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NIPE WP 16 / 2007

NÚCLEO DE INVESTIGAÇÃO EM POLÍTICAS ECONÓMICAS
UNIVERSIDADE DO MINHO

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URL:
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[*] NIPE – *Núcleo de Investigação em Políticas Económicas* – is supported by the Portuguese Foundation for Science and Technology through the *Programa Operacional Ciência, Teconologia e Inovação* (POCI 2010) of the *Quadro Comunitário de Apoio III*, which is financed by FEDER and Portuguese funds.

Using cross-wavelets to decompose the time-frequency relation between oil and the macroeconomy*

Luís Aguiar-Conraria† Maria Joana Soares‡

September 12, 2007

Abstract

A large body of empirical literature has suggested that oil price shocks have an important effect on economic activity. But in most of the literature the analysis is exclusively done in the time domain. However, interesting relations exist at different frequencies. We use (cross) wavelet analysis to uncover some of these relations, estimating the spectral characteristics of the time-series as a function of time. Our analysis suggests that the volatility of both the inflation rate and the output growth rate started to decrease in the decades of 1950 and 1960, suggesting that the great moderation started then, but that it was temporarily interrupted due to the oils crisis of the 1970s, whose effects extend until the mid 1980s. We also show that while at business cycle frequencies oil prices lead industrial production, in the very long run production increases lead oil price increases. The exception to this long-run relation occurred between the mid 1970s and mid 1980s. Our analysis also suggests that monetary policy became much more efficient after 1980 to deal with the inflationary pressures of oil shocks.

*Luís Aguiar-Conraria acknowledges financial support from Fundação para a Ciência e a Tecnologia, project "Oil shocks and the Macroeconomy: Econometric estimation, economic modeling and policy implications", PTDC/ECO/64750/2006

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Keywords: Business cycles, time-frequency analysis, non-stationary time series, wavelets, cross wavelets, wavelet coherency.

JEL Classification: C10, E32.

1 Introduction

A large body of empirical literature has suggested that oil price shocks have an important effect on economic activity. Hamilton (1983, 1985, 1996), Burbidge and Harrison (1984), Santini (1985), Gisser and Goodwin (1986), Rotemberg and Woodford (1996), Bernanke et al. (1997), Aguiar-Conraria and Wen (2007) and many others have provided empirical evidence that oil prices were significant determinants of U.S. economic activity.

All these papers follow the VAR tradition to identify the oil shocks and to study its effects on macroeconomic variables (imposing some kind of short run or long run restrictions). Still, some authors are suspicious of these conclusions. As Hooker (1996) showed, the correlation between oil prices and economic activity is much less clear after 1985.

Mork (1989) attributed this instability of the empirical relation between oil prices and output to misspecification of the functional form. As Mork noted, until 1980 almost all oil price changes were upward. Only after 1980 we observed consistent oil price decreases. Mork (1989) showed that the reaction of output to positive and negative changes of oil prices was asymmetric.

To model this asymmetry, Hamilton (1996 and 2003) introduced the concept of net oil price increases. According to Hamilton it is not the oil price increase per se that causes output disruptions but net oil price increases. His measure of the net oil price increase is defined as the amount by which oil prices in quarter t exceed their peak value over the previous 12 or 36 months. Raymond and Rich (1997) used a Markov switching framework to model the asymmetric impact of oil shocks.

More recently, other approaches have been followed. For example, Kilian (2007) looks at historical accounts and industry sources to identify exogenous oil production shortfalls. A similar approach, which looks at prices instead of quantities, is followed by Cavallo and Wu (2006). Their measure of oil shocks is based on exogenous events that trigger substantial fluctuations in spot oil prices and are constructed to be free of endogenous and anticipatory movements. Once these nonlinear methods are considered, the basic results obtained in the

1980s with linear methods are replicated.

The cited works share a common feature. The analysis is done exclusively in the time domain. The frequency domain is left out. However, some interesting relations may exist at different frequencies. For example, it is possible that oil prices may act like a supply shock in the short or medium-run (high and medium frequencies), therefore affecting industrial production, while, in the longer run (lower frequencies) it is the industrial production, through a demand effect, that affects oil prices. These types of relations are difficult to uncover using time-domain methods.

To uncover relations at different frequencies, it is common to utilize Fourier analysis. However, under the Fourier transform, the time information of a time-series is completely lost. Because of this loss of information it is hard to distinguish transient relations or to identify when structural changes do occur. Moreover, these techniques are not appropriate to deal with non-stationary time-series. To overcome the problems of analyzing non-stationary data, Gabor (1946) introduced the Short Time Fourier Transform. The basic idea is to break a time series into smaller sub-samples and apply the Fourier transform to each sub-sample. However, as Raihan et al. (2005) pointed out, this approach is inefficient because the frequency resolution is the same across all different frequencies.

As an alternative, wavelet analysis has been proposed. Wavelet analysis performs the estimation of the spectral characteristics of a time-series as a function of time revealing how the different periodic components of the time-series change over time. While the Fourier transform breaks down a time series into constituent sinusoids of different frequencies and infinite duration in time, the wavelet transform expands the time series into shifted and scaled versions of a function – the so-called *mother-wavelet* – that has limited spectral band and limited duration in time.

One major advantage afforded by the wavelet transform is the ability to perform natural local analysis of a time series in the sense that the length of wavelets varies endogenously. It stretches into a long wavelet function to measure the low frequency movements; and it compresses into a short wavelet function to measure the high frequency movements. In order to capture abrupt changes, for example, one would like to have very short functions (narrow windows). At the same time, in order to isolate slow and persistent movements, one would like to have very long functions (wide windows). This is exactly what can be achieved with the wavelet transform.

We know from the Heisenberg uncertainty principle that there is always a trade-off between localization in time and localization in frequency; in particular, we cannot ask for a function to be, simultaneously, band and time limited. However, a mother wavelet can be chosen with a fast decay in time and frequency which, for all practical purposes, corresponds to an effective band and time limiting; see Daubechies (1992).

As a coherent mathematical body, wavelet theory was born in the mid-1980s (Grossmann and Morlet 1984, Goupillaudand et al. 1984). After 1990, the literature rapidly expanded and wavelet analysis is now used extensively in physics, geophysics, astronomy, epidemiology, signal processing, oceanography, etc. Interestingly, and in spite of all its potential advantages, this technique is very rarely used in Economics. The pioneering work of Ramsey and Lampart (1998a and 1998b) and Ramsey (1999) is unknown to most of the economists, who reveal a strong preference for traditional econometric methods, overlooking the potential for using wavelets to analyze economic data. Notable exceptions to this rule are Raihan et al. (2001 and 2005), Gençay et al. (2001 and 2005), Wong et al. (2003).

Probably, one of the reasons why wavelets are not more popular in the economics literature is related to the difficulty to simultaneously analyze two (or more) time-series. In Economics, these techniques have either been applied to a single time-series (e.g. Raihan et al. 2005) or used to individually analyze two time-series (one each time), whose decompositions are then studied using traditional time domain methods, such us correlation analysis or Granger causality (see Ramsey and Lampart, 1998a and 1998b).

In this paper, we present three tools, Cross Wavelet Transform, Cross Wavelet Coherence, and the phase difference, proposed by Hudgins et al. (1993), Torrence and Compo (1998), and Jevrejeva et al. (2003) that overcome this problem. Cross wavelet tools generalize wavelet methods, allowing the analysis of time-frequency dependencies between two time-series. With these tools, we are able to use wavelet analysis to directly study the interactions between two time-series at different frequencies and how they evolve over time. We will develop time-frequency concepts that are analogous to measures typically used by economists, such as covariance, correlation and causality.

We use these tools to analyze the impact of oil price changes in two macroeconomic variables: Industrial Production and Inflation.

This paper proceeds as follows. In section 2, we briefly present some of the properties that a proper wavelet must satisfy, and introduce the reader to the most popular complex wavelet: the Morlet wavelet. We also give a brief description of the Continuous Wavelet Transform.

This paper proceeds as follows. In section 2, we discuss the Continuous Wavelet Transform (CWT),[1] its localization properties and discuss in some detail the optimal characteristics of the Morlet wavelet. Section 3 describes the Cross Wavelet Transform (XWT), the Cross Wavelet Coherence (WTC), and the phase difference and discusses how to assess their statistical significance. In section 4, we use the wavelet power spectrum, wavelet coherency and wavelet phase-difference to analyze our data. Section 5 concludes.

2 Wavelets: The dynamical decomposition of time

2.1 The wavelet

We start by introducing some mathematical notation. In what follows, $L^2(\Re)$ denotes the set of square integrable functions, i.e. the set of functions defined on the real line such that

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty. \tag{1}$$

Since the above quantity is usually referred to as the energy of the function x, this space is also known as the space of functions with finite energy. As it is well known, one can define

[1] For a review of the discrete wavelet transform and some of its applications to economic data, the reader is referred to Crowley (2007).

in $L^2(\Re)$ an inner product

$$\langle x, y \rangle := \int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt \tag{2}$$

and an associated norm $\|x\| := \langle x, x \rangle^{\frac{1}{2}}$. Here, and throughout the paper, the asterisk superscript will be used to denote complex conjugation and the symbol := means "by definition".

Given a function $x(t) \in L^2(\Re)$, we will denote by $X(f)$ the Fourier transform of $x(t)$:

$$X(f) := \int_{-\infty}^{\infty} x(t)\, e^{-i2\pi ft} dt. \tag{3}$$

We recall the well-known Parseval relation, valid for all $x(t), y(t) \in L^2(\Re)$:

$$\langle x(t), y(t) \rangle = \langle X(f), Y(f) \rangle, \tag{4}$$

from which the Plancherel identity (which sates that the energy of a function is preserved by the Fourier transform) immediately follows:

$$\|x(t)\|^2 = \|X(f)\|^2; \tag{5}$$

see, for example, Körner (1988).

The minimum requirements imposed on a function $\psi(t)$ to qualify for being a *mother (admissible or analyzing) wavelet* are that $\psi \in L^2(\Re)$ and also fulfills a technical condition, known as the *admissibility condition*, which reads as follows:

$$0 < C_\psi := \int_{-\infty}^{\infty} \frac{|\Psi(f)|}{|f|} df < \infty. \tag{6}$$

The wavelet ψ is usually normalized to have unit energy: $\|\psi\|^2 = \int_{-\infty}^{\infty} |\psi(t)|^2 dt = 1$.

The square integrability of ψ is a very mild decay condition; the wavelets used in practice have much faster decay; typical behavior will be exponential decay ($|\psi(t)| \leq Me^{-C|t|}$,for some constants C and M) or even compact support.

For functions with sufficient decay it turns out that the admissibility condition (6) is equivalent to requiring

$$\Psi(0) = \int_{-\infty}^{\infty} \psi(t)\, dt = 0. \tag{7}$$

This means that the function ψ has to wiggle up and down the t-axis, i.e. it must behave like a wave; this, together with the decaying property, justifies the choice of the term wavelet (originally, in French, *ondelette*) to designate ψ.

2.2 The continuous wavelet transform

Starting with a mother wavelet ψ, a family $\psi_{s,\tau}$ of "wavelet daughters" can be obtained by simply scaling ψ by s and translating it by τ

$$\psi_{s,\tau}(t) := \frac{1}{\sqrt{|s|}} \psi\left(\frac{t-\tau}{s}\right), \quad s, \tau \in \mathcal{R}, s \neq 0. \tag{8}$$

The parameter s is a scaling or dilation factor that controls the length of the wavelet (the factor $1/\sqrt{|s|}$ being introduced to guarantee preservation of the unit energy, $\|\psi_{s,\tau}\| = 1$) and τ is a location parameter that indicates where the wavelet is centered. Scaling a wavelet simply means stretching it (if $|s| > 1$), or compressing it (if $|s| < 1$).[2]

Given a function $x(t) \in L^2(\Re)$ (a time series), its continuous wavelet transform (CWT), with respect to the wavelet ψ, is a function $W_x(s,\tau)$ obtained by projecting $x(t)$, in the L^2 sense, onto the over-complete family $\{\psi_{s,\tau}\}$:

$$W_x(s,\tau) = \langle x, \psi_{s,\tau} \rangle = \int_{-\infty}^{\infty} x(t) \frac{1}{\sqrt{|s|}} \psi^*\left(\frac{t-\tau}{s}\right) dt. \tag{9}$$

The importance of the admissibility condition (6) comes from the fact that it guarantees that it is possible to recover $x(t)$ from its wavelet transform; see e.g. Daubechies (1992):

$$x(t) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} W_x(s,\tau)\, \psi_{s,\tau}(t)\, d\tau \right] \frac{ds}{s^2}. \tag{10}$$

[2] Note that for negative s, the function is also reflected.

Since we can go from $x(t)$ to its wavelet transform, and from the wavelet transform back to $x(t)$,[3] we can conclude that both are representations of the same mathematical entity. They just present information in a different manner, allowing us to gain insights that would, otherwise, remain hidden. It is also important to observe that the energy of $x(t)$ is preserved by the wavelet transform, in the sense that

$$\|x\|^2 = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} |W_x(s,\tau)|^2 \, d\tau \right] \frac{ds}{s^2} \tag{11}$$

and that a Parseval type identity also holds

$$\langle x, y \rangle = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \left[W_x(s,\tau) \, W_y^*(s,\tau) \, d\tau \right] \frac{ds}{s^2} \tag{12}$$

for $x, y \in L^2(\mathfrak{R})$.

Because the wavelet function $\psi(t)$ may, in general, be complex, the wavelet transform W_x may also be complex. The transform can then be divided into its real part, $\mathcal{R}\{W_x\}$, and imaginary part, $\mathcal{I}\{W_x\}$, or in its amplitude, $|W_x|$, and phase, $\phi_x(s,\tau) = \tan^{-1}\left(\frac{\mathcal{I}\{W_x\}}{\mathcal{R}\{W_x\}}\right)$. The phase of a given time-series $x(t)$ can be viewed as the position in the pseudo-cycle of the series and it is parameterized in radian ranging from $-\pi$ to π. For real-valued wavelet functions the imaginary part is zero and the phase is undefined. Therefore, in order to separate the phase and amplitude information of a time series it is important to make use of complex wavelets. In particular, it is convenient to choose $\psi(t)$ to be *progressive* or *analytic*, i.e. to be such that $\Psi(f) = 0$ for $f < 0$; in this case, if $x(t)$ is real, a variant of the reconstruction formula, in which the parameter s can be restricted to positive values only, is possible:

$$x(t) = \frac{2}{C_\psi} \int_{0}^{\infty} \left[\int_{-\infty}^{\infty} \mathcal{R}\left(W_x(s,\tau) \, \psi_{s,\tau}(t) \right) d\tau \right] \frac{ds}{s^2}, \tag{13}$$

[3] One can also limit the integration over a range of scales, performing a band-pass filtering of the original series.

one also has

$$\|x\|^2 = \frac{2}{C_\psi} \int_0^\infty \left[\int_{-\infty}^\infty |W_x(s,\tau)|^2 \, d\tau \right] \frac{ds}{s^2} \tag{14}$$

and

$$\langle x, y \rangle = \frac{2}{C_\psi} \int_0^\infty \left[W_x(s,\tau) \, W_y^*(s,\tau) \, d\tau \right] \frac{ds}{s^2}; \tag{15}$$

see Daubechies (1992), pp. 27-28, Kaiser 1994, pp. 70-73 or Mallat (1998), pp.82-83 for more details about analytic wavelets. Throughout the rest of the paper, since, in the practical economic applications, we will use an analytic wavelet, we always assume that the scaling parameter s takes positive values only.

In view of the energy preservation formula (14), and in analogy with the terminology used in the Fourier case, the function $|W_x(s,\tau)|^2$ is usually referred to as the wavelet power spectrum (sometimes also called the scalogram, see Flandrin 1988).

2.3 Localization properties

Let the wavelet ψ be normalized so that $\|\psi\| = 1$ and define its center μ_t by

$$\mu_t = \int_{-\infty}^\infty t \, |\psi(t)|^2 \, dt. \tag{16}$$

In other words, the center of the wavelet is simply the mean of the probability distribution obtained from $|\psi(t)|^2$. As a measure of concentration of ψ around its center one usually takes the variance σ_t:

$$\sigma_t = \left\{ \int_{-\infty}^\infty (t - \mu_t)^2 \, |\psi(t)|^2 \, dt \right\}^{\frac{1}{2}}. \tag{17}$$

In a total similar manner, one can also define the center μ_f and variance σ_f of the Fourier transform $\Psi(f)$ of ψ.

The interval $[\mu_t - \sigma_t, \mu_t + \sigma_t]$ is the set where ψ attains its "most significant" values whilst the interval $\left[\mu_f - \sigma_f, \mu_f + \sigma_f\right]$ plays the same role for $\Psi(f)$ of ψ. The rectangle $[\mu_t - \sigma_t, \mu_t + \sigma_t] \times \left[\mu_f - \sigma_f, \mu_f + \sigma_f\right]$ in the (t, f) $-$plane is called the Heisenberg box or window in the time-frequency plane. We then say that ψ is localized around the point (μ_t, μ_f) of the time-frequency plane with uncertainty given by $\sigma_t \sigma_f$.

The uncertainty principle, first established by Werner Karl Heisenberg, gives a lower bound on the product of the standard deviations of position and momentum for a system, implying that it is impossible to have a particle that has an arbitrarily well-defined position and momentum simultaneously. In our context, the Heisenberg uncertainty principle establishes that the uncertainty is bounded from below by the quantity $1/4\pi$:

$$\sigma_t \sigma_f \geq \frac{1}{4\pi}. \tag{18}$$

It is also known that the equality in (18) is attained if and only if the function ψ is a (translated and modulated) Gaussian: $\psi(t) = ae^{i\mu_f t}e^{-b(t-\mu_t)^2}$; see Messiah (1961).

It follows from the Parseval relation (4) that

$$\begin{aligned} W_x(s,\tau) &= \langle x(t), \psi_{s,\tau}(t) \rangle \\ &= \langle X(f), \Psi_{s,\tau}(f) \rangle \end{aligned} \tag{19}$$

where $X(f)$ and $\Psi_{s,\tau}(f)$ are the Fourier transforms of $x(t)$ and $\psi_{s,\tau}(t)$,respectively.

If the mother wavelet ψ is centered at μ_t and has variance σ_t and its wavelet transform $\Psi(f)$ is centered at μ_f with a variance σ_f, then one can easily show that the daughter wavelet $\psi_{\tau,s}$ will be centered at $\tau + s\mu_t$ with variance $s\sigma_t$, whilst its Fourier transform $\Psi_{s,\tau}$ will have center $\frac{\mu_f}{s}$ and variance $\frac{\sigma_f}{s}$. Hence, (19) shows that the continuous wavelet transform $W_x(s,\tau)$ gives us local information within a time-frequency window

$$[\tau + s\mu_t - s\sigma_t, \tau + s\mu_t + s\sigma_t] \times \left[\frac{\mu_f}{s} - \frac{\sigma_f}{s}, \frac{\mu_f}{s} + \frac{\sigma_f}{s}\right] \tag{20}$$

In particular, if ψ is chosen so that $\mu_t = 0$ and $\mu_f = 1$,then the window associated with $\psi_{\tau,s}$ becomes

$$[\tau - s\sigma_t, \tau + s\sigma_t] \times \left[\frac{1}{s} - \frac{\sigma_f}{s}, \frac{1}{s} + \frac{\sigma_f}{s}\right] \tag{21}$$

In this case, the wavelet transform $\{\mathcal{W}_{\psi}f\}(s,\tau)$ will give us information on $x(t)$ for t near the instant $t=\tau$, with precision $s\sigma_t$, and information about $X(f)$ for frequency values near the frequency $f=\frac{1}{s}$, with precision $\frac{\sigma_f}{s}$. Therefore:

- small values of s correspond to information about $x(t)$ in a fine scale and about $X(f)$ in a broad scale,
- large values of s correspond to information in a broad scale about $x(t)$ and in a fine scale about $X(f)$,
- although the area of the windows is constant at all scales, $A=4\sigma_t\sigma_f$, their dimensions change according to the scale; the windows stretch for large values of s (broad scales s – low frequencies $f=1/s$) and compress for small values of s (fine scale s – high frequencies $f=1/s$).

2.4 The Morlet wavelet: optimal joint time-frequency concentration

There are several types of wavelet functions available with different characteristics, such as, Morlet, Mexican hat, Haar, Daubechies, etc.; see, e.g. Daubechies (1992), Mallat (1998) or Meyer (1993). Since the wavelet coefficients $W_x(s,\tau)$ contain combined information on both the function $x(t)$ and the analyzing wavelet $\psi(t)$, the choice of the wavelet is an important aspect to be taken into account. This will depend mainly on the particular application one has in mind. In this paper we choose a complex wavelet, as it yields a complex transform, with information on both the amplitude and phase, which is essential for the analysis we want to perform. One of the most popular wavelets used is the Morlet wavelet, first introduced in Goupillaudand (1984), which is defined as

$$\psi_\eta(t)=\pi^{-\frac{1}{4}}\left(e^{i\eta t}-e^{-\frac{\eta^2}{2}}\right)e^{-\frac{t^2}{2}}, \tag{22}$$

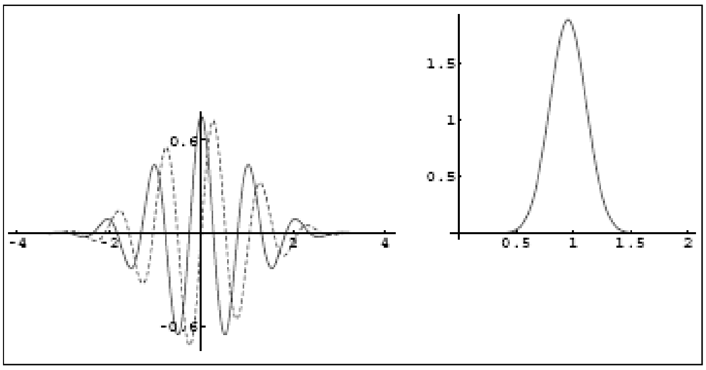

Figure 1: On the left: the Morlet wavelet $\psi_6(t)$ — real part (solid line) and imaginary part (dashed line). On the right: its Fourier transform.

the term $e^{-\frac{\eta^2}{2}}$ being introduced to guarantee the fulfillment of the admissibility condition; however, for $\eta \geq 5$ this term becomes negligible. The simplified version

$$\psi_\eta(t) = \pi^{-\frac{1}{4}} e^{i\eta t} e^{-\frac{t^2}{2}} \tag{23}$$

of (22) is normally used (and still referred to as a Morlet wavelet). Our results in the next section, were obtained with the particular choice $\eta = 6$.

This wavelet has interesting characteristics. First of all, it is (almost) analytic. The Fourier transform of the "true" Morlet wavelet (22) is, in fact, supported in $(0, \infty)$, but that of (23) has some mass on $(-\infty, 0)$:

$$\Psi_\eta(f) = \pi^{\frac{1}{4}} \sqrt{2} e^{-\frac{1}{2}(2\pi f - \eta)^2} \tag{24}$$

For $\eta > 5$, this mass is, however, negligible, so, for all practical purposes, the wavelet can be considered as analytic; see Foufoula-Gergiou and Kumar (1993).

The wavelet (23) is centered at the point $\left(0, \frac{\eta}{2\pi}\right)$ of the time-frequency plane; hence, for the particular choice $\eta = 6$, one has that the frequency center is

$$\mu_f = \frac{6}{2\pi} \approx 1 \tag{25}$$

and the relationship between the scale and frequency is simply

$$f = \frac{\mu_f}{s} \approx \frac{1}{s}. \tag{26}$$

It is also very simple to verify that the time variance is $\sigma_t = 1/\sqrt{2}$ and the frequency variance is $\sigma_f = 1/\left(2\pi\sqrt{2}\right)$. Therefore, the uncertainty of the corresponding Heisenberg box attains the minimum possible value $\sigma_t \sigma_f = \frac{1}{4\pi}$ and one can thus say that the Morlet wavelet has optimal joint time-frequency concentration.[4]

2.5 Transform of finite discrete data

If one is dealing with a discrete time series $\{x_n, n = 0, ..., N-1\}$ of N observations with a uniform time step δt, the integral in (9) has to be discretized and is, therefore, replaced by a summation over the N time steps; the CWT of the time series $\{x_n\}$ is thus given by

$$W_m^x(s) = \frac{\delta t}{\sqrt{s}} \sum_{n=0}^{N-1} x_n \psi^* \left((n-m) \frac{\delta t}{s} \right), \qquad m = 0, 1, \ldots, N-1. \tag{27}$$

Although it is possible to calculate the wavelet transform using the above formula for each value of s and m, one can also identify the computation for all the values of m simultaneously as a simple convolution of two sequences; in this case, one can follow the standard procedure and calculate this convolution as a simple product in the Fourier domain, using the fast Fourier transform (FFT) algorithm to go forth and back from time to spectral domain; this is precisely the technique prescribed by Torrence and Compo (1998).[5]

As with other types of transforms, the CWT applied to a finite length time series inevitably suffers from border distortions; this is due to the fact that the values of the transform at the beginning and the end of the time series are always incorrectly computed, in the sense that they involve "missing" values of the series which are then artificially prescribed; the most

[4]This could be antecipated by noting that ψ_η is a simple modulated Gaussian.

[5]A program code based on the above procedure is also available at the site http://paos.colorado.edu/research/wavelets/.

common choices are zero padding – extension of the time series by zeros – or periodization. Since the "effective support" of the wavelet at scale s is proportional to s, these edge-effects also increase with s. The region in which the transform suffers from these edge effects is called the cone of influence (COI). In this area of the time-frequency plane the results are unreliable and have to be interpreted carefully. In this paper, the cone of influence is defined, following Torrence and Compo (1998), as the e-folding time of the wavelet at the scale s, that is, so that the wavelet power of a Dirac δ at the edges decreases by a factor of e^{-2}. In the case of the Morlet wavelet this is given by $\sqrt{2}s$, and in all the pictures is marked as a shadow in the wavelet plot.

3 Data analysis with wavelets[6]

3.1 Wavelet Power Spectrum

We simply define the wavelet power as $|W_n^x|^2$. Following Torrence and Compo (1998), the statistical significance of wavelet power can be assessed relative to the null hypotheses that the signal is generated by a stationary process with a given background power spectrum (P_f). Torrence and Compo assumed a first order auto-regressive model and, using Monte Carlo simulations, showed that on average, the local wavelet power spectrum is indistinguishable from the Fourier power spectrum. They then derive, under the null, the corresponding distribution for the local wavelet power spectrum,

$$D\left(\frac{|W_n^x(s)|^2}{\sigma_x^2} < p\right) = \frac{1}{2}P_f\chi_v^2, \tag{28}$$

at each time n and scale s. The value of P_f in (28) is the mean spectrum at the Fourier frequency f that corresponds to the wavelet scale s (in our case $s \approx \frac{1}{f}$, see equation (26)) and v is equal to 1 for real and 2 for complex wavelets.

[6] We thank Bernard Cazelles for letting us use his MatLab software package in Cazelles, et al. (2007). The MATLAB functions used can be downloaded at http://ecologie.snv.jussieu.fr/cazelles/wavelets/.

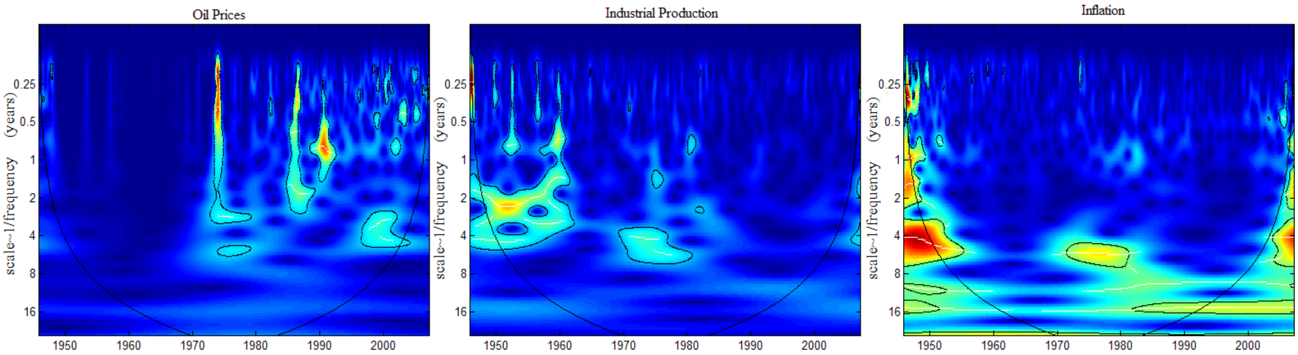

Figure 2: Wavelet Power Spectrum — The dashed black contour designates the 5% significance level against an AR1 (see Aguiar-Conraria et al. 2007 for details) and the cone of influence, which indicates the region not affected by edge effects is also shown. The color code for power range from blue (low power) to red (high power). The white lines indicate the maxima of the undulations of the wavelet power spectrum.

In Figure 2, we can see the estimated power spectrum for several time series for the United States economy: inflation (based on the Consumer Price Index), Oil Prices (growth rate) and Industrial Production Index (growth rate). The dashed black contour designates the 5% significance level against an AR1 (see Cazelles et al. 2007 for details) and the cone of influence, which indicates the region not affected by edge effects, is also shown. The color code for power range from blue (low power) to red (high power). The white lines indicate the maxima of the undulations of the wavelet power spectrum.

It is clear that the different time series have different characteristics in the time-frequency domain. During the late 1940s and early 1950s, the inflation rate variance was quite high both at low and high scales. Again in the 1970s and 1980s, probably as a consequence of very active oil shocks, the variance of the inflation rate became higher, but in this case, the effect is clearer at medium and high scales, suggesting that we were facing medium and long term shocks to inflation. The power, at all scales, of the industrial production was quite high until 1950s. After that, it has been steadily decreasing, with an exception between mid 1970s and mid 1980s, when the variance at the business cycle frequency (3 to 8 years) was quite high. It has become common in the literature to argue that we have been observing, in the last two decades, a decrease in the volatility of GDP in the United States (e.g. see Blanchard and Simon 2001). Some authors call it the "Great Moderation". In reality, we can observe that this is a secular, and not decadal, trend. Imediatly after World War II, the volatility was quite high at business cucly frequencies. In the 1960s, the volatility decreased at all scales. It then increased again, probably due to the oil shocks, at the business cycle frequency in the 1970s, however this increase was temporary.

If we look at the power spectrum of the Oil Prices growth rate, we observe that until 1970s there was not much action, between 1975 and 1980, both low and medium scales $\frac{1}{12} \sim 6$ years (high and medium frequencies) show high power We observe similar effects in late 1980s and early 1990s, and again in 2000. A structural change occurred in the Oil Price in the mid 1970s. These changes are related to the oil crisis that occurred in the 1970s, after which oil prices became market based and much more volatile.

3.2 The Cross Wavelet and the Phase Difference

Probably, one of the reasons why wavelets are not more popular in the Economics literature is because it has been a difficult task to use wavelets to analyze two, or more, time series together. Hudgins et al. (1993), Torrence and Compo (1998), and Jevrejeva et al. (2003) showed how the Cross Wavelets can be used to quantify relationships between two time series in the time-frequency space.

The cross wavelet transform of two time series, $x = \{x_n\}$ and $y = \{y_n\}$, first introduced by Hudgins et al. (1993) is simply defined as

$$W_n^{xy} = W_n^x W_n^{y*}, \tag{29}$$

where W_n^x and W_n^y are the wavelet transforms of x and y, respectively. The cross wavelet power is given by $|W_n^{xy}|$.

While the wavelet power spectrum depicts the variance of a time series, with times of large variance showing large power, the cross–wavelet power of two time series depicts the covariance between these time series at each scale or frequency. Therefore, cross–wavelet power gives us a quantified indication of the similarity of power between two time series.

As in the Fourier spectral approaches, the cross wavelet coherence can be defined as ratio of the cross-spectrum to the product of the spectrum of each series, and can be thought of as the local correlation between two CWTs. Here, again, we follow Jevrejeva et al. (2003) and define the wavelet coherence between two time series $x = \{x_n\}$ and $y = \{y_n\}$ as follows:

$$R_n^2(s) = \frac{\left|S\left(s^{-1}W_n^{xy}(s)\right)\right|}{S\left(s^{-1}\left|W_n^x\right|\right)^{\frac{1}{2}} S\left(s^{-1}\left|W_n^y\right|\right)^{\frac{1}{2}}}, \tag{30}$$

where S denotes a smoothing operator in both time and scale.

Smoothing is a necessary step, because, without that step, coherence is identically one at all scales and times. In Fourier analysis we overcome this problem by smoothing the cross-spectrum before normalizing. For wavelet analysis and, in particular with the Morlet wavelet, one can follow Torrence and Webster (1998): the smoothing is achieved by a convolution in time and scale. The time convolution is done with a Gaussian and the scale convolution is performed by a rectangular window; see Grinsted et al. (2004) for details.

Theoretical distributions for WTC have not been derived yet. So to assess the statistical significance of the estimated wavelet coherence we follow Grinsted et al. (2004) and use Monte Carlo methods. Again, see Grinsted et al. (2004) for details.

Phase differences are useful to caracterize phase relationships between two time series, $x = \{x_n\}$ and $y = \{y_n\}$. As we said before, the phase of a given time-series, ϕ_x, can be viewed as the position in the pseudo-cycle of the series.The phase difference, $\phi_{x,y}$, caracterizes phase relationships between the two time-series. The phase difference is defined as

$$\phi_{x,y} = \tan^{-1}\left(\frac{\mathcal{I}\left\{W_n^{xy}\right\}}{\mathcal{R}\left\{W_n^{xy}\right\}}\right), \qquad \text{with } \phi_{x,y} \in [-\pi, \pi]. \tag{31}$$

A phase difference of zero indicates that the time series move together (analogous to positive covariance). If $\phi_{x,y} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ then the series move in-phase, with the time-series y leading x.if $\phi_{x,y} \in \left(-\frac{\pi}{2}, 0\right)$ On the other hand, if $\phi_{x,y} \in \left(-\frac{\pi}{2}, 0\right)$ then it is x that is leading.

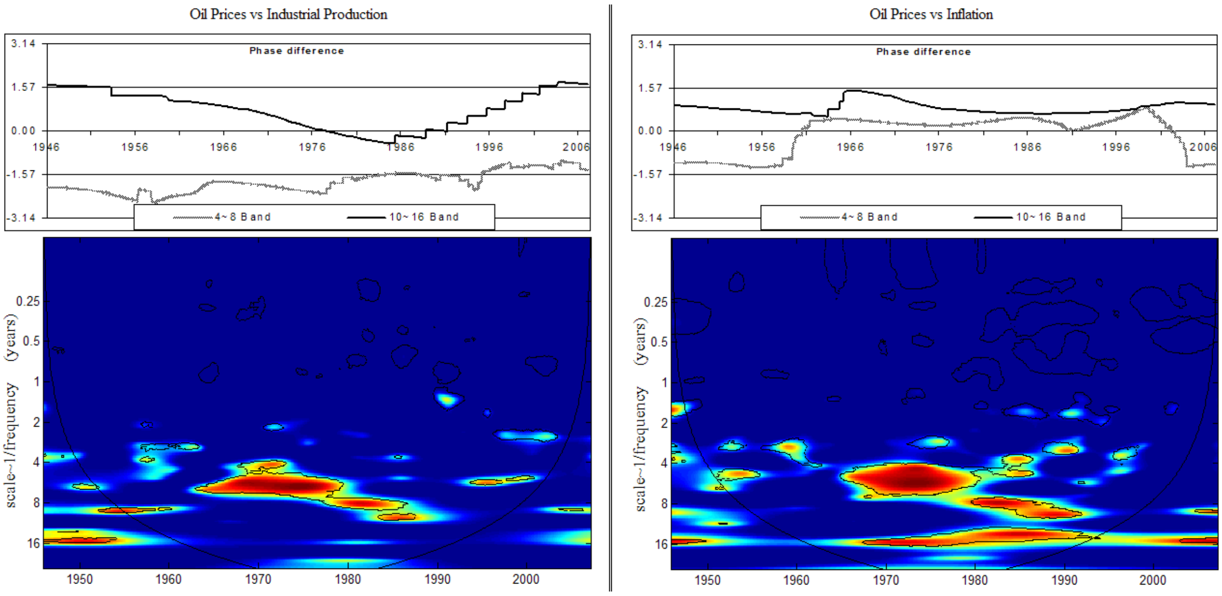

Figure 3: **On top:** Cross Wavelet Coherency. The dashed black contour designates the 5% significance level against an AR1 (obtained by Monte Carlo simulations) and the cone of influence, which indicates the region not affected by edge, effects is also shown. The color code for power range from blue (low coherency – close to zero) to red (high coherency – close to one). **In the bottom:** Phase Difference between the two series computed with the wavelet transform in the indicated periodic band.

We have an anti-phase relation (analogous to negative covariance) if If $\phi_{x,y} \in \left(\frac{\pi}{2}, \pi\right] \cup \left(-\pi, \frac{\pi}{2}\right]$. If $\phi_{x,y} \in \left(\frac{\pi}{2}, \pi\right)$ then x is leading. Time-series y is leading if $\phi_{x,y} \in \left(-\pi, -\frac{\pi}{2}\right)$.

On the left, figure 3 gives us the coherence and phase relations between oil prices and industrial production. Several structural changes occurred. In the 1950s, there is high coherency at large scales (in the $10 \sim 16$ years band).[7] Between mid 1960s and 1990, we can see a high coherence at medium scales ($4 \sim 8$ years). Looking at the phase difference in the $10 \sim 16$ years band, we can see a positive relation between industrial production and oil prices, with industrial production leading, for all periods, except between 1975 and 1985. This suggests that in the very long run, increases in the industrial production lead to increases in the oil price, suggesting that these oil price increases are demand-driven. The exception to this rule happened between 1975 and 1985, a period during which oil markets were quite turbulent and successive supply crises occurred (for example, we had the Arab oil embargo in late 1973 early

[7]Note however that this band is affected by edge effects, so the results should be interpreted conservatively)

1974 and the Iranian Revolution in 1979). If we look at the $4 \sim 8$ years band, we see that the phase difference is contained between $-\pi$ and $-\frac{\pi}{2}$ for most of the time, suggesting an inverse relation between oil prices and industrial production, with oil prices leading. This means that the Industry reacts to increases in the oil prices, and hence in the production costs, decreasing output. This relation seems to have changed after 1996, but this region is affected by edge effects, so it is to early to draw serious conclusions.

On the right of figure 3 , we see that the relation between oil prices and inflation is even stronger and more stable. The phase differences reveal a very stable relation. At large scales ($10 \sim 16$ years band) the phase difference has consistently been between zero and $\pi/2$. The same happens at medium scales ($4 \sim 8$ years band), after 1960 and until 2002 (after this year edge effects are no longer irrelevant). This suggests that oil price increases lead the consumer price index increases. Looking at coherency some different patterns emerge. There is a structural change in the late 1960s, coinciding with the six-day war of 1967. Before that time, there were not many periods of high coherence. In the 1970s there is high coherency at both medium ($4 \sim 8$ years band) and large scales ($12 \sim 16$ years band). During the 1980 decade, we observe high coherence in the $8 \sim 16$ years band After 1990, only at very high scales do we observe strong coherency. This suggests that monetary authorities became more proficient on avoiding the inflationary effects of oil price increases. Some political economy major events that happened during these decades may explain this evolution. The decade of 1970 is the decade of the big oil shocks. Then in 1980 there was a strong shift in the American monetary policy. In July 1979, Paul Volcker had been nominated, by President Carter, the Chairman of the Federal Reserve Board. Volker announced a fierce fight against inflation and implemented a very restrictive monetary policy as a reaction to the inflationary pressures of the second oil shock. In 1987, and during entire decade of 1990, when Alan Greenspan was the chairman of the Federal Reserve, inflation was under control.

4 Conclusion

Wavelet analysis is an important addition to time-series methods with practical applications in Economics, which allows us to decompose relationships in the time-frequency domain. We illustrated how wavelet analysis can naturally be applied to the study of business cycles (given its periodic nature), or to any field of economics, or finance, especially when there is a distinction between short and long-run relations. Wavelet analysis can help us to interpret multi-frequency, non-stationary time-series data, revealing features we could not see otherwise. We have argued that the wavelet transform is much better suited for economic data than the Fourier transform. The main advantage of the wavelet approach is the ability to analyze transient dynamics, both for single time-series or for the association between two time-series.

We showed that some of the shortcomings that economists have found when applying wavelet techniques to study two or more time series disappears once the concept of cross wavelet is introduced. We used tools that, to our knowledge, have not been used yet by economists: the Cross Wavelet Coherence and the phase difference. While the wavelet power spectra quantifies the main periodic component of a given time-series and its time evolution, the Cross Wavelet Transform and the Cross Wavelet Coherence Wavelet are used to quantify the degree of linear relation between two non-stationary time-series in the time–frequency domain. Phase analysis is a nonlinear technique that makes possible to study the phase synchronization of two time-series.

We have studied the relation between oil and output and uncovered an interesting relation: while at business cycle frequencies (3-8 years) oil prices lead industrial production, with oil price increases having negative effects on production, in the very long run we observe a different causal relation, with production changes leading oil price changes, suggesting that these are demand driven. The exception to this long-run relation occurred between the mid 1970s and mid 1980s, a time during which oil crisis were clearly a supply problem.

The relation between oil price increases and inflation was also studied. This relation proved to be more stable with oil price increases leading inflation increases across all timescales. But an interesting feature was also apparent, the tight monetary policy of the 1980s proved to be successful, with a decrease of the inflationary impact of oil price shocks. During the 1990s, monetary policy was also very efficient on controlling the inflationary impacts of oil price increases.

We have also shown that the volatility of both the inflation rate and the output growth rate started to decrease in the decades of 1950 and 1960, suggesting that the great moderation started then, but that it was temporarily interrupted due to the oils crisis of the 1970s, whose effects were felt until the mid 1980s.

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