Physics Letters B 752 (2016) 306–310

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Rényi entropy and the thermodynamic stability of black holes

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ARTICLE INFO

Article history:
Received 3 September 2015
Received in revised form 19 November 2015
Accepted 21 November 2015
Available online 26 November 2015
Editor: J.-P. Blaizot

Keywords:
Non-extensive entropy
Black hole thermodynamics
Stability

ABSTRACT

Thermodynamic stability of black holes, described by the Rényi formula as equilibrium compatible entropy function, is investigated. It is shown that within this approach, asymptotically flat, Schwarzschild black holes can be in stable equilibrium with thermal radiation at a fixed temperature. This implies that the canonical ensemble exists just like in anti-de Sitter space, and nonextensive effects can stabilize the black holes in a very similar way as it is done by the gravitational potential of an anti-de Sitter space. Furthermore, it is also shown that a Hawking–Page-like black hole phase transition occurs at a critical temperature which depends on the q-parameter of the Rényi formula.

1. Introduction

The aim of this Letter is to investigate the thermodynamic stability problem of a Schwarzschild black hole based on a recent approach [1], where the equilibrium compatible entropy function of the black hole is considered to be the Rényi one [2].

The nonextensive nature of the Bekenstein–Hawking entropy of black hole event horizons has been noticed [3] very early on after the thermodynamic theory of black holes had been formulated [4], and the corresponding thermodynamic stability problem has also been investigated many times with various approaches. The standard stability analysis of extensive systems however (with the criteria that the *Hessian* of the entropy function has no positive eigenvalues), is not applicable for black holes, as it strongly depends on the *additive* property of the entropy function, which condition clearly fails to hold in this case.

The standard thermodynamic functions of a Schwarzschild black hole are given by

$$S_{BH} = 4\pi M^2, \qquad \frac{1}{T_H} = \frac{\partial S_{BH}(M)}{\partial M} = 8\pi M, \tag{1}$$

and

$$C_{BH} = \frac{-S'^2_{BH}(M)}{S''_{BH}(M)} = -8\pi M^2, \tag{2}$$

where S_{BH} is the Bekenstein–Hawking entropy, T_H is the Hawking temperature and C_{BH} is the corresponding heat capacity of the black hole. In the classical approach (concluding from a *Hessian* analysis), Schwarzschild black holes appear to be thermodynamically unstable in the canonical treatment, since the heat capacity of the hole is always negative. On the other hand, this approach is clearly not reliable, as the Bekenstein–Hawking entropy is not additive, and the corresponding Hawking temperature is also not compatible with thermal equilibrium requirements [5]. For a better understanding on the problem, one needs to consider the consequences of nonadditive thermodynamic effects as well.

To circumvent this issue, Kaburaki et al. [6] have used an alternative approach, and investigated the thermodynamic stability of black holes by the Poincaré turning point method [7], which is a topological approach, and does not depend on the additivity of the entropy function. Later on, this method has been used to study critical phenomena of higher dimensional black holes and black rings as well [8].

In [9], we investigated the Bekenstein–Hawking entropy problem of a Schwarzschild black hole by considering the so-called formal logarithm approach [5] (discussed below), and found that (if the classical picture can be taken seriously without any quantum corrections in the small energy limit), the equilibrium compatible entropy function of the black hole is linear in the hole's mass, and

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http://dx.doi.org/10.1016/j.physletb.2015.11.061

the corresponding zeroth law compatible temperature is constant, i.e. it is independent of the hole's energy. We also analyzed the thermodynamic stability of the problem, and showed that isolated Schwarzschild black holes are stable against spherically symmetric perturbations within this approach.

In the present Letter however, we are focusing on the direction that we proposed in [1], where we regarded the Bekenstein–Hawking formula as a nonextensive Tsallis entropy [10]. This model was motivated by the requirement of the existence of an empirical temperature in thermal equilibrium, or in other words, by the satisfaction of the zeroth law of thermodynamics. By applying the formal logarithm method [5], we showed that the zeroth law compatible entropy function of black holes in this model is the Rényi one [2], and the corresponding temperature function has an interesting similarity to the one of an AdS black hole in standard thermodynamics [11].

In the general case, both the Tsallis and the Rényi entropies contain a constant free parameter, whose physical meaning may depend on the concrete physical situation. In particular, for the problem of black hole thermodynamics, it may arise e.g. from quantum corrections to micro black holes (a semi-classical approach has been obtained from the Bekenstein bound [12] in [1]), or from finite size reservoir corrections in the canonical ensemble [13,14]. Many other parametric situations are also possible.

The purpose of this Letter is to extend our study on the Tsallis–Rényi problem by investigating the corresponding thermodynamic stability of black holes. In the stability analysis we consider both the Poincaré turning point and the *Hessian* methods because the Rényi entropy is additive for factorizing probabilities, and hence the standard approach is also applicable. In the obtained results we find perfect agreement from both directions. Throughout the paper we use units such as $c = G = \hbar = k_B = 1$.

2. The Tsallis–Rényi approach

Nonextensive approaches to black hole thermodynamics have been investigated several times with various methods (see e.g. [15] and references therein), on the other hand, a zeroth law compatible formulation of nonextensive thermodynamics is a long standing problem, and a possible solution has been proposed only very recently. Based only on the concept of composability, Abe showed [16] that the most general nonadditive entropy composition rule which is compatible with homogeneous equilibrium has the form

$$H_\lambda(S_{12}) = H_\lambda(S_1) + H_\lambda(S_2) + \lambda H_\lambda(S_1)H_\lambda(S_2), \tag{3}$$

where H_λ is a differentiable function of S, $\lambda \in \mathbb{R}$ is a constant parameter, and S_1, S_2 and S_{12} are the entropies of the subsystems and the total system, respectively. By extending this result, Biró and Ván investigated non-homogeneous systems as well [5], and developed a formulation to determine the most general functional form of those nonadditive entropy composition rules that are compatible with the zeroth law of thermodynamics. They found that the general form is additive for the formal logarithms of the original quantities, which in turn, also satisfy the familiar relations of standard thermodynamics. They also showed, that for homogeneous systems the most general, zeroth law compatible entropy function has the form

$$L(S) = \frac{1}{\lambda}\ln[1 + \lambda H_\lambda(S)], \tag{4}$$

which is additive for composition, i.e.

$$L(S_{12}) = L(S_1) + L(S_2), \tag{5}$$

and the corresponding zeroth law compatible temperature function can be obtained as

$$\frac{1}{T} = \frac{\partial L(S(E))}{\partial E}, \tag{6}$$

if one assumes additivity in the energy composition.

For the classical black hole case, it is easy to show from the area law of the entropy function, that the Bekenstein–Hawking formula satisfies the equilibrium compatibility condition of (3), as it follows the nonadditive composition rule

$$S_{12} = S_1 + S_2 + 2\sqrt{S_1}\sqrt{S_2}, \tag{7}$$

which is equivalent with the choices of $H_\lambda(S) = \sqrt{S}$ and the $\lambda \to 0$ limit in the Abe formula [16]. The corresponding thermodynamic and stability problem for the case of a Schwarzschild black hole (applying also the formal logarithm method) has been studied in [9]. In the more general case however, when the parameter $\lambda \neq 0$, (originating e.g. from finite size reservoir corrections in the canonical approach [13,14], or from quantum corrections to micro black holes (see e.g. [17] and references therein)), the Rényi entropy formula may arise quite generally, when the conditions $L(0) = 0$ and $L'(0) = 1$ are also imposed, due to the consequence of some natural physical requirements (e.g. triviality, and leading order additivity for small energies) [5].

The Rényi entropy [2], defined as $S_R = \frac{1}{1-q}\ln\sum_i p_i^q$, is equivalent with the choices of $H_\lambda(S) = S$ and $\lambda = 1 - q$ in (4), if the original entropy functions follow the nonadditive composition rule

$$S_{12} = S_1 + S_2 + \lambda S_1 S_2, \tag{8}$$

which is known as the Tsallis composition rule, and $q \in \mathbb{R}$ is the so-called nonextensivity parameter. The Tsallis entropy is defined as $S_T = \frac{1}{1-q}\sum_i(p_i^q - p_i)$ [10], and it is easy to show that the formal logarithm of the Tsallis formula provides the Rényi entropy

$$S_R \equiv L(S_T) = \frac{1}{1-q}\ln[1 + (1-q)S_T]. \tag{9}$$

In the limit of $q \to 1$ ($\lambda \to 0$), both the Tsallis and the Rényi formulas reproduce the standard Boltzmann–Gibbs entropy, $S_{BG} = -\sum p_i \ln p_i$.

3. Schwarzschild black holes

Based on the parametric Tsallis–Rényi model, we investigated the thermodynamic properties of a Schwarzschild black hole in [1]. We found that the temperature-horizon radius relation is identical to the one obtained from a black hole in AdS space by using the original entropy formula, in both cases the temperature has a minimum. According to (9) the Rényi entropy function of black holes can be obtained by taking the formal logarithm of the Bekenstein–Hawking entropy, which – in the leading order of the λ parameter – follows the nonadditive Tsallis composition rule (8). Therefore, the Rényi entropy of a black hole can be computed as

$$S_R = \frac{1}{\lambda}\ln[1 + \lambda S_{BH}], \tag{10}$$

and for the Schwarzschild solution it results

$$S_R = \frac{1}{\lambda}\ln\left(1 + 4\pi\lambda M^2\right), \tag{11}$$

$$T_R = \frac{1}{8\pi M} + \frac{\lambda}{2}M, \quad C_R = \frac{8\pi M^2}{4\pi\lambda M^2 - 1}. \tag{12}$$

For comparison (not presented in [1]), on Fig. 1 we plot the temperature functions versus the black hole mass for the Schwarzschild–Rényi, the AdS–Boltzmann and the standard

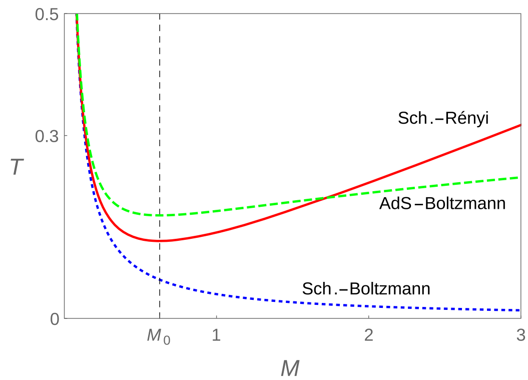

Fig. 1. The figure shows the temperature of a Schwarzschild black hole as a function of its mass-energy parameter in the asymptotically flat case with Boltzmann (blue, dotted) and Rényi (red, continuous) entropies, and also in AdS space with Boltzmann entropy (green, dashed).

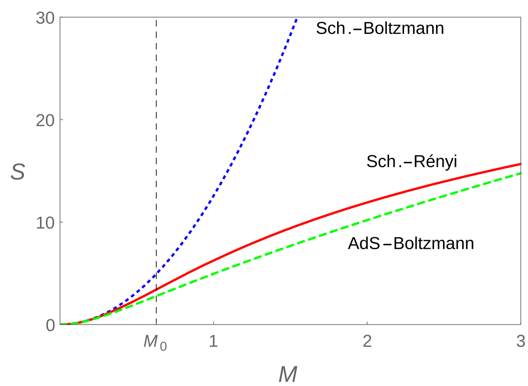

Fig. 2. The figure shows the entropy of a Schwarzschild black hole as a function of its mass-energy parameter in the asymptotically flat, Boltzmann (blue, dotted) and Rényi (red, continuous) cases, and also the Boltzmann case in AdS space (green, dashed).

Schwarzschild–Boltzmann cases. On Fig. 2 the corresponding entropy functions are also plotted. On all plots in this paper we use the parameter value $\lambda = 0.2$, and the corresponding curvature parameter of the AdS space is chosen such as to obtain the same M_0.

In the following section we discuss the thermodynamic stability of this problem first by considering pure, isolated black holes in the microcanonical treatment, and then by focusing on the canonical ensemble, where the black holes are surrounded by a bath of thermal radiation. For the purpose of generality, our analysis will be based on the Poincaré turning point method [7] following the works of Kaburaki et al. [6], however the canonical approach is also considered from the *Hessian* point of view.

4. Stability analysis

To separate stable and unstable configurations for cases of a one-parameter series of equilibria, Poincaré developed a powerful analytic approach [7]. It has been applied several times to problems in astrophysical and gravitating systems, in particular for the study of the thermodynamic stability of black holes in standard four [6] and also in higher dimensions [8]. Let us only quote here the main results of this method and omit all the details and proofs which can be found in the original references.

Suppose $Z(x^i, y)$ is a distribution function whose extrema $\partial Z/\partial x^i = 0$ define stable equilibrium configurations if the extremal value of Z is a maximum. Consider now the equilibrium value $Z(y) = Z[X^i(y), y]$, where $X^i(y)$ is a solution of $\partial Z/\partial x^i = 0$. If the derivative function dZ/dy, plotted versus y has the topology of a continuous and differentiable curve, it can be shown that changes of stability will occur only at points where the tangents are vertical. The Z distribution function is called *Massieu* function, the points with vertical tangents are called *turning points*, and the points with negative tangents near the turning points are a branch of *unstable* configurations, while the points with positive slopes near the turning points are a branch of *more stable* configurations.

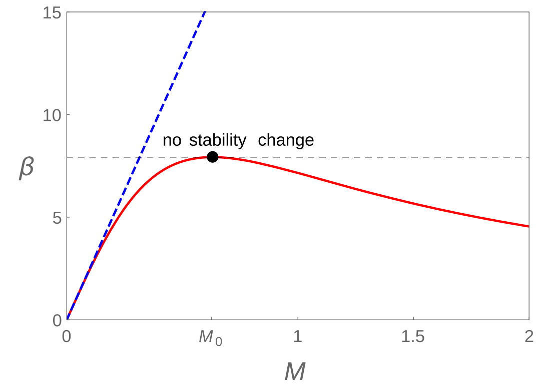

Fig. 3. The figure shows the Poincaré stability curves of a Schwarzschild black hole within the Boltzmann (blue, dashed) and the Rényi (red, continuous) approaches in the microcanonical treatment. No vertical tangent occurs in either case.

Pure, isolated black holes are described by Kaburaki et al. [6] when a perfectly reflecting, spherical mirror covers the hole just above its event horizon. In this idealistic case, the black hole can be described without radiation in the microcanonical treatment with $Z = Z(x^i, M)$ being the Bekenstein–Hawking entropy. A thermodynamic variable, y, is the total mass-energy, M, and the conjugate variable of M with respect to the entropy is the derivative $\beta = \frac{\partial S}{\partial M}$, which is the inverse temperature.

In the standard picture of black hole thermodynamics, β is the inverse Hawking temperature, and the stability curve of a Schwarzschild black hole is the linear function $\beta(M) = 8\pi M$. This straight line represents all the equilibrium configurations without any turning point, and hence the isolated Schwarzschild solution in vacuum is stable against spherically symmetric perturbations in the classical (microcanonical) treatment [6].

For the parametric Rényi case, the equilibrium compatible entropy function is given in (11), and β is the inverse Rényi temperature

$$\beta \equiv \frac{1}{T_R} = \frac{8\pi M}{1 + 4\pi\lambda M^2} \, . \tag{13}$$

The corresponding stability curve $\beta(M)$ is plotted on Fig. 3 together with the classical Schwarzschild curve for comparison. As it can be seen, similarly to the standard result, the Rényi curve has no vertical tangent, and since this curve represents all the equilibrium configurations, we can conclude that isolated Schwarzschild black holes are thermodynamically stable against spherically symmetric perturbations (in the microcanonical treatment) in the Rényi approach as well.

When the black hole is surrounded by an infinite bath of thermal radiation, the appropriate thermodynamic approach to consider is the canonical one. In the traditional picture, the radiation is treated as an ideal reservoir with infinite size and an infinitely large heat capacity, so it can emit or absorb all the heat that is needed by any change of the black hole without modifying its temperature. Recently, more realistic approaches have also been developed by considering large but finite size reservoirs instead of the infinite approximation. In particular, Biró showed that finite size reservoir corrections can provide modifications to the standard canonical theory in the form of nonadditive thermodynamics [13].

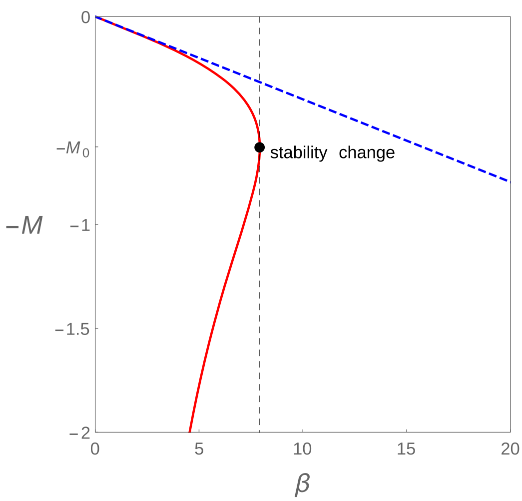

Fig. 4. The figure shows the Poincaré stability curves of a Schwarzschild black hole within the Boltzmann (blue, dashed) and the Rényi (red, continuous) approaches in the canonical treatment. For the Rényi curve, a vertical tangent occurs at M_0, which is the sign of a stability change.

These effects are usually neglected in the classical thermodynamic limit (when an infinite number of degrees of freedom is present), however they can provide relevant modifications to the behavior of finite size systems. In [13,14], it has been shown, that in the case of a finite heat bath with a large but finite and constant heat capacity, the inverse heat capacity of the bath may serve as the λ parameter in the equilibrium compatible entropy composition rule (3), and provide the corresponding Tsallis and Rényi entropy formulas in the leading order. These results provide additional motivation to investigate the Rényi approach for black hole thermodynamics in the canonical treatment.

Let us now consider the black hole in the canonical approach. The black hole entropy is no longer the appropriate Massieu function, Z, which takes its maximum at a stable equilibrium, rather it is

$$Z = S - \beta M \equiv -\beta F, \tag{14}$$

where F is the Helmholtz free energy [6]. The parameter y now is β, and the conjugate variable, dZ/dy, is

$$\frac{d(S-\beta M)}{d\beta} = -M. \tag{15}$$

The corresponding stability curve is then $-M(\beta)$, which we plotted on Fig. 4.

The canonical stability curves are simply the $\pi/2$ clockwise rotated versions of the microcanonical ones, and it is immediate to see that a vertical tangent appears at $M = M_0$, which belongs to the minimum temperature $T_0 = \frac{1}{2}\sqrt{\frac{\lambda}{\pi}}$. Black holes with mass parameter smaller than M_0 are *unstable* against spherically symmetric perturbations within this approach, however larger black holes with $M > M_0$ are *stable*, as opposed to the standard result where all solutions appear to be unstable.

The stability change at M_0 can also be confirmed by the *Hessian* analysis, i.e. in this case simply by checking the signature of the heat capacity of the Schwarzschild black hole in the bath. The corresponding curves are plotted on Fig. 5.

It can be seen (as shown in [1]), that the heat capacity has a pole at M_0, analogous to the AdS–Boltzmann case, and black holes with larger mass parameter have positive heat capacities (i.e. negative *Hessian*) and hence these solutions are thermodynamically stable. Black holes with smaller masses however are unstable solutions.

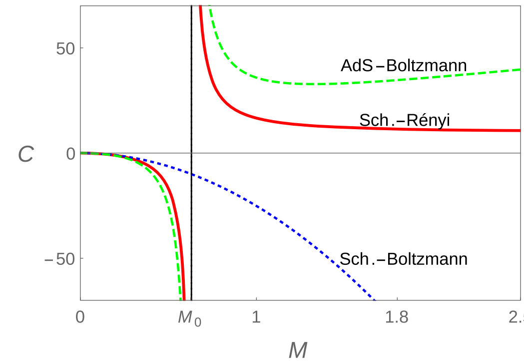

Fig. 5. The figure shows the heat capacity of a Schwarzschild black hole as a function of its mass-energy parameter in the standard (blue, dotted), Rényi (red, continuous) and AdS (green, dashed) cases.

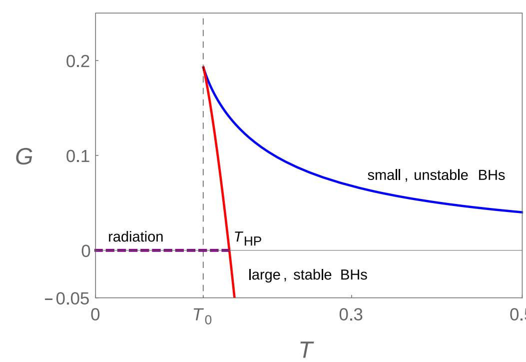

Fig. 6. The figure shows the free energy of a Schwarzschild black hole in the Rényi model as a function of the temperature in the canonical treatment.

5. Phase transition

In their classic paper [11], Hawking and Page investigated the thermodynamic properties of black holes in asymptotically AdS spacetimes. They showed that due to the presence of the gravitational potential of the AdS space, the stability properties of black holes are different from their corresponding ones in asymptotically flat spacetimes. In particular, the canonical ensemble exists for Schwarzschild black holes, and they can be in stable equilibrium with thermal radiation at a fixed temperature. In addition, it has been found that a thermodynamic phase transition occurs between the radiation and the black hole phases at a critical temperature which depends on the AdS curvature parameter only. Based on the similarity of our findings to the AdS problem, it is motivated to discuss the question of a possible phase transition in the Tsallis–Rényi approach as well.

By plotting the free energy of the black hole versus the temperature (for asymptotically flat black holes, i.e. at zero pressure, the Gibbs free energy is equivalent with the Helmholtz one), a Hawking–Page-like transition can be revealed. On Fig. 6, the blue, dotted curve represents the branch of black holes with $M < M_0$, which are unstable at any temperature. Black holes with $M_0 < M$ (red, continuous curve) are stable above the minimum temperature, but they posses positive free energy in the $T_0 \leq T < T_{HP}$ region, and therefore the pure radiation phase (purple, dashed line) is the thermodynamically preferred state with its approximately zero free energy. On the other hand, for $T_{HP} < T$, the large (and stable) black hole configuration becomes the thermodynamically preferred state because its free energy is smaller than zero, and hence, a phase transition occurs at T_{HP} from the radiation to the black hole phase. From the $F \equiv M - TS = 0$ condition we get $T_{HP} \approx 0.62\sqrt{\frac{\lambda}{\pi}}$, which depends on the λ parameter only.

This picture is completely analogous to the Hawking–Page transition of Schwarzschild black holes in AdS space, where the curvature parameter plays a similar role that is played by the λ ($\equiv 1-q$) parameter in the Rényi model. A relevant difference however, compared to the AdS case, is that in our model the Rényi entropy is additive, and hence the corresponding stability results are reliable.

6. Summary and discussion

In this Letter we studied the thermodynamic stability problem of Schwarzschild black holes described by the Rényi formula as equilibrium and zeroth law compatible entropy function. First we considered the question of a pure, isolated black hole in the microcanonical approach, and showed that these configurations are stable against spherically symmetric perturbations, just like in the classical picture. We also considered the problem when the black hole is surrounded by a bath of thermal radiation in the canonical treatment, and found that, as opposed to the standard picture, asymptotically flat, Schwarzschild black holes can be in stable equilibrium with thermal radiation at a fixed temperature, and a stability change occurs at a certain value of the mass-energy parameter which belongs to the minimum temperature solution. Black holes with smaller masses are unstable in this model, however larger black holes become stable. These results are very similar to the ones obtained by Hawking and Page in AdS space within the standard Boltzmann entropy approach [11]. Based on this similarity, we also analysed the question of a possible phase transition in the canonical picture, and showed that a Hawking–Page-like black hole phase transition occurs in a very similar fashion as in AdS space, and the corresponding critical temperature depends only on the q-parameter of the Rényi formula.

Our findings are relevant in many aspects. Parametric corrections to the Bekenstein–Hawking entropy formula may arise in various kind of physical situations, most importantly from quantum considerations (stemming either from string theory, loop quantum gravity, or other semi-classical theories). In these corrections, the perturbation parameters are small, and by connecting them to the λ parameter in Abe's formula (3), it can be expected that the Tsallis composition rule (7) is obtained quite generally in the leading order. In addition, from the requirement of the existence of an empirical temperature in thermal equilibrium, the Rényi entropy arises very naturally via the formal logarithmic method [5]. Based on these lines, our obtained results seem to be quite generic for parametric corrections to the Bekenstein–Hawking model in the small parameter and small energy limit.

As a different direction, we also mentioned that finite size reservoir corrections can result the same Tsallis–Rényi model in the canonical picture [13,14], and we expect that many other parametric situations are also possible. One of the motivations of this approach has been to satisfy the zeroth law of thermodynamics, and it is an important question whether the model is also in accordance with the remaining laws. In particular, Bekenstein's generalized second law [18] states that the sum of the black hole entropy and the common (ordinary) entropy in the black hole exterior never decreases on a statistical average. In obtaining this result, Bekenstein considered the Boltzmann–Gibbs formula for estimating the entropy of the system. Within our approach, Bekenstein's computations may be repeated by using the Rényi formula instead (see Sec. 2 for the definition), and due to the properties of the Rényi entropy measure [2], we expect that the generalized second law remains valid in this approach as well. We postpone this study for a future work.

As for the third law, the question is even more open. Recent results suggest (see e.g. [19] and references therein), that the generalized entropy formulas, in particular the Rényi entropy, violate the third law of thermodynamics even when the q-parameter is close to 1. The validity and applicability of the generalized entropies is therefore clearly an unsettled problem, and it is in the focus of active investigations today. There are many open questions in this field and also many research directions to consider. In this Letter, we studied the simplest problem of a standard $3+1$ dimensional, Schwarzschild black hole, and our plan is to extend our work to more general settings, e.g. rotating, Kerr black holes, or dynamic black holes formed by collapsing shells. Due to its similarity to the AdS problem, our present results might have some relevance from the AdS/CFT correspondence [20,21] point of view as well. Further studies to address these questions are in progress and we hope to report on them in a forthcoming publication.

Acknowledgements

V.G.Cz. is grateful for discussions with Prof. T.S. Biró. The research leading to this result has received funding from: the European Union Seventh Framework Programme (FP7/2007-2013) under the grant agreement No. PCOFUND-GA-2009-246542; the FCT project SFRH/BCC/105835/2014; the Japanese Ministry of Education, Science, Sports, and Culture Grant-in-Aid for Scientific Research (C) (No. 23540319), and also from the Japan Society for the Promotion of Science L14710 grant.

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