $(\mathcal{C})$ *(resp.* $(\mathcal{C}_l)$*,* $(\mathcal{C}_r)$*). Then,* $\mathcal{L}\mathbf{I}\vee\mathbf{V}=\mathcal{U}\mathbf{V}$ *(resp.* $\mathbf{K}\vee\mathbf{V}=\mathcal{U}_l\mathbf{V}$*,* $\mathbf{D}\vee\mathbf{V}=\mathcal{U}_r\mathbf{V}$*).*

Proof. We show the result for $\mathbf{K}$. The beginning of the proof is similar to the proof of Theorem 4.1. So we only consider the case when $\mathbf{K}\vee\mathbf{V}$ satisfies a pseudoidentity $x=y$, with x and y both not explicit, and prove that $\mathcal{U}_l\mathbf{V}$ satisfies $x=y$. We show that this is a consequence of the assumption that $\mathbf{V}$ satisfies $(\mathcal{C}_l)$.

Indeed, in that case, since $\mathbf{K}\vee\mathbf{V}$ satisfies $x=y$ and x and y are both not explicit, then, by $(\mathcal{C}_l)$, $\mathcal{U}_l\mathbf{V}$ satisfies $x=rs^\omega w$ and $y=rs^\omega w'$ for some $r,s,w,w'\in\hat{F}_n(\mathbf{S})$ such that $\mathbf{V}$ satisfies $s^\omega w=s^\omega w'$. Now, by definition of $\mathcal{U}_l\mathbf{V}$, it is clear that $\mathcal{U}_l\mathbf{V}$ also satisfies $s^\omega w=s^\omega w'$. Therefore $\mathcal{U}_l\mathbf{V}$ satisfies $x=rs^\omega w=rs^\omega w'=y$. □

Let $\mathbf{H}$ be a pseudovariety of groups. This last result applies, for instance, to the pseudovarieties $\mathbf{H}$, $\mathbf{Com}$, $\mathbf{J}$, $\mathcal{D}\mathbf{RH}$ (and so, in particular, to $\mathbf{R}$), $\mathcal{D}\mathbf{RH}\cap\mathcal{L}\mathcal{E}\mathbf{Com}$, $\mathcal{D}\mathbf{H}\cap\mathcal{L}\mathcal{E}\mathbf{Com}$, $\mathcal{D}\mathbf{RH}\cap\mathcal{L}\mathbf{ZE}$ and $\mathcal{D}\mathbf{H}\cap\mathcal{L}\mathbf{ZE}$. It applies also to $\mathbf{Com}\vee\mathbf{H}$ and so, in particular, to $\mathbf{ZE}$.

Let us show that $\mathbf{J}$ satisfies $(\mathcal{C})$. Suppose that $\mathbf{LI}\vee\mathbf{J}$ satisfies a pseudoidentity $x=y$, with x and y not explicit. In particular $\mathbf{J}$ satisfies $x=y$. Then, by Proposition 2.4, $\mathcal{D}\mathbf{S}$ (and so also $\mathcal{U}\mathbf{J}$ since it is contained in $\mathcal{D}\mathbf{S}$) satisfies $x=u_0x_1u_1\cdots x_ru_r$ where: each u_i is a word and each x_i is regular when restricted to $\mathcal{D}\mathbf{S}$; if u_i is empty, then $c(x_i)$ and $c(x_{i+1})$ are $\subseteq$-incomparable; if u_i is not empty, its first letter is not in $c(x_i)$ and its last letter is not in $c(x_{i+1})$. Moreover, $\mathcal{D}\mathbf{S}$ satisfies $y=u_0y_1u_1\cdots y_ru_r$ with y_i regular when restricted to $\mathcal{D}\mathbf{S}$ and $c(x_i) = c(y_i)$. Now, since $\mathcal{L}\mathbf{I}$ satisfies $x = y$, one can show that $\mathcal{D}\mathbf{S}$ satisfies $x_1 = t^\omega x_1'$, $y_1 = t^\omega y_1'$, $x_r = x_r' z^\omega$ and $y_r = y_r' z^\omega$, for some $x_1', y_1', x_r', y_r', t, z \in \hat{F}_n(\mathbf{S})$, so that $\mathcal{D}\mathbf{S}$ satisfies $x = u_0 t^\omega x_1' u_1 \cdots x_r' z^\omega u_r$ and $y = u_0 t^\omega y_1' u_1 \cdots y_r' z^\omega u_r$. Moreover, we deduce from Proposition 2.4 that $\mathbf{J}$ satisfies $t^\omega x_1' u_1 \cdots x_r' z^\omega = t^\omega y_1' u_1 \cdots y_r' z^\omega$. That is, $\mathbf{J}$ satisfies $(\mathcal{C})$. Analogously, $\mathbf{J}$ satisfies $(\mathcal{C}_l)$ and $(\mathcal{C}_r)$.

That $\mathcal{D}\mathbf{RH}$ satisfies $(\mathcal{C})$ can be proved analogously, using the characterization of the implicit operations on $\mathcal{D}\mathbf{RH}$ given by Almeida and Weil [11]. The same can be done for $\mathcal{D}\mathbf{RH}\cap\mathcal{L}\mathcal{E}\mathbf{Com}$, $\mathcal{D}\mathbf{H}\cap\mathcal{L}\mathcal{E}\mathbf{Com}$, $\mathcal{D}\mathbf{RH}\cap\mathcal{L}\mathbf{ZE}$ and $\mathcal{D}\mathbf{H}\cap\mathcal{L}\mathbf{ZE}$ using the characterization of the implicit operations on each of these pseudovarieties given by the author [15, 17].

To show that $\mathbf{Com}$ satisfies $(\mathcal{C}_l)$, for instance, it suffices to note that $\mathbf{Com} \subseteq \mathcal{D}\mathbf{S}$ and that, if $\mathbf{Com}$ satisfies a pseudoidentity of the form $ax = ay$, where a is a letter, then $\mathbf{Com}$ satisfies $x = y$. This fact is an almost direct consequence of the considerations of [6, pp. 91-92]. That $\mathbf{H}$ satisfies $(\mathcal{C}_l)$ is clear since $\hat{F}_n(\mathbf{H})$ satisfies the cancellation law. We then deduce that also $\mathbf{Com} \vee \mathbf{H}$ satisfies $(\mathcal{C}_l)$ since a pseudoidentity is satisfied by $\mathbf{Com} \vee \mathbf{H}$ if and only if it is satisfied by both $\mathbf{Com}$ and $\mathbf{H}$.

6 Conclusion and open questions

The examples of pseudovarieties satisfying $(\mathcal{C})$, presented after Theorem 5.1, suggest that this property must be valid frequently. On the other hand, Theorem 5.1 shows that these pseudovarieties satisfy the formula

$$\mathcal{L}\mathbf{I} \vee \mathbf{V} = \mathcal{U}\mathbf{V} \cap \mathcal{L}_l(\mathbf{D}_1 \vee \mathbf{V}) \cap \mathcal{L}_r(\mathbf{K}_1 \vee \mathbf{V})$$

which is also valid for subpseudovarieties of $\mathbf{CR} \circledm \mathbf{N}$ as shown in Theorem 4.1. One can, therefore, ask if that formula is satisfied by any pseudovariety. Similar questions can be made for the one-sided cases. We do not know the answer to these questions.

We also suspect that the equality

$$\mathbf{N} \vee \mathbf{V} = (\mathbf{K} \vee \mathbf{V}) \cap (\mathbf{D} \vee \mathbf{V})$$

proved in Proposition 4.4 for certain pseudovarieties, is valid in general.

Acknowledgements

The results of this article were obtained, in most part, during my Ph.D. Thesis under the supervision of Pascal Weil. My thanks to him. I am also grateful to Assis Azevedo, as always, for helpful discussions during the preparation of this paper.

References

[1] D. Albert, R. Baldinger and J. Rhodes. Undecidability of the identity problem for finite semigroups, *J. Symbolic Logic* **57** (1992) 179-192.

[2] J. Almeida. Power pseudovarieties of semigroups I, *Semigroup Forum* **33** (1986) 357-373.

[3] J. Almeida. Some pseudovariety joins involving the pseudovariety of finite groups, *Semigroup Forum* **37** (1988) 53-57.

[4] J. Almeida. The algebra of implicit operations, *Algebra Universalis* **26** (1989) 16-72.

[5] J. Almeida. Implicit operations on finite $\mathcal{J}$-trivial semigroups and a conjecture of I. Simon, *J. Pure and Applied Algebra* **69** (1990) 205-218.

[6] J. Almeida. *Finite semigroups and universal algebra*, World Scientific, Singapore, 1994.

[7] J. Almeida and A. Azevedo. The join of the pseudovarieties of $\mathcal{R}$-trivial and $\mathcal{L}$-trivial semigroups, *J. Pure and Applied Algebra* **60** (1989) 129-137.

[8] J. Almeida and A. Azevedo. On regular implicit operations, *Portugaliæ Mathematica* **50** (1993) 35-61.

[9] J. Almeida, A. Azevedo and M. Zeitoun. Pseudovariety joins involving $\mathcal{J}$-trivial semigroups and completely regular semigroups, *Int. J. Algebra and Computation*, to appear.

[10] J. Almeida and P. Weil. Reduced factorizations in free profinite groups and join decompositions of pseudovarieties, *Int. J. Algebra and Computation* **4** (1994) 375-403.

[11] J. Almeida and P. Weil. Free profinite $\mathcal{R}$-trivial semigroups, *Int. J. Algebra and Computation* **7** (1997) 625-671.

[12] A. Azevedo. *Operações implícitas sobre pseudovariedades de semigrupos. Aplicações*, Ph.D. Thesis, Universidade do Porto, 1989.

[13] A. Azevedo. The join of the pseudovariety **J** with permutative pseudovarieties, in J. Almeida and al (Eds.), *Lattices, Semigroups and Universal Algebra*, Plenum, New York (1990) 1-11.

[14] A. Azevedo and M. Zeitoun. Three examples of join computations, *Semigroup Forum* **57** (1998) 249-277.

[15] J. Costa, *Quelques intersections de variétés de semigroupes finis et de variétés de langages, opérations implicites*, Ph.D. Thesis, Université Paris 6, 1998.

[16] J. Costa. Free profinite $\mathcal{R}$-trivial, locally idempotent and locally commutative semigroups, *Semigroup Forum* **58** (1999) 423-444.

[17] J. Costa. Free profinite semigroups over some classes of semigroups locally in $\mathcal{D}$**G**, *Int. J. Algebra and Computation*, to appear.

[18] S. Eilenberg. *Automata, languages and machines*, vol. B, Academic Press, New York, 1976.

[19] J.-E. Pin. *Varieties of formal languages*, Plenum, New York and North Oxford, London, 1986.

[20] J.-E. Pin and P. Weil. Profinite semigroups, Mal'cev products and identities, *J. Algebra* **182** (1996) 604-626.

[21] N. R. Reilly and S. Zhang. Operators on the lattice of pseudovarieties of finite semigroups, *Semigroup Forum* **57** (1998) 208-239.

[22] J. Reiterman. The Birkhoff theorem for finite algebras, *Algebra Universalis* **14** (1982) 1-10.

[23] W. Taylor. Equational logic, *Houston J. Math.* (1979) i-iii, 1-83. Survey issue.

[24] P. Trotter and M. Volkov. The finite basis problem in the pseudovariety joins of aperiodic semigroups with groups, *Semigroup Forum* **52** (1996) 83-91.

[25] M. Volkov. On a class of semigroup pseudovarieties without finite pseudoidentity basis, *Int. J. Algebra and Computation* **5** (1995) 127-135.

[26] M. Zeitoun. On the decidability of the membership problem of the pseudovariety $\mathbf{J} \vee \mathbf{B}$, *Int. J. Algebra and Computation* **4** (1994).

[27] M. Zeitoun. The join of the pseudovarieties of idempotent semigroups and locally trivial semigroups, *Semigroup Forum* **50** (1995) 367-381. Corrigendum, *Semigroup Forum* **52** (1996) 249.

Dep. Matemática
Universidade do Minho
Campus de Gualtar
4700-320 Braga
Portugal
email: jcosta@math.uminho.pt