An Integral-type Constraint Qualification for Optimal Control Problems with State Constraints *

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Abstract

Standard necessary conditions of optimality (NCO) for constrained optimal control problems – Maximum Principle type conditions – may fail to provide useful information to select candidates to minimizers among the overall set of admissible solutions. This phenomenon is known as the degeneracy phenomenon and there has been continuing interest in the literature in proposing stronger forms of NCO that can be informative in such cases: the so-called nondegenerate NCO. The nondegenerate NCO proposed here are valid under a different set of hypothesis and under a constraint qualification of an integral-type that, in relation to some previous literature, can be verified for more problems.

1 Introduction

In this report we are interested in Necessary Conditions of Optimality – Maximum Principle type conditions – for optimal control problem with pathwise state constraints. We consider the following problem

$$
(P)\quad
\begin{array}{ll}
\text{Minimize} & g(x(1)) \qquad (1)\\
\text{subject to} & \\
& \dot{x}(t) = f\left(t, x(t), u(t)\right) \quad \text{a.e. } t \in [0,1] \qquad (2)\\
& x(0) = x_0\\
& x(1) \in C\\
& u(t) \in \Omega(t) \qquad \text{a.e. } t \in [0,1]\\
& h\left(t, x(t)\right) \le 0 \qquad \text{for all } t \in [0,1], \qquad (3)
\end{array}
$$

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for which the data comprises functions $g : \mathbb{R}^n \mapsto \mathbb{R}$, $f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$, $h : [0,1] \times \mathbb{R}^n \mapsto \mathbb{R}$, a set C and a multifunction $\Omega : [0,1] \rightrightarrows \mathbb{R}^m$.

The set of *control functions* for (P) is

$$\mathcal{U} := \{u : [0,1] \mapsto \mathbb{R}^m : u \text{ is a measurable function}, u(t) \in \Omega(t) \text{ a.e. } t \in [0,1]\}.$$

The *state trajectory* is an absolutely continuous function which satisfies (2). The domain of the above optimization problem is the set of *admissible processes*, namely pairs (x, u) comprising a control function u and a corresponding state trajectory x which satisfy the constraints of (P). We say that an admissible process $(\bar{x}, \bar{u})$ is a *strong local minimizer* if there exists $\delta > 0$ such that

$$g(\bar{x}(1)) \leq g(x(1))$$

for all admissible processes (x, u) satisfying

$$\|x(t) - \bar{x}(t)\|_{L^\infty} \leq \delta.$$

It is well known that, the necessary conditions for such problems appears in the form of Maximum Principle (MP). The original formulation of the MP (see [1]) applied to problems with very regular properties for the involved functions. However, over the last three decades continuous development allow to reformulate the MP for "nonsmooth" data (data that can be non differentiable), free endtime constraints, a broad description of endpoint constraints, and other refinements. Se for example [2], [3], [4], [5], [6]. We consider here a nonsmooth version of the MP, as in [6].

We assume that problem (P) satisfies the following set of hypothesis:

There exists a positive scalar δ' such that:

H1 The function $(t, u) \mapsto f(t, x, u)$ is $\mathcal{L} \times \mathcal{B}$ measurable for each x. ($\mathcal{L} \times \mathcal{B}$ denotes the product σ-algebra generated by the Lebesgue subsets $\mathcal{L}$ of $[0,1]$ and the Borel subsets of $\mathbb{R}^m$.)

H2 There exists a $\mathcal{L} \times \mathcal{B}$ measurable function $k(t, u)$ such that $t \mapsto k(t, \bar{u}(t))$ is integrable and

$$\|f(t, x, u) - f(t, x', u)\| \leq k(t, u)\|x - x'\|$$

for $x, x' \in \bar{x}(t) + \delta'\mathbb{B}$, $u \in \Omega(t)$ a.e. $t \in [0,1]$. Furthermore there exist scalars $K_f > 0$ and $\epsilon' > 0$ such that

$$\|f(t, x, u) - f(t, x', u)\| \leq K_f\|x - x'\|$$

for $x, x' \in \bar{x}(0) + \delta'\mathbb{B}$, $u \in \Omega(t)$ a.e. $t \in [0, \epsilon']$.

H3 The function g is Lipschitz continuous on $\bar{x}(1) + \delta'\mathbb{B}$.

H4 The end-point constraint set C is closed.

H5 The graph of Ω is $\mathcal{L} \times \mathcal{B}$ measurable.

H6 The function h is upper semicontinuous and there exists a scalar $K_h > 0$ such that the function $x \mapsto h(t, x)$ is Lipschitz of rank K_h for all $t \in [0,1]$.

Define the Hamiltonian

$$H(t,x,p,u) = p \cdot f(t,x,u).$$

The Necessary Conditions assert existence of an absolutely continuous function p, a measurable function γ, a non-negative measure μ representing an element in $C^*([0,1] : \mathbb{R})$ and $\lambda \geq 0$ such that

$$-\dot{p}(t) \in \mathrm{co}\, \partial_x H(t,\bar{x}(t),q(t),\bar{u}(t)) \quad \text{a.e. } t \in [0,1], \tag{4}$$

$$-q(1) \in N_C(\bar{x}(1)) + \lambda \partial g(\bar{x}(1)), \tag{5}$$

$$\gamma(t) \in \partial_x^{>} h(t,\bar{x}(t)) \quad \mu \text{ a.e. }, \tag{6}$$

$$\mathrm{supp}\{\mu\} \subset \{t \in [0,1] : h\left(t,\bar{x}(t)\right) = 0\}, \tag{7}$$

for almost every $t \in [0,1]$, $\bar{u}(t)$ maximizes over $\Omega(t)$

$$u \mapsto H(t,\bar{x}(t),q(t),u) \tag{8}$$

and,

$$\mu\{[0,1]\} + \|p\|_{L^\infty} + \lambda > 0, \tag{9}$$

where

$$q(t) = \begin{cases} p(t) + \int_{[0,t)} \gamma(s)\mu(ds) & t \in [0,1) \\ p(t) + \int_{[0,1]} \gamma(s)\mu(ds) & t = 1. \end{cases}$$

Here, ∂f denotes the limiting subdifferential of f, N_C denotes the limiting normal cone and $\partial_x^{>} h$ denotes the hybrid partial subdifferential . (The definitions are in Section 2)

Stronger forms of NCO for optimal control problems that guarantee nondegeneracy and/or normality were discussed in [7]. In this report we are interested, in the strong form of the MP introduced in [8], which ensures that the MP can be written with the nontriviality condition

$$\mu\{(0,1]\} + \|q\|_{L^\infty} + \lambda > 0,$$

if the problem satisfies the following additional condition - constraint qualification:

(CQ) If $h(0,x_0) = 0$, then there exist positive constants ϵ,ϵ_1, δ, and a control $\tilde{u} \in \Omega(t)$ such that for a.e. $t \in [0,\epsilon)$

$$\zeta \cdot [f(t,x_0,\tilde{u}(t)) - f(t,x_0,\bar{u}(t))] < -\delta$$

for all $\zeta \in \partial_x^{>} h(s,x),\ s \in [0,\epsilon),\ x \in \{x_0\} + \epsilon_1 \mathbb{B}$.

In this report, the main result consist of a stronger MP, as above, but requiring a weaker constraint qualification of an integral-type:

(CQ^*) If $h(0,x_0)=0$, then there exist positive constants K_u, ϵ_1, δ, $\bar{\tau}^* \in (0,1]$ and a control $\tilde{u} \in \Omega(t)$ such that for all $\tau \in [0,\bar{\tau}^*]$

$$\int_0^\tau \zeta \cdot [f(t,x_0,\tilde{u}(t)) - f(t,x_0,\bar{u}(t))]dt < -\delta\tau$$

for all $\zeta \in \partial_x^> h(s,x),\ s \in [0,\bar{\tau}^*),\ x \in \{x_0\} + \epsilon_1 \mathbb{B}$

Note that CQ implies CQ^*, so the new constraint qualification CQ^* is more interesting as a condition since it can be verified for more problems.

However, the NCO given here (valid under CQ^*) require a convex velocity set as an additional hypothesis.

2 Preliminaries

Throughout, B will denote the closed unit ball in Euclidean space and $\text{co}\, S$ the convex hull of a set S.

The *limiting normal cone* of a closed set $C \subset \mathbb{R}^n$ at $\bar{x} \in C$, denoted by $N_C(\bar{x})$, is the set

$$N_C(\bar{x}) := \{\eta \in \mathbb{R}^n : \exists \text{ sequences } \{M_i\} \in \mathbb{R}^+, x_i \to \bar{x},\ \eta_i \to \eta \text{ such that } x_i \in C \text{ and } \eta_i \cdot (y - x_i) \le M_i |y - x_i|^2 \text{ for all } y \in \mathbb{R}^n, i = 1,2,...\}.$$

The *limiting subdifferential* of a lower semicontinuous function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ at a point $\bar{x} \in \mathbb{R}^n$ such that $f(\bar{x}) < +\infty$, denoted by $\partial f(\bar{x})$, is defined to be

$$\partial f(\bar{x}) = \{\eta \in \mathbb{R}^n : (\eta, -1) \in N_{\text{epi} f}(\bar{x}, f(\bar{x}))\};$$

where $\text{epi} f = \{(x,\alpha) \in \mathbb{R}^{n+1} : \alpha \ge f(x)\}$ denotes the epigraph of a function f.

We define also $\partial_x^> h(t,x)$, to be the following *the hybrid partial subdifferential* of h in the x-variable

$$\partial_x^> h(t,x) := \text{co}\{\xi : \text{ there exist } (t_i,x_i) \to (t,x) \text{ s.t. } h(t_i,x_i) > 0,\ h(t_i,x_i) \to h(t,x), \text{ and } \nabla_x h(t_i,x_i) \to \xi\}$$

See [6] for a review of Nonsmooth Analysis and related concepts using a similar notation.

3 Main Result

In addition to (H1)-(H6), we assume that

H7 There exists positive constants $\bar{\tau}^*$ and ϵ_1 such that $f(t,x,\Omega(t))$ is convex for all $t \in [0,\bar{\tau}^*)$ and for all $x \in \{x_0\} + \epsilon_1 \mathbb{B}$,

and

(CQ^*) (*constraint qualification*) If $h(0,x_0)=0$ then there exist positive constants K_u, ϵ_1, δ, $\bar{\tau}^* \in (0,1]$ and a control $\tilde{u} \in \Omega(t)$ such that for all $\tau \in [0,\bar{\tau}^*]$

$$\|f(t,x_0,\bar{u}(t))\| \le K_u, \quad \|f(t,x_0,\tilde{u}(t))\| \le K_u,$$

and

$$\int_0^{\tau} \zeta \cdot [f(t, x_0, \tilde{u}(t)) - f(t, x_0, \bar{u}(t))]dt < -\delta\tau$$

for all $\zeta \in \partial_x^> h(s,x),\ s \in [0, \bar{\tau}^*),\ x \in \{x_0\} + \epsilon_1 \mathbb{B}$.

Proposition 3.1 *Assume hypotheses (H1)-(H7) and (CQ^*). Then the nontriviality condition (9) of the Maximum Principle can be replaced by the stronger condition*

$$\mu\{(0,1]\} + \|q\|_{L^\infty} + \lambda > 0. \tag{10}$$

As in ([8]), the proof consists of making an appropriate modification of the data "near" to the left endpoint and then apply the standard Maximum Principle.

4 Proof of the Main Result

We assume that $h(0, x_0) = 0$, since otherwise the Maximum Principle cannot be satisfied by the trivial multipliers.

Step 1: Consider, for $\alpha \in (0, 1]$, the system of equations (S)

$$(S) \begin{cases} \dot{x}(t) = f(t, x(t), \bar{u}(t)) + y(t) \cdot \Delta f(t, x(t)) & \text{a.e. } t \in [0, \alpha] \\ x(0) = x_0, & \\ \dot{y}(t) = 0 & \text{a.e. } t \in [0, \alpha] \\ y(0) \in [0, 1] & \end{cases} \tag{11}$$

where we define

$$\Delta f(t, x) := f(t, x, \tilde{u}(t)) - f(t, x, \bar{u}(t)). \tag{12}$$

Here $\tilde{u}$ is the control function featuring in the constraint qualification CQ^*.

Since $\dot{y} = 0$ and the function y is absolutely continuous, it will be constant. We will here denote the value of the function simply by y in $[0, \alpha]$ instead of $y(t)$, when it is not ambiguous.

Step 2: By reducing the size of α, we can ensure that

$$h(t, x(t)) \leq 0 \quad \text{for all } t \in [0, \alpha], \tag{13}$$

for all trajectories x solving system (S).

For that, we start by introducing the following lemma, which is a simple consequence of the hypotheses imposed on the data and standard Gronwall-type estimates. A similar proof can be found in [9].

Lemma 4.1 *Consider the functions x, y solving the system of equations (S), and $\bar{x}$ solving (P). There exist positive constants A, and B such that for α small enough*

$$\begin{aligned} \|x(t) - x_0\| &\leq At \\ \|x(t) - \bar{x}(t)\| &\leq Byt \end{aligned}$$

for all $t \in [0, \alpha]$.

Choose an α satisfying

$$\alpha < \min\left\{\frac{2\delta}{K_h K_f(2A+B)}, \frac{\epsilon_1}{A}, \bar{\tau}^*\right\}. \tag{14}$$

Suppose, in contradiction, that for some fixed $t \in [0, \alpha]$

$$h(t, x(t)) > 0. \tag{15}$$

Define for $\beta \in [0, 1]$

$$r(\beta) := h(t, \bar{x}(t) + \beta(x(t) - \bar{x}(t))).$$

In view of the properties of h as a function of x, r is continuous. We have also that

$$\begin{aligned} r(0) &= h(t, \bar{x}(t)) \leq 0 \\ r(1) &= h(t, x(t)) > 0 \end{aligned}$$

It follows that the set

$$D := \{\beta \in [0, 1] : r(\beta) = 0\}$$

is non-empty, closed and bounded. We can therefore define

$$\beta_m := \max_{\beta \in D} \beta.$$

Since $r(1) > 0$, we have $\beta_m < 1$.
Take any $\beta \in (\beta_m, 1]$.
Applying the Lebourg Mean-Value Theorem ([2]), we obtain

$$\begin{aligned} h(t, x(t)) - r(\beta) &= \zeta_t \cdot [x(t) - \bar{x}(t) - \beta(x(t) - \bar{x}(t))] \\ &= (1-\beta)\zeta_t \cdot [x(t) - \bar{x}(t)] \end{aligned}$$

for some $\zeta_t \in \mathrm{co}\, \partial_x h(t, \hat{x})$, and $\hat{x}$ in the segment $(x(t), \bar{x}(t) + \beta[x(t) - \bar{x}(t)])$.

As $r(\beta) > 0$ for all $\beta \in (\beta_m, 1]$, we have that $h(t, \hat{x}) > 0$, which implies that $\mathrm{co}\partial_x h(t, \hat{x}) \subset \partial_x^> h(t, \hat{x})$. It follows that $\zeta_t \in \partial_x^> h(t, \hat{x})$.

Expanding the expression above yields

$$\begin{aligned} & h(t, x(t)) - r(\beta) \\ & = (1-\beta)\,\zeta_t \cdot \int_0^t [f(s, x(s), \bar{u}(s)) + y\Delta f(s, x(s)) - f(s, \bar{x}(s), \bar{u}(s))]\, ds \\ & \leq (1-\beta)\left(\zeta_t \cdot \int_0^t y\Delta f(s, x(s))\, ds\ + \|\zeta_t\| K_f \int_0^t \|x(s) - \bar{x}(s)\|\, ds\right) \\ & \leq (1-\beta)\left(\int_0^t \zeta_t \cdot y\Delta f(s, x_0)\, ds + 2K_f\|\zeta_t\| y \int_0^t \|x(s) - x_0\|\, ds \right. \\ & \qquad \left. + K_h K_f \int_0^t \|x(s) - \bar{x}(s)\|\, ds\right) \\ & \leq (1-\beta) \textstyle\int_0^t \zeta_t \cdot y\Delta f(s, x_0) ds + 2K_f K_h y \int_0^t \|x(s) - x_0\| ds \\ & \qquad + K_h K_f \textstyle\int_0^t \|x(s) - \bar{x}(s)\| ds \\ & \leq (1-\beta)\,(-y\delta t + K_h K_f y(A + B/2)t^2) \\ & \leq 0 \quad \text{for all } \beta \in (\beta_m, 1]. \end{aligned}$$

Here we have used the fact that the norm of every element of the subdifferential is bounded by the Lipschitz rank of the function. In the last two inequalities we have used CQ^* and (14).

Since r is continuous and $r(\beta_m) = 0$, it follows that

$$h(t, x(t)) \leq 0.$$

Step 3: Take a decreasing sequence $\{\alpha_i\}$ on $(0, \alpha)$, converging to zero. Associate with each α_i the following problem (P_i), in which satisfaction of the state constraint is enforced only on the subinterval $[\alpha_i, 1]$.

$$(P_i) \quad \begin{array}{ll} \text{Minimize} & g(x(1)) \\ \text{subject to} & \\ & \dot{x}(t) = f(t, x(t), \bar{u}(t)) + y(t) \cdot \Delta f(t, x(t)) \\ & \qquad\qquad \text{a.e. } t \in [0, \alpha_i) \\ & \dot{x}(t) = f(t, x(t), u(t)) \\ & \qquad\qquad \text{a.e. } t \in [\alpha_i, 1] \\ & \dot{y}(t) = 0 \qquad \text{a.e. } t \in [0, \alpha_i) \\ & x(0) = x_0 \\ & x(1) \in C \\ & y(0) \in [0, 1] \\ & u(t) \in \Omega(t) \quad \text{a.e. } t \in [\alpha_i, 1] \\ & h(t, x(t)) \leq 0 \text{ for all } t \in [\alpha_i, 1]. \end{array}$$

By convexity hypotheses (**H7**) all possible components x trajectories of (P_i) are contained in the set of possible trajectories of (P). Moreover the trajectory for (P_i) $y \equiv 0$ and $x \equiv \bar{x}$ will lead to a cost $g(\bar{x}(1))$ identical to the optimal cost (P). We have proved the following Lemma.

Lemma 4.2 *The trajectory $y \equiv 0$ and $x \equiv \bar{x}$ solves all problems (P_i).*

The necessary conditions for problem (P_i) assert the existence of an arc $(p_i, c_i) : [0, 1] \mapsto \mathbb{R}^n \times \mathbb{R}$, a measurable function γ_i, a nonnegative Radon measure $\mu_i \in C^*([\alpha_i, 1], \mathbb{R})$, and a scalar $\lambda_i \geq 0$ such that

$$\mu_i\{[\alpha_i, 1]\} + \|(p_i, c_i)\| + \lambda_i > 0, \tag{16}$$

$$-\dot{p}_i(t) \in \begin{cases} p_i(t) \cdot co\partial_x f(t, \bar{x}(t), \bar{u}(t)), & \\ \qquad\qquad\qquad \text{a.e. } \ t \in [0, \alpha_i); & \\ \left(p_i(t) + \int_{[\alpha_i, t)} \gamma_i(s)\mu_i(ds)\right) \cdot co\partial_x f(t, \bar{x}(t), \bar{u}(t)), & \\ \qquad\qquad\qquad \text{a.e. } \ t \in [\alpha_i, 1]. & \end{cases} \tag{17}$$

$$-\dot{c}_i(t) = \begin{cases} p_i(t) \cdot \Delta f(t, \bar{x}(t)), & \text{a.e. } \ t \in [0, \alpha_i); \\ 0, & \text{a.e. } \ t \in [\alpha_i, 1]. \end{cases} \tag{18}$$

$$-\left(p_i(1) + \int_{[\alpha_i, 1]} \gamma_i(s)\mu_i(ds) + \lambda_i \xi_i\right) \in N_C(\bar{x}(1)) \tag{19}$$

where $\xi_i \in \partial_x g(\bar{x}(1))$

$$-c_i(1) = 0\{0 = \lambda_i \partial_y g(\bar{x}(1))\} \tag{20}$$

$$c_i(0) \in N_{[0,1]}(0)$$

$$\gamma_i(t) \in \partial_x^{>} h(t, \bar{x}(t)) \quad \mu \text{ a.e. },$$

$$\text{supp}\{\mu_i\} \subset \{t \in [\alpha_i, 1] : h(t, \bar{x}(t)) = 0\}, \tag{21}$$

and for almost every $t \in [\alpha_i, 1]$, $\bar{u}(t)$ maximizes over $\Omega(t)$

$$u \mapsto \left(p_i(t) + \int_{[\alpha_i,t)} \gamma_i(s)\mu_i(ds)\right) \cdot f(t, \bar{x}(t), u). \tag{22}$$

It remains to pass to the limit as $i \to \infty$ and thereby to obtain a set of nondegenerate multipliers for the original problem.

Without changing the notation, we extend μ_i as a regular Borel measure on $[0, 1]$

$$\mu_i(\mathcal{B}) = \mu_i(\mathcal{B} \cap [\alpha_i, 1]) \text{ for all Borel set } \mathcal{B} \subset [0, 1].$$

Extend also γ_i, originally defined on $[\alpha_i, 1]$, arbitrarily to the interval $[0, 1]$ as a Borel measurable function. With theses extensions, noting that $\mu([0, \alpha_i)) = 0$, we can write

$$-\dot{p}_i(t) \in \left(p_i(t) + \int_{[0,t)} \gamma_i(s)\mu_i(ds)\right) \cdot co\partial_x f(t, \bar{x}(t), \bar{u}(t)) \quad \text{a.e. } t \in [0, 1].$$

It can be easily seen that c_i can be omitted from (16), since $p_i \equiv 0$ implies $c_i \equiv 0$. By scaling the multipliers we can then ensure that

$$\|\mu_i\{[\alpha_i, 1]\}\|_{T.V.} + \|p_i\|_{L_\infty} + \lambda_i = 1.$$

The multifunction $\partial_x^{>}$ is uniformly bounded, compact, convex, and has a closed graph. As $\{p_i\}$ is uniformly bounded and $\{\dot{p}_i\}$ is uniformly integrally bounded, we can arrange by means of subsequence extraction [2, Thm 3.1.7, Prop. 3.1.8] that

$$p_i \to p \text{ uniformly}, \quad \gamma_i d\mu_i \to \gamma d\mu \text{ weak}^*, \quad \lambda_i \to \lambda, \xi_i \to \xi,$$

where μ is the weak* limit of μ_i in the space of nonnegative-valued functions in $C^*([0, 1], \mathbb{R})$, γ is a measurable selection of $\partial_x^{>} h(t, \bar{x}(t))$ μ a.e., and $\xi \in \partial g(\bar{x}(1))$. To obtain ξ we have used the fact that $\partial g(\bar{x}(1))$ is a compact set.

It follows that the conditions (7), (9), (4) for problem (P) are satisfied and as $N_C(\bar{x}(1))$ is closed, (5) also holds.

Consider the set $S_i = [\alpha_i, 1] \backslash \Omega_i$ where Ω_i is a null Lebesgue measure set in $[\alpha_i, 1]$ containing all times where the maximization of (22) is not achieved at $\bar{u}$. We can then write

$$\left(p_i(t) + \int_{[\alpha_i,t)} \gamma_i(s)\mu_i(ds)\right) \cdot f(t, \bar{x}(t), u) \leq \left(p_i(t) + \int_{[\alpha_i,t)} \gamma_i(s)\mu_i(ds)\right) \cdot f(t, \bar{x}(t), \bar{u}(t)),$$

for all $t \in S_i$ and for all $u \in \Omega(t)$.

Now consider the full measure set $S = (0,1] \setminus \bigcup_i \Omega_i$. Fix some t in S. Then for all $i > N$, where N is such that $\alpha_N \leq t$ we have

$$\left(p_i(t) + \int_{[0,t)} \gamma_i(s)\mu_i(ds) \right) \cdot f(t, \bar{x}(t), u) \leq \left(p_i(t) + \int_{[0,t)} \gamma_i(s)\mu_i(ds) \right) \cdot f(t, \bar{x}(t), \bar{u}(t)) .$$

for all $u \in \Omega(t)$. Applying limits to both sides of this inequality we obtain (8).

At this point we have established that the set of multipliers (p, μ, λ), obtained as limit of (p_i, μ_i, λ_i) satisfy the necessary conditions of optimality for the original problem (P).

Step 4: It remains to verify

$$\mu\{(0,1]\} + ||q||_{L^\infty} + \lambda > 0. \tag{23}$$

We start by proof the following lemma:

Lemma 4.3 *The adjoint vector p_i in the necessary conditions of optimality for problem (P_i) satisfies*

$$\int_0^{\alpha_i} p_i(t) \cdot \Delta f(t, \bar{x}(t)) dt \leq 0. \tag{24}$$

As the cost function g does not depend on y, $c_i(1) = 0$. The set $N_{[0,1]}(0)$ is $(-\infty, 0]$, so $c_i(0) \leq 0$. Now, by integrating the differential equation involving c_i (18) we get

$$c_i(1) = c_i(0) + \int_0^{\alpha_i} -p_i(t) \cdot \Delta f(t, \bar{x}(t)) = 0.$$

The result easily follows.

In view of the constraint qualification, there exists a constant $\delta > 0$ and $\bar{\tau}^* \in (0,1]$ such that, $\forall \tau \in [0, \bar{\tau}^*]$

$$\int_0^{\tau} \zeta \cdot [f(t, x_0, \tilde{u}(t)) - f(t, x_0, \bar{u}(t))] dt < -\delta\tau$$

for all $\zeta \in \partial_x^> h(s,x),\ s \in [0, \bar{\tau}^*),\ x \in \{x_0\} + \epsilon_1 B$

Suppose, in contradiction, that

$$\mu\{(0,1]\} + ||q||_{L^\infty} + \lambda = 0. \tag{25}$$

Since $(\lambda, \mu, p) \neq 0$, we must have

$$\begin{aligned} &\lambda = 0, \\ &\mu = \beta\delta_{\{0\}}, \\ &p(t) = -\beta\zeta \quad \text{for some } \beta > 0 \text{ and } \zeta \in \partial_x^> h(0, x_0). \end{aligned} \tag{26}$$

The constraint qualification (CQ^*) implies

$$\int_0^{\tau} -p(t) \cdot \Delta f(t, x_0) dt = \int_0^{\tau} \beta\zeta \cdot \Delta f(t, x_0) dt < -\delta\beta\tau \quad \forall \tau \in [0, \bar{\tau}^*].$$

But expanding this last expression we can write

$\int_0^{\alpha_i} p_i(t) \cdot \Delta f(t, \bar{x}(t))dt$
$= \int_0^{\alpha_i} p(t) \cdot \Delta f(t, x_0) + (p_i(t) - p(t))\Delta f(t, x_0) + p_i(t)[\Delta f(t, \bar{x}(t)) - \Delta f(t, x_0)]dt$
$\geq \delta\beta\alpha_i - \int_0^{\alpha_i} 2K_u\|p_i(t) - p(t)\| + 2K_f\|\bar{x}(t) - x_0\|\|p_i(t)\|dt$
$\geq \delta\beta\alpha_i - \int_0^{\alpha_i} 2K_u\|p_i(t) - p(t)\| + 2K_f At\|p_i(t)\|dt$

By the uniform convergence of p_i, we can make $\|p_i - p\| < \bar{\epsilon}$ for any $\bar{\epsilon} > 0$ of our choice provided we choose a sufficient large i. Moreover $\|p_i\| \leq 1$.

It follows that

$$\int_0^{\alpha_i} p_i(t) \cdot \Delta f(t, \bar{x}(t))dt \geq \delta\beta\alpha_i - (2K_u\bar{\epsilon}\alpha_i + K_f A\alpha_i^2) > \delta\beta/2\alpha_i > 0$$

if $\bar{\epsilon} < \dfrac{\delta\beta}{8K_u}$ and $\alpha_i < \dfrac{\delta\beta}{4K_f A}$.

So, we would have $\int_0^{\alpha_i} p_i(t) \cdot \Delta f(t, \bar{x}(t))dt > 0$ contradicting Lemma (4.3). We deduce (23).

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