Adaptive Resource Allocation in Multimodal Activity Networks

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Abstract

In practice, project managers must cope with *uncertainty,* and must manipulate the *allocation* of their resources *adaptively* in order to achieve their ultimate objectives. Yet, treatments of the well-known 'resource constrained project scheduling problem' have been deterministic and static, and have addressed mostly unimodal activities. We present an approach to resource allocation under stochastic conditions for multimodal activity networks. Optimization is *via* dynamic programming, which proves to be demanding computationally. We suggest approximation schemes that do not detract significantly from optimality, but are modest in their computational requirements.

Key Words: Activity Networks, Resource Allocation, Multimodal Activities, Dynamic Programming

1 PROBLEM DEFINITION

This paper is concerned with the optimal resource allocation to the activities of a project in order to optimize an economic objective in face of uncertainty. We assume that the structure of the project is given (the definition of the activities and their precedence relataions, hence the activity network (AN), are specified). For simplicity we assume that there is only one resource that is of abundant availability; subsequent research shall deal with limited resource availability that may be variable over time.

The point of departure of this work from other contributions to the theory of AN's resides in our perspective concerning *uncertainty*.

We suspect that we may have been adopting the wrong perspective when we assume that the uncertainty in project execution resides exclusively in the activity *duration* (thanks to PERT [18] which initiated this perspective). We submit that the uncertainty is caused by two independent factors. The first is *external* to the activity (such as the weather; the possible breakdown of equipment; *etc.*). The second is *internal* to the activity, and relates to the estimation of the activity's *work content.* When one says that a task would take 4 days, one is already taking into account one's other commitments, so that the effort devoted to the task is limited to 2 hrs per day, which, in turn, results in the estimate of 4 days. The key to that decision on the duration is the estimate of the work content of 8 man-hours. If the task is required to be completed in less than 4 days, one's immediate reaction is to relegate other tasks to the "back burner" – in other words, one modifies the resource allocation (one's time) in order to cope with the new requirement. A request to complete the task in 2 days would immediately result in "doubling the original effort". Uncertainty stems from the lack of definiteness in the knowledge of the task's work content – it may require between 8 to 12 man-hours. This mode of uncertainty is separate and apart from any uncertainty in the "external" factors that may interfere with the conduct of the activity. In the same vein, a project manager manipulates the resources at his disposal to achieve a specific goal (or perhaps several goals) under the knowledge of the activity's total work content, whether deterministically of stochastically estimated. Given the resources allocated to an activity, the duration becomes a known entity. If the work content were deterministically known, and the resources are deterministically allocated, then the duration is easily derivable. In the manager's mind, the work content is the fixed entity, the resource is the *decision variable*, and the duration is the result. This perspective sheds a completely different light on the issues of concern to researchers in the field of AN's.

An activity is said to be *unimodal* if the resource requirements, and hence its duration, are fixed *a priori* (*e.g.*, it takes 2 men working 6 days). It is said to be *multimodal* if it can be performed with different allocations of resources, with its duration being a function of such allocation (*e.g.*, the total work content of 12 man-days may be performed by 1 man in 12 days, 2 men in 6 days, or even by 1.5 men in 8 days). We shall assume that the total work content of an activity is infinitely divisible.

As stated before, the main departure of our approach from conventional treatments is that we accord randomness to the total *work content* of each activity, generically denoted by W_a for activity a. (We are adopting the Activity-on-Arc mode of representation of the project.) In this paper we shall assume that the activity's work content follows the exponential distribution;

$$W_a \sim \exp(\lambda_a), \text{ for all } a \in A,$$

where A is the set of activities. The rationale for such choice shall become clearer as the discussion progresses. We consider only one resource, and denote the resource allocation to the activity by x_a, which is supposed to lie within a specified interval;

$$0 \leq l_a \leq x_a \leq u_a < \infty, \; \forall a \in A, \tag{1}$$

where l_a is the lower bound of the resource allocation and u_a is the upper bound. In this treatise we take the resource allocation to be deterministic; stochasticity in the resource shall be the subject of future work. An allocation of x_a units of the resource to activity a incurs a resource cost equal to[1]

$$C_a = x_a W_a, \tag{2}$$

and results in a duration

$$Y_a = \frac{W_a}{x_a}. \tag{3}$$

Observe that both C_a and Y_a are also random variables (*r.v.*).

We further assume that the project has a specified due date T_s, and a tardiness penalty that is a function of the actual completion of the project, denoted by $C(\Upsilon_n - T_s)$, where Υ_n is the time of realization of node n, a *r.v.*[2]. For the sake of simplicity, we shall take the penalty function to be proportional to the tardiness, with proportionality constant c_L (cost of tardiness per unit time).

$$C(\Upsilon_n - T_s) = c_L \cdot \max\{0, \Upsilon_n - T_s\}. \tag{4}$$

We are now in a position to specify our problem more precisely. The objective of the analysis is to determine the resource allocation vector X_a to all the activities of the project $a \in A$ such that the total expected cost (of undertaking the activities and of being penalized for tardiness) is minimized,

$$\min_X \mathcal{E}\left\{\sum_{a\in\mathcal{A}} [C(X) + c_L \cdot \max\{0, \Upsilon_n - T_s\}]\right\}, \tag{5}$$

where X is the vector $\{x_a\}_{a\in A}$, A is the set of activities, and $C(X)$ is a *r.v.* given by

$$C(X) = \sum_{a\in\mathcal{A}} x_a W_a, \tag{6}$$

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[1] The rationale for this expression lies in the assumption that the cost of the resource is quadratic in the allocation, incurred throughout the duration of the activity. Therefore the cost is given by bx^2Y, where b is a constant of proportionality and Y is the duration of the activity. But for resource allocation x the duration Y is given by W/x, which results in the cost being equal to bxW. We normalize the resource so that $b = 1$, which results in the expression $C_a = x_a W_a$.

[2] We assume that the nodes of the project network are topologically ordered so that node 1 is the start node and node n is the last node.

and each x_a is constrained by (1).

The reaminder of the paper is organized as follows. In §2 we present the proposed dynamic programming (DP) model of this problem which has general applicability to any probability distribution of the activities' work content. We follow, in §3 with an illustration of the proposed model to an example AN in which the activities possess a particular probability distribution, namely the exponential distribution which was selected for ease of mathematical manipulation, and we exemplify the interesting issue of 'sensitivity analysis'. The computer code for the solution of the illustrative example is available on request. The computer code for a general network, together with an extensive experimental investigation, is the subject of a second report [21] by the same authors. As will become amply evident, the computational burden of the proposed DP model is quite heavy, and in §4 we propose two approximation schemes, one based on switching the order of expectation and optimization, and the other based on 'aggregation' of activities, and one bounding scheme. The approximation schemes are also analyzed in the second report [21]. Finally, the Appendix details some technical aspects relating to the DP approach, namely the determination of the uniformly directed cutsets and the cutset intersection index which is a measure of the complexity of the DP approach.

2 THE DYNAMIC PROGRAMMING MODEL

Suppose that we have been successful in identifying a subset of the activities (preferably a minimal number of activities) to be 'conditioned upon', in the sense that their resource allocation is *fixed* (which removes them from the set of decision variables), in such a way that each uniformly directed cutset (*udc*) of the network[3] contains exactly one decision variable. Denote this set of activities by $\mathcal{F}$. Naturally, the 'con-

[3] A uniformly directed cutset is a cutset in which all the arrows are directed from the subset of nodes containing the start node (number 1) into the subset of nodes containing the terminal node (number n).

ditioning' must later be removed by enumerating all possible allocations to the subset $\mathcal{F}$ and selecting the best; hence our interest in the *minimal* cardinality of $\mathcal{F}$. Such 'conditioning' depends on the identification of the *udc*'s of the network and their intersection; which are given in the Appendix. Once the resource allocation to an activity is known, its duration and cost become also known (being *r.v.*'s of the same probability distribution as their respective W's, appropriately scaled by the resource allocation). We define the *stage* of the DP recursion as the epoch of decision on the value of x_a for some activity a in the set of decision variables, denoted by $\mathcal{D}$, which is the complementary set $\overline{\mathcal{F}}$, $\mathcal{D} \equiv \overline{\mathcal{F}} = \mathcal{A} - \mathcal{F}$. At each stage of the DP iterations we optimize over one decision variable. In this way there shall be as many stages as the size of the set $\mathcal{D}$. Let us suppose there are K stages; hence, $K = |\mathcal{D}| = |A| - |\mathcal{F}|$. We define the *state* as the (vector of) times of realization of a subset of the nodes that enable the decision x_a to be made and the stage 'reward' be evaluated, $a \in \mathcal{D}$. The state in stage k shall be generically denoted by s_k. Typically, $s_k = (t_{i_1}, \cdots, t_{i_r})$ for some subset of nodes $(i_1, \cdots, i_r)$. We shall number the stages backwards so that 'stage k' means 'k stages to go' to complete the project, and identify the decision variable in stage k as $x_{[k]}$, $k = 1, \cdots, K$. Hence 'stage 1' is the stage containing the terminal node n and has decision variable $x_{[1]}$, and stage K is the stage containing the initial node 1 and decision variable $x_{[K]}$. For all stages except stage 1 (the one containing the terminal node n) the *stage reward* is simply the resource cost, a *r.v.* equal to $x_{[k]} W_{[k]}$. In stage 1 the stage reward is the sum of the resource utilization cost $(= x_{[1]} W_{[1]})$ and the cost of tardiness, if any; $(= c_L \cdot \max\{0, \Upsilon_n - T_s\})$, with both costs being *r.v.*'s. Let $f_k(s_k \,|\, \mathcal{F})$ denote the minimal cost at stage k when the state is s_k conditioned on the (fixed) allocation in $\mathcal{F}$. Then, in typical DP fashion,

$$f_k(s_k \,|\, \mathcal{F}) = \min_{x_{[k]} \in D} \mathcal{E}\left\{C_{[k]}\left(\left[x_{[k]}\right], s_k\right) + \mathcal{E} f_{k-1}(S_{k-1} \,|\, \mathcal{F})\right\}, \tag{7}$$

S_{k-1} a *r.v.*, represents the realization time of state s_{k-1}, $k = 2, \cdots, K$,

where $x_{[k]}$ is the decision variable in stage k, and $f_1(s_1 \,|\, \mathcal{F})$ is determined from (10) below. The optimum is secured by removing the conditioning, and the final solution is achieved when we evaluate

$$f(s_K = 0) = \min_{\mathcal{F}} f_K(s_K \mid \mathcal{F}). \tag{8}$$

The solution via dynamic programming (DP) yields a policy that prescribes the optimal resource allocation under every conceivable state of the project as it progresses over time.

A straightforward (but not necessarily optimal) process to select the activities to be 'conditioned upon' (the set $\mathcal{F}$) is the following[4]:

1. Determine the longest path (in the sense of *number of activities*) in the network. In case of ties, select one that is closest to the 'boundary' of the network. The activities on the longest path will be the decision variables (set $\mathcal{D}$).

2. The other activities, $A - \mathcal{D}$, will be the activities to be fixed (set $\mathcal{F}$).

At the outset we evaluate the expected cost of the resource for the fixed variables, denoted by *rcf*[5],

$$rcf = \mathcal{E} \sum_{i \in \mathcal{F}} x_i W_i = \sum_{i \in \mathcal{F}} x_i \cdot \mathcal{E}(W_i). \tag{9}$$

The first stage begins in the state s_1 that leads to the last node of the network. We evaluate

$$f_1(s_1 \mid \mathcal{F}) = rcf + \min_{x_{[1]} \in D} \mathcal{E}\left\{x_{[1]} W_{[1]} + c_L \cdot \mathcal{E}(U)\right\} \tag{10}$$

where

$$U = \max\{0, \Upsilon_n - T_s\} \tag{11}$$

After the expected cost of stage 1 is evaluated we proceed to implement the extremal equation of (7) until the last stage is reached (the initial node of the network) where the state is $s_K = t_1 = 0$.

[4] An optimal set of decision variable $\mathcal{D}$ is determined as the solution of a 'set covering' problem as explained in the Appendix.

[5] For "resource cost, fixed activities".

This process yields the best allocation to the first activity $x_{[K]}$ defined by the *udc* at node 1. The whole procedure is then repeated for all possible fixed allocations to the activities in the set $\mathcal{F}$. Since x_a is defined the interval $[l_a, u_a]$, we must resort to *discretization.* Let m be the number of different allocation values of an activity in the set $\mathcal{F}$. Then there are $m^{|\mathcal{F}|}$ possible allocations, from which the optimum is selected. This determines the resource allocation to the activities emanating from the origin node, node 1. The policy to follow afterwards depends, naturally enough, on the state of the process when the decision is called for. This is the essence of the 'adaptive' nature of the DP approach: later allocations must await the realization of these activities as the project evolves over time.

It is not difficult to observe the computing burden of such a DP model.

3 EXAMPLE NETWORK

Consider the network in figure 1.

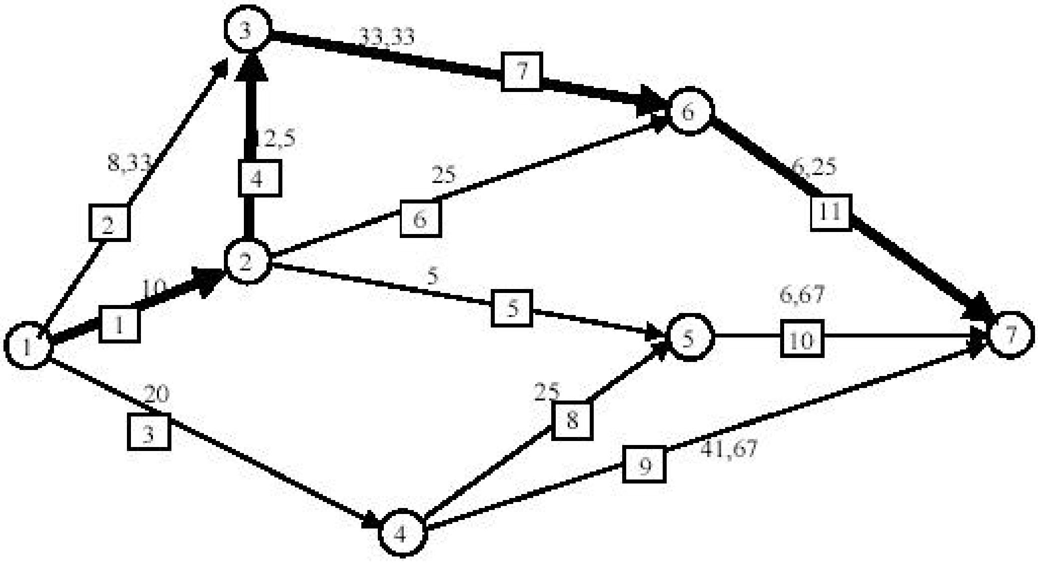

Figure 1: Example Network

The work content for each activity is exponentially distributed $W_a \sim \exp(\lambda_a)$, with the parameters given in table 1.

Activity a	**AoA Designation** (i,j)	**Parameter** λ_a	**Expected Work Content**
1	$(1,2)$	0.10	10.00
2	$(1,3)$	0.12	8.33
3	$(1,4)$	0.05	20.00
4	$(2,3)$	0.08	12.50
5	$(2,5)$	0.20	5.00
6	$(2,6)$	0.04	25.00
7	$(3,6)$	0.03	33.33
8	$(4,5)$	0.04	25.00
9	$(4,7)$	0.024	41.67
10	$(5,7)$	0.15	6.67
11	$(6,7)$	0.16	6.25

Table 1 - Parameters of the exponential distributions.

The due date of the project is $T_s = 65$[6] and the cost of tardiness is $c_L = 5$ per unit time (week) tardy. Any resource allocation x_a is bounded by $0.5 \leq x_a \leq 1.5$.

The cost of each resource allocation will be $c(a) = x_a W_a$ and the duration of the activity $Y_a = \frac{W_a}{x_a}$.

First we determine the longest path in the network shown in heavy lines in figure 1.

The activities along the longest path are the decision variables; set $\mathcal{D} = \{x_1, x_4, x_7, x_{11}\}$. The set of fixed activities is the set $\mathcal{F} = \{\hat{x}_2, \hat{x}_3, \hat{x}_5, \hat{x}_6, \hat{x}_8, \hat{x}_9, \hat{x}_{10}\}$,

[6] The PERT estimate of the expected duration of the project at this resource allocation to the activities in $\mathcal{F}$, and assuming nominal resource allocation (equal to 1) to the activities in the set $\mathcal{D}$, is 63.33, which we know underestimates the true expected duration. Hence the specified due date is slightly larger than the expected project duration.

in which we have highlighted the fixed nature of the allocations by the hat '^' on the x's. For illustrative purposes the feasible interval of the resource allocation $[0.5, 1.5]$ was discretized at five values: $0.5, 0.75, 1.00, 1.25, 1.50$ and the resource allocation confined to only these values. Guided by the expected durations of the activities at the 'nominal' resource allocation $\hat{x}_a = 1.00$ for all $a \in \mathcal{F}$, the allocation to the activities in the set $\mathcal{F}$ was fixed at

$$\{\hat{x}_2, \hat{x}_3, \hat{x}_5, \hat{x}_6, \hat{x}_8, \hat{x}_9, \hat{x}_{10}\} = \{1.0, 1.5, 0.5, 0.5, 1.0, 1.5, 1.0\}.$$

meaning that the allocation to activity 2 was fixed at $\hat{x}_2 = 1.0$, to activity 3 at $\hat{x}_3 = 1.5$, *etc.* The evaluation of the *expected* resource cost for the fixed variables is easily deduced to be

$$rcf = \sum_{a \in \mathcal{F}} \hat{x}_a \cdot \mathcal{E}(W_a) = 147.5. \tag{12}$$

Again, for the sake of illustration, each activity's work content W_a was discretized at only four values, each with probability 0.25, with the same expected value as $\mathcal{E}(W_a)$. For instance, $W_1 \sim \exp(0.10)$ was assumed to take only four values: $1.3695, 4.7675, 10.00, 23.8629$, all with equal probability. To be sure; the average of these four values is 10, which is the expected value of the *r.v.*.

The DP iterations are initiated at **stage 1** which is defined by the decision variable $x_{[1]} = x_{11}$, the allocation to activity 11. The state may be defined by the triplet $s_1 = (t_4, t_5, t_6)$, the times of realization of nodes 4, 5 and 6. We have that

$$f_1(t_4, t_5, t_6 \,|\, \mathcal{F}) = rcf + \min_{x_{11}} \mathcal{E}\left\{x_{11} W_{11} + c_L \mathcal{E}(U)\right\}, \tag{13}$$

where

$$U = \max\{0, \Upsilon_7 - T_s\}, \text{ a } r.v., \tag{14}$$

and

$$\Upsilon_7 = \max\{t_6 + \frac{W_{11}}{x_{11}}, t_4 + \frac{W_9}{\hat{x}_9}, t_5 + \frac{W_{10}}{\hat{x}_{10}}\}. \tag{15}$$

Stage 2 is defined by the decision variable $x_{[2]} = x_7$. Its state is defined by the triplet $s_2 = (t_2, t_3, t_4)$, and we have,

$$f_2(t_2, t_3, t_4 \,|\, \mathcal{F}) = \min_{x_7} \mathcal{E}\{x_7 W_7 + \mathcal{E}[f_1(t_4, \Upsilon_5, \Upsilon_6)]\} \tag{16}$$

where

$$\Upsilon_5 = \max\{t_2 + \frac{W_5}{\hat{x}_5}, t_4 + \frac{W_8}{x_8}\} \quad \text{and} \quad \Upsilon_6 = \max\{t_2 + \frac{W_8}{\hat{x}_8}, t_3 + \frac{W_7}{x_7}\}. \tag{17}$$

Stage 3 is defined by the decision variable $x_{[3]} = x_4$, and its state is defined by (t_2),

$$f_3(t_2 \,|\, \mathcal{F}) = \min_{x_4} \mathcal{E}\{x_4 W_4 + \mathcal{E}[f_2(t_2, \Upsilon_3, \Upsilon_4)]\} \tag{18}$$

$$\Upsilon_3 = \max\{t_2 + \frac{W_4}{x_4}, \frac{W_2}{\hat{x}_2}\} \quad \text{and} \quad \Upsilon_4 = \frac{W_3}{x_3}. \tag{19}$$

Finally, **stage 4**, which is defined by the decision variable $x_{[4]} = x_1$, has its state is defined by $t_1 = 0$:

$$f_4(t_1 = 0 \,|\, \mathcal{F}) = \min_{x_1} \mathcal{E}\{x_1 W_1 + \mathcal{E}[f_3(\Upsilon_2)]\} \tag{20}$$

$$\Upsilon_2 = \frac{W_1}{x_1} \tag{21}$$

The results are as follows:

$$\begin{aligned} x_1^*|\mathcal{F} &= 1.25, \quad \text{at total expected cost of } 348.28, \\ \text{with } \mathcal{F} &= \left\{ \begin{array}{c} \{\hat{x}_2,\hat{x}_3,\hat{x}_5,\hat{x}_6,\hat{x}_8,\hat{x}_9,\hat{x}_{10}\} = \\ \{1.0, 1.5, 0.5, 0.5, 1.0, 1.5, 1.0\} \end{array} \right\} \end{aligned}$$

Observe that about 148 of this cost is due to the resource cost allocated to the´fixed' activities, which implies that, under this allocation to the ´fixed' activities, the expected cost of resource allocation to the four decision variables$\{x_1,x_4,x_7,x_{11}\}$ plus the cost of tardiness is about 201. Since the expected cost of resource allocation to the decision variables varies between $(0.5 \times 62.083) \simeq 31$ and $(1.5 \times 62.083) \simeq 93$, we deduce that the expected cost of tardiness is between 108 and 170. Since the marginal cost of tardiness is 5, we therefore anticipate that the project shall be tardy at least 21.6 and at most 34 weeks.

In order to remove the conditioning on $\mathcal{F}$ we must evaluate the expected cost and the optimal value of the decision variables for every possible combination of the fixed variables, and there are $5^7 = 78125$ of them. The result of such exhaustive search is as follows:

$$x_1^*|\mathcal{F}^* = 1.25, \quad \text{at total expected cost of } 280.85, \tag{22}$$

$$\text{with} \quad \mathcal{F}^* = \left\{ \begin{array}{c} \{x_2^*, x_3^*, x_5^*, x_6^*, x_8^*, x_9^*, x_{10}^*\} \\ = \{0.5, 1.5, 0.5, 1.0, 1.0, 1.5, 1.0\} \end{array} \right\} \tag{23}$$

Thus the optimal allocation at the initial *udc*, which is composed of activities *1,2,3*, is given by

$$\{x_1^*, x_2^*, x_3^*\} = \{1.25, 0.5, 1.5\}.$$

Indeed, we also have in hand the complete vector of allocations for the 'fixed' activities in the set $\mathcal{F}$ that yielded the optimum (given by eq.(23)), as well as the corresponding optimal policies (at this allocation to the 'fixed' variables) at all remaining stages of the project. These latter have not been exhibited for the sake of economy in space.

Values of the allocations to the other decision variables (activities *4,7*, and *11*) must depend, naturally, on the outcome of the three activities in the first *udc*. In particular,

(i) If activity *1* completes first then node 2 will be realized. This, together with the specified allocations to the 'fixed' activities in $\mathcal{F}$ would enable us to locate the optimal allocation x_4^* in the previously developed optimal policy for stage 3.

(ii) If activity *2* completes first then it shall remain 'dormant' awaiting the completion of activity *4* which, in turn, cannot be initiated before the completion of activity *1*.

(iii) If activity *3* completes first then node 4 will be realized. We may initiate activities *8* and *9* at the specified allocation in the vector of 'fixed' allocations in $\mathcal{F}$.

The optimal allocation[7] has a resource cost of at least 168.34[8]. A lower bound on the resource cost of the remaining three decision variables is

$$0.5\left(\mathcal{E}\left[W_4\right]+\mathcal{E}\left[W_7\right]+\mathcal{E}\left[W_{11}\right]\right)=26.04,$$

and an upper bound is

$$1.5\left(\mathcal{E}\left[W_4\right]+\mathcal{E}\left[W_7\right]+\mathcal{E}\left[W_{11}\right]\right)=78.12.$$

Therefore, the lower bound on the resource cost is 194.38, and the upper bound is 246.46. Then the cost of tardiness ranges between $(280.85-246.46)=34.39$ and $(280.85-194.38)=86.47$. Since the unit penalty for tardiness is 5, this, in turn, implies that the anticipated tardiness under this optimal allocation is between 6.88 and 17.29 weeks. Observe that the drastic reduction in cost is mainly due to the reduction in the expected penalty for tardiness.

3.1 Sensitivity Analysis

One may view the full factorial experimentation (with 5^7 different allocations to the activities in the set $\mathcal{F}$) as a form of sensitivity analysis. But having achieved the 'optimal' vector $(x_1^*, x_2^*, x_3^*, x_5^*, x_6^*, x_8^*, x_9^*, x_{10}^*)$ one may be interested in testing the sensitivity of the total cost to small variations in the decision variables. An elementary

[7]Note that this is not the absolute optimum since the search over the spaces of W and X was limited to the selected discretized values, and even then we did not search over all 5^7 possible allocations to the *fixed* activities.

[8]$168.34=x_1^*\cdot\mathcal{E}\left[W_1\right]+\sum_{k\in\mathcal{F}}\hat{x}_k\cdot\mathcal{E}\left[W_k\right]$.
The actual resource cost is higher than this value because of the allocation to activities *4,7,* and *11.*

form of sensitivity analysis is the "one-at-a-time" variation, which may be described as follows: for each x_k^* evaluate the performance of the project cost at each of the 'neighboring' values. There are four possible outcomes of the cost. It is

1. monotone increasing: then decrease x^*, if feasible[9];
2. monotone decreasing: then increase x^* if feasible;
3. it is '$\vee$'-shaped: then test at both sides of x^*;
4. it is '$\wedge$'-shaped: then test at both sides of x^*.

This analysis was conducted and resulted in the following changes:

$$\begin{aligned}\{x_1^*, x_2^*, x_3^*, x_5^*, x_6^*, x_8^*, x_9^*, x_{10}^*\} &= \{1.25, 0.5, 1.0, 0.5, 1.0, 1.25, 1.5, 1.0\} \\ \text{with expected cost} &= 274.86.\end{aligned}$$

The decrease in total expected cost (of 5.99 units) may appear insignificant, but analysis of its constituent elements is illuminating. The resource cost varies between

$$x_1^* \cdot \mathcal{E}[W_1] + \sum_{a \in \mathcal{F}^*} x_k^* \cdot \mathcal{E}[W_k] + 0.5\left(\mathcal{E}[W_4] + \mathcal{E}[W_7] + \mathcal{E}[W_{11}]\right) = 190.63$$

and

$$\min\left\{274.86;\ x_1^* \cdot \mathcal{E}[W_1] + \sum_{a \in \mathcal{F}^*} x_k^* \cdot \mathcal{E}[W_k] + 1.5\left(\mathcal{E}[W_4] + \mathcal{E}[W_7] + \mathcal{E}[W_{11}]\right)\right\} = 242.71.$$

Therefore, the cost of tardiness ranges between 0 and 32.15, and tardiness itself ranges between 0 and 6.43 weeks. It is then evident that the reduction in total expected cost was realized because of the reduction in the expected tardiness.

[9] infeasibility of decreasing the value would occur if $x_a^* = l_a$. Infeasibility of increasing the value would occur if $x_a^* = u_a$.

4 SUMMARY AND DIRECTIONS FOR FUTURE RESEARCH

We have presented an argument for an alternative intepretation of *uncertainty* in project networks based on the inherent *work content* of the activities, rather than on external factors. This optic led us to the issue of optimal resource allocation in order to achieve a stated objective which incorporated the cost of the resource and the penalty for tardiness. We have also presented a DP approach to its resolution which has general applicability to any distribution of the work content (albeit we exemplified it by application to exponentially distributed *r.v.*'s).

The onerous burden of computing encountered in the 'straightforward' DP model detailed above is evident. In a large activity network of 300 activities, say, the longest path may be of length 50, leaving some 250 activities in the set $\mathcal{F}$. Complete enumeration over the 'fixed' activities is now impossible even if each assumes only two values. This provides the main driving force behind the search for 'good' approximating schemes ('good' in the sense of minimal degradation in performance), that are computationally more frugal. There are two main avenues of such approximations which are currently under investigation:

1. Replace the search for the optimum of the expected values at selected stages of iteration with the search for the expected value of the optima. This approach is in the spirit of the original PERT model, in which the determination of the expected value of the longest path in the network was replaced with the determination of the longest path when the expected values of the activity durations were taken as the certainty equivalents of their respective *r.v.*'s.

2. Forfeit some 'managerial flexibility' in the adaptive optimization by 'aggregating' activities. The very act of 'aggregation' combines two or more activities into a larger 'aggregate activity', which will necessarily delete nodes in the original network. The aggregated activities shall all be treated as a single activity

to which a resource allocation shall be made. This robs the manager of the flexibility of varying the allocation to individual activities according to progress to date, which was manifested in the realization of the (deleted) nodes. For instance, in the example of figure 1 if managerial flexibility is forfeited with respect to activities 5,10, and to activities 3, 8, 9, 10, and we 'reduce' node 5 (in the sense of [2]), then the network would appear as shown in figure 2 in which the resource allocations to compound activities 12 ($=\{5,10\}$) and 13 ($=\{3,8,9,10\}$) are fixed. Clearly, this is a much simpler network to analyze than 1, with only 4 activities in the set $\mathcal{F}$.

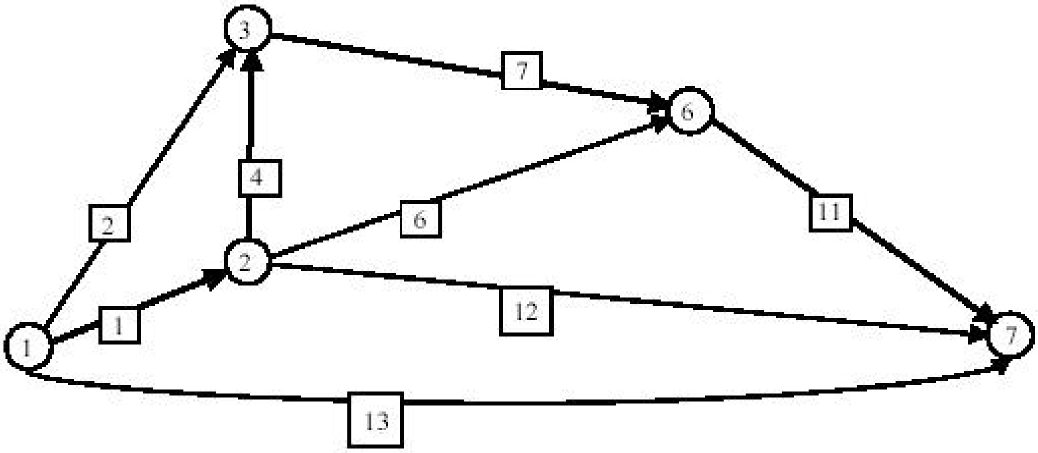

Figure 2: The aggregation of some activities.

3. Forfeit the generality of the DP approach in favor of the specialized treatment of exponentially distributed *r.v.*'s which would lead to the interpretation of the project as a continuous time Markov chain. Such analysis may provide upper bound on the expected cost of the project, which may be useful in the budgeting/bidding process.

Finally, the DP approach is very demanding computationally, in whichever form it is used. So one should try to apply other compu-search approaches to this problem, such as simulated annealing, tabu search, genetic algorithms, and a global optimization technique based on the 'Electromagnetism Algorithm' designed by Birbil and Fang [3]. These approaches are currently under investigation..

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5 APPENDIX

5.1 The Identification of The UDC's

The procedure to identify the uniformly directed cutsets (*udc*) in an AoA network representation of the project proceeds iteratively from the terminal node n.[10], utilizing the following proposition (compare with Proposition 4 in Dodin and Elmaghraby [9]). We first have,

Definition 1 *A "terminal subset" (TS) is a subset of nodes with all arcs directed into its nodes.*

Proposition 2 *For each TS of nodes U, the udc of activities, CU, is given by*

$$C_U = \left\{\overline{U} \times U\right\} \subset \mathcal{A},$$

with the initial condition

$$C_{U=\{n\}} = \{(i,n) \in \mathcal{A}\}.$$

In words, C_U= set of arcs connecting the nodes in the complementary set $\overline{U}$ to the nodes in the set U.

Rank the udc's in order of ascending activity number, and within the udc's containing activity a, rank them in increasing number of origin nodes. The earliest cutset of an activity is defined as the udc of smallest index that contains the activity.

As example, consider the project network of figure 3. The procedure generates the udc's shown in table A1.

[10] Alternatively, one may proceed forward from node 1. Construct 'initial subsets' of activities which are defined in a dual fashion to the terminal subsets; call each X, and for each subset define $\bar{X} = N - X$. The *udc* is then the set of arcs linking X to $\bar{X}$.

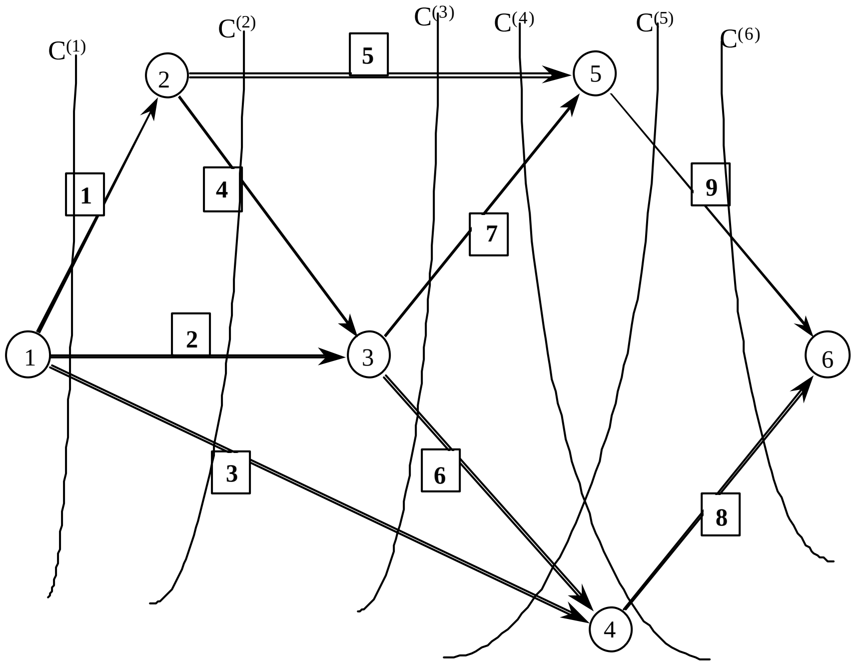

Figure 3: Example project network and its udc's

Table A1. The generation of *udc's* for the project of figure 3.

TS $U^{\ddagger}$	**Cutset**†	**Identification**
$\{6\}$	$\{8,9\}$	$\mathcal{C}^{(6)}$
$\{5,6\}$	$\{5,7,8\}$	$\mathcal{C}^{(5)}$
$\{4,6\}$	$\{3,6,9\}$	$\mathcal{C}^{(4)}$
$\{4,5,6\}$	$\{3,5,6,7\}$	$\mathcal{C}^{(3)}$
$\{3,4,5,6\}$	$\{2,3,4,5\}$	$\mathcal{C}^{(2)}$
$\{2,3,4,5,6\}$	$\{1,2,3\}$	$\mathcal{C}^{(1)}$

‡ Identification by the nodes.
† Identified by the activities

Observe that we did not identify subsets $\{3,4,6\}$, $\{3,5,6\}$, etc., as terminal subsets since there are arcs directed out of them.

The *earliest cutset* for each activity is as follows:

Activity:	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
Earliest *udc:*	$\mathcal{C}^{(1)}$	$\mathcal{C}^{(1)}$	$\mathcal{C}^{(1)}$	$\mathcal{C}^{(2)}$	$\mathcal{C}^{(2)}$	$\mathcal{C}^{(3)}$	$\mathcal{C}^{(3)}$	$\mathcal{C}^{(5)}$	$\mathcal{C}^{(4)}$

5.2 The Determination of the *cii*

To determine the *cutset intersection index* (*cii*) one needs to have generated the uniformely directed cutsets (*udc*'s) of the network. One then constructs the activity-*udc* incidence matrix M, with entry $m_{ij} = 1$ if activity i lies on *udc* j, and $= 0$ otherwise. The objective now is to retain 'unfixed' the maximal subset of activities such that each *udc* contains exactly one activity. This translates into retaining the maximal number of rows in the matrix M, such that the residual matrix contains exactly one 1 in each column. This is the well known *set partitioning problem* (*SPP*). That is, we wish to

$$\max \quad z = \sum_i x_i \tag{24}$$

such that,

$$\sum_i m_{ij} x_i = 1, \ \forall j, \tag{25}$$

$$x_i \in \{0,1\}. \tag{26}$$

It is well known that the *SPP* is NP-complete. Fortunately, for this case of *dag*'s, the solution of this *SPP* is easily secured as all the activities that do not lie on the path of longest number of activities.[11] Applying this procedure to the graph of figure 3 one easily finds that there are two paths of 'length' 4: *1,4,7,9* and *1,4,6,8.* Selecting the first path one secures the *IS* as

$$IS = \{2, 3, 5, 6, 8\} \Rightarrow cii = 5.$$

[11] We are grateful to Professor Bert Deryck of the London School of Business for suggesting this elementary procedure to secure the conditioned activities.

The IS is identified in figure 3 with double lines. This IS leaves only one activity in each udc: activity *1* in $\mathcal{C}^{(1)}$, activity *4* in $\mathcal{C}^{(2)}$, activity *7* in $\mathcal{C}^{(3)}$ and $\mathcal{C}^{(4)}$, and activity *9* in $\mathcal{C}^{(5)}$ and $\mathcal{C}^{(6)}$.

Note that this project network has node reduction index (nri)[2] of only 2.