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Making Sense of Fractions at Primary School — the Case of Teacher João

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Abstract. The most recent Portuguese guidelines for the Primary School Mathematics preconize an in-depth contact with fractions, dealing with quotient, part-whole, measure and operator interpretations for fractions. It is well known that teachers often struggle on teaching such matters. Since fractions are traditionally approached mainly in part-whole and operator interpretations, it seems pertinent to investigate whether the actual teaching practices reflect those innovative guidelines. This study focuses on teaching practices on fractions and aims to understand primary school teachers constrains and difficulties when teaching fractions. It addresses three questions: 1) How does the teacher make sense of fractions in the classroom? 2) Does the teacher properly promote the connections between fractions and everyday situations? 3) How does the teacher articulate distinct interpretations of fractions in the classroom? Four primary school teachers participated in a collaborative working program about fractions with the researcher (one of the authors of this paper) and their classes were observed. This paper presents the results related to the observed classes of one of the participating teachers – João (fictitious name). A qualitative analysis of the observed lessons suggests some fragilities regarding the teaching of the different interpretations of fractions, namely: on approaching the equivalence and the ordering of fractions in quotient interpretation; on marking fractions on the number line; on articulating the interpretations of fractions. Therefore, in-service teacher training should be regularly promoted for primary school teachers in order to ensure greater convergence between curriculum and teaching practices, improving the quality of the latter.

Résumé. Les directives portugaises les plus récentes sur les mathématiques dans les écoles primaires préconisent un contact approfondi avec les fractions, traitant des interprétations quotient, partie-ensemble, mesure et opérateur pour les fractions. Il est bien connu que les enseignants ont souvent du mal à enseigner de telles matières. Étant donné que les fractions sont généralement abordées principalement sous forme d'interprétations partielles et d'opérateurs, il semble pertinent de rechercher si les pratiques pédagogiques actuelles reflètent ces directives novatrices. Cette étude se concentre sur les pratiques pédagogiques sur les fractions et vise à comprendre les contraintes et les difficultés rencontrées par les enseignants du primaire lors de l'enseignement des fractions. Il aborde trois questions: 1) Comment l'enseignant donne-t-il un sens aux fractions dans la classe? 2) L'enseignant favorise-t-il correctement les liens entre les fractions et les situations de la vie courante? 3) Comment l'enseignant articule-t-il des interprétations distinctes des fractions dans la classe? Quatre enseignants du primaire ont participé à un programme de travail collaboratif sur les fractions avec le chercheur (l'un des auteurs de cet article) et leurs classes ont été observées. Cet article présente les résultats relatifs aux classes observées de l'un des enseignants participants - João (nom fictif). Une analyse qualitative des leçons observées laisse entrevoir certaines fragilités quant à l'enseignement des différentes interprétations des fractions, à savoir: sur l'approche de l'équivalence et le classement des fractions dans l'interprétation du quotient; sur les fractions de marquage sur la droite numérique; sur articuler les interprétations des fractions. Par conséquent, la formation continue des enseignants devrait être régulièrement encouragée pour les enseignants du primaire afin d'assurer une plus grande convergence entre les programmes et les pratiques pédagogiques, améliorant ainsi la qualité de ces derniers.

1. Introduction

The concept of fraction is considered fundamental for a successful and proper development of children's mathematical thought. It is also assumed as a rich basis for intellectual development and as a powerful tool to understand and deal with problems within real world's daily life (Behr, Lesh, Post & Silver, 1983). Nevertheless, it is also known as a complex concept to teach and likewise difficult to learn (Behr *et al.*, 1983; Cardoso, 2016; Mamede & Nunes, 2008; Nunes & Bryant, 2007). Its high complexity and comprehensiveness lie in its different interpretations, i.e., in the set of situations or interpretations that make the concept useful and meaningful — quotient, part-whole, measure, ratio and operator (Behr *et al.*, 1983, Nunes & Bryant, 2007).

In Portugal, the most recent curricular guidelines anticipate a more in-depth approach to the concept of fraction in the primary school levels (6-10-years-old). According to such guidelines, the introduction to the concept of fraction in these levels should be made using several interpretations of fractions: measure, quotient, part-whole and operator (see MEC-DGE 2012a, 2012b, 2013). Additionally, these documents propose the learning of operations with non-negative rational numbers on the 3rd and 4th grades. Such a curriculum implies significant changes if one takes into account that, previously (ME-DEB 2004), only the operator interpretation was regarded and, having this prior curriculum been used since the early 90s, it naturally underwent a deep rooting. Thus, several teachers might be barely acquainted with a comprehensive teaching of fractions, as desirable and as demanded by the current guidelines.

Therefore, and since the traditional approach to fractions relies on part-whole and operator interpretations, it seems pertinent to investigate whether the current teaching practices reflect the current guidelines. Are the teachers comfortable and fully prepared to teach fractions? Within this scope, and particularly regarding the Portuguese reality, research has scarcely been developed, especially regarding teachers' classroom practices.

1.1 The interpretations of fractions

To master a complete concept of fraction implies to know how to represent and operate with all interpretations fort fractions. Several authors have distinguished interpretations that might offer a full and fruitful understanding of the concept of fraction (see Behr *et al.*, 1983; Kieren, 1976, 1993; Mack, 2001; Nunes *et al.*, 2004). Given their inclusion in the most recent Portuguese curricular guidelines for primary school, quotient, part-whole, measure and operator interpretations were selected for approach in the present study. Within this paper, and regarding the quotient interpretation, the denominator designates the number of recipients and the numerator designates the number of items being shared. In this situation, a fraction may indicate the relation between the number of items to share and the number of recipients but also the amount of an item that each recipient gets. In part-whole interpretation, the denominator designates the number of parts into which a whole has been cut and the numerator designates the number of parts taken. In measure interpretation, the fraction $1/b$ ($b \neq 0$) is used repeatedly to determine a distance; it is often accompanied by a number line or an image of a measuring instrument, allowing students to measure the distance from one point to another in terms of $1/b$ unities. Finally, in an operator interpretation, the denominator designates the number of equal groups into which a set of discrete quantities was divided and the numerator designates the number of groups taken.

1.2 Teacher's knowledge on rational numbers

Studies focused on teachers' knowledge of rational numbers suggest that teachers have difficulties with the concept of fraction. As part of the Rational Number Project (RNP), Post *et al.* (1991) conducted a study involving 218 teachers (grades 4-6), that intended to draw a profile regarding their knowledge of rational numbers. The authors identified several difficulties, namely with the interpretations of fractions and with the ordering and equivalence between fractions. Post *et al.* (1991) emphasised that teachers have difficulties in presenting pedagogical explanations for computations with rational numbers performed by themselves.

Tirosh et al. (1998), as researchers from the Conceptual Adjustments in Progress to Non-Negative Rational Numbers (CAPWN) project, carried out a diagnostic questionnaire to 147 prospective primary teachers in order to examine formal, algorithmic and intuitive understanding of rational numbers. Prospective teachers' mathematical knowledge was found to be rigid and segmented. For most of them, Mathematics was a mere collection of computational techniques not well mastered, unjustified formally, indeed often even intuitively. Their results also showed that the prospective teachers tended to over generalise their knowledge of whole numbers when working in the domain of rational numbers.

In Portugal, the results obtained by Pinto and Ribeiro (2013) by carrying out a questionnaire on 27 prospective teachers for Primary School (grades 1-4) suggest that these ones possess a limited knowledge of rational numbers. The results particularly suggest difficulties with: the quotient, part-whole and operator interpretations of fractions; the understanding of the role of the reference unit; the order and equivalence between fractions; and the density of rational numbers.

Mamede and Pinto (2015) carried out a questionnaire on 86 pre-service teacher training for Primary School (grades 1-4) to know their ideas about fractions. The results indicate difficulties of prospective teachers with the understanding of the reference unit; weak domain of the interpretations of fractions, mainly in the scope of problems involving the quotient interpretation and in the scope of problems involving the representation of rational numbers on the number line when numbers different than one are used as reference and when it is necessary a redefinition of the scale; weak domain of the property of density of rational numbers; and difficulties with the ordering and equivalence between fractions.

Specifically concerning the Portuguese teaching practices on fractions, little is known. Aware of the recent Portuguese mathematics curriculum, that preconize an in-depth contact with fractions, the research presented through the present paper was conducted with Portuguese primary school teachers and focused on their teaching practices. It addresses the following questions: 1) How does the teacher make sense of fractions in his class? 2) Does the teacher properly promote the connections between fractions and everyday situations? 3) How does the teacher articulate distinct interpretations of fractions in the classroom?

This paper deals with, and expands, a case-study that is part of a larger study and pioneer research on Portuguese primary school teachers' practices on fractions.

2. Methodology

This study used qualitative methods since it is intended to have a description and interpretation of an educational phenomena in their natural environment (see Bogdan & Biklen, 2001; Merriam, 1998). A multiple case studies design was used, according to Yin (2010) such option is particularly appropriate, both to answer questions of the type 'how?' and 'why?', and to seek for a deep thorough understanding of the phenomena.

2.1. Participants

Four primary school teachers of the district of Braga, in Portugal, participated in this study. The present paper presents the results concerning only one of the cases — teacher João (fictitious name), with nine years of teaching practice. His 3rd grade class had 17 students (aged 8 and 9 years). By the time this data collection, the students had not had any formal contact with fractions before the lessons presented here.

2.2. Design

While introducing the concept of fraction to his students, João was involved in a collaborative working program with a researcher - one of the authors of this paper. This program was organized into cycles of activities, each consisting in the following sequence: working meeting, with all the participants, for reflection on the observed lessons and preparation of the next ones; observation of the lessons of each participant by the researcher only; individual interview on the observed lesson occurring immediately after each lesson to assess teacher's critical view of his/her practices (Figure 1).

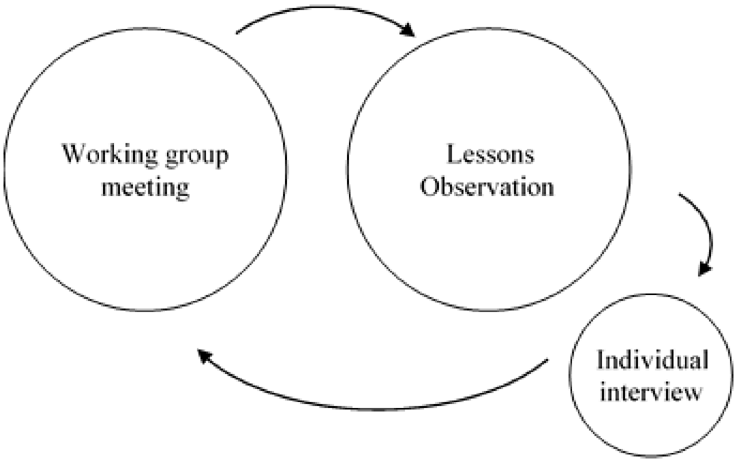

Figure 1. Standard cycle of the collaborative work program.

Five cycles of the collaborative program were carried out. Each cycle comprised one or two observed lessons. Each working meeting comprised: a) discussion on different interpretations of fractions referred in the official guidelines; b) discussion on teachers' suggestions for introduction of the concept of fraction in the classroom; and c) presentation of suggestions of the researcher on the topic. The selection and implementation of tasks in the classroom was teacher's responsibility. Tasks presented at the working meetings focused on the interpretations of fractions (quotient, part-whole, measure and operator) and on representation, equivalence and ordering of fractions in these interpretations.

The collaborative work aimed to help teachers to improve their practices in a reflective way, and in agreement with Saraiva and Ponte (2003), it can help them to accomplish the desire to innovate and do better. The researcher and teachers acted as pairs, discussing mathematical and didactical doubts according to the rhythm, needs and teachers' interests when teaching in the natural context of the school.

2.3. Data collection and data analysis

Data collection comprised digital audio records, photos and field notes taken by the researcher, one of the authors of this paper. Photos were also taken but only during the lesson observation. A large and varied set of data was collected in order to guarantee validity. During the lessons, the researcher was a non-participant observer, acting as an observer only. The lessons were observed in locus only by the researcher (one of the authors of this paper). The researcher did not intervene in any lesson development.

Data analysis was based on the model about knowledge base for teaching presented by Ball, Thames, and Phelps (2008). Thus, in order to interpret the data, a categorisation of the analysed aspects was made, according to the different parameters of the above-mentioned model: aspects of content knowledge and aspects of pedagogical content knowledge for teachers regarding the concept of fraction teaching.

3. Results

Concerning the observed lessons, the results suggest some difficulties of teacher João when introducing the concept of fraction – some of them are summarized below. The results presented here concern seven consecutive observed lessons on fractions.

From now on, in the transcriptions of classroom dialogues presented in this section of results, the letter S represents the intervention of a student — numbered according to the order in which different students appear in each dialogue, T represents the intervention of the teacher, and Sv represents the simultaneous intervention of several students.

Ordering and equivalent of fractions in quotient interpretation

To approach the equivalence of fractions in quotient interpretation, the teacher presented, for example, a task on the fair share of 3 cheeses between 6 friends. Despite one of the students’ immediate answer of “$\frac{1}{2}$ of cheese for each friend”, the teacher induced the students to answer “$\frac{3}{6}$”. The following transcription illustrates this situation:

S1 — [Answers to a task about sharing 3 cheeses between 6 boys] It’s two boys for each cheese.
Teacher — Your colleague has already seen there a relationship of a cheese for two boys. But I just want you to present the fraction. Follow the logic of what we did before…
S1 — [Answers $\frac{3}{6}$] (Figure 1)
Teacher — It is three cheeses for six boys. Is that right?
Sv — Yes!

In reaction to some students’ insistence on answering “$\frac{1}{2}$”, the teacher made sequences of the values of the fraction (numerator and denominator) to produce equivalent fractions of “$\frac{1}{2}$”, in order to show that “$\frac{1}{2}$” and “$\frac{3}{6}$” are equivalent fractions. The following transcription and figure illustrate this situation:

Teacher — Does anyone have something else to say?
S1 — Three sixths is half.
Teacher — Why?
S1 — Because three is half of six.
Teacher — Very well! Write another fraction that represents half.
S1 — [At the request of the teacher, the student writes on the white-board: $\frac{1}{2}=\frac{2}{4}=\frac{3}{6}=\frac{4}{8}=\frac{5}{10}=\frac{6}{12}=\frac{7}{14}$ (Figure 1)]
[…]
Teacher — Above [pointing to the numerators] it goes one by one and below [pointing to the denominators] ...?
Sv — It goes two by two.

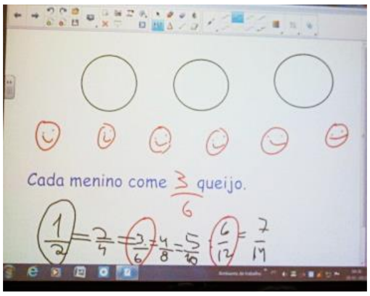

Figure 1. Answering a task about the fair share of 3 cheeses between 6 friends

Therefore, within the quotient interpretation, the teacher promoted a mechanized learning by reducing an eventual approach to the equivalence of fractions to the production of sequences of natural numbers. In other words, the teacher draws students’ attention to relationships of addition between the numerators and between the denominators. Such relationships ignore a fundamental trait of the concept of fraction that is its representation as a part of a whole.

In another moment of João´s classes, a student that easily used relationships of addition to produce equivalent fractions could not answer correctly to a question like $\frac{1}{5}=\frac{3}{?}$. Thus, this student did not fully understand the idea of equivalent fractions. The teacher should have explored the quotient interpretation in-depth to promote students understanding of the idea of equivalent fraction, as the quotient interpretation is intrinsically connected to the proportional reasoning.

Within tasks about the ordering of fractions in quotient interpretation, the teacher induced the students to divide the items, consequently reducing that interpretation to the part-whole one, and leaving it undesirably unexplored. The following transcription illustrates such a classroom situation, specifically involving the ordering of $\frac{2}{6}$ and $\frac{3}{6}$.

Teacher — Now, look over here [pointing to $\frac{3}{6}$] and then over there [pointing to $\frac{2}{6}$]: which children eat the most?
S1 — [Silence]
Teacher — How many pieces do you eat in this one [pointing to $\frac{3}{6}$]?
Sv — Three.
Teacher — And, how many pieces do you eat in this one [pointing to $\frac{2}{6}$]?
Sv — Two.
Teacher — So, where do you eat more?
Sv — In the first.
Teacher — It's in the first. If we divide something into six, here we eat three parts [pointing to $\frac{3}{6}$] and here we eat two [pointing to $\frac{2}{6}$]. Where do you eat more?
Sv — In the first [pointing to $\frac{3}{6}$].
Teacher — Look at this example [the teacher writes $\frac{11}{20}$ and $\frac{8}{20}$]. In the first case I have 11 parts of 20 and in the second I have 8 parts of 20. So I eat more in the first one.

Again, the teacher seemed unaware of the fact that the quotient interpretation promotes the understanding of the ordering and equivalence of fractions, as it calls for the use of correspondences between portions and recipients. Indeed, children are quite good at making correspondences to produce equal shares — thus thinking about a direct relation between the quantities. Such kind of reasoning is easier for the students than thinking about an inverse relation between the quantities involved in the problem – typical reasoning of the part-whole interpretation.

Marking fractions on the number line

In order to represent fractions on a number line, the teacher proposed to the students the use of correspondent decimal numbers. Figures 2 and 3 illustrate this type of approach.

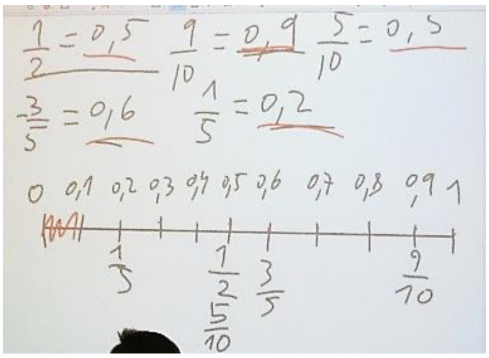

Figure 2. Marking $\frac{1}{10}, \frac{1}{5}, \frac{1}{2}, \frac{5}{10}$ and $\frac{3}{5}$ on the number line

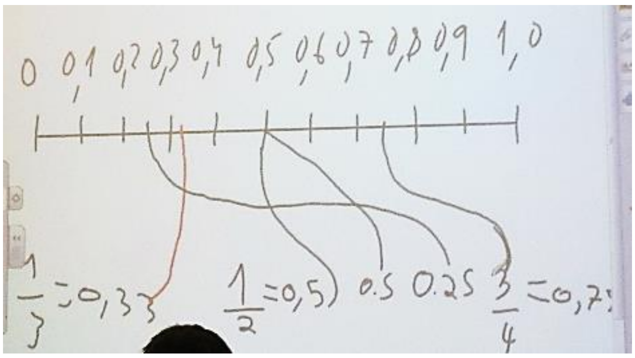

Figure 3. Marking $\frac{1}{3}, \frac{1}{2}, \frac{5}{10}$ and $\frac{3}{4}$ on the number line

Hence, the results also suggest weaknesses concerning the measure interpretation: within a task that most properly would have benefitted from the direct use of that interpretation, the teacher firstly converted the fractions involved in the task to decimals, representing the latter on the number line. Such a tendency of approach prevents the important and useful capability of conceptualising a fraction as a point on a line. By the end of the collaborative program, the teacher was already marking fractions on the number line, using the fraction $\frac{1}{b}$ $(b\neq0)$, repeatedly, to determine a distance to the origin equal to $a\times\frac{1}{b}$.

Articulation of the interpretations of fractions

It is also important to provide the students opportunities to make connections between the several forms of representation of fractions. However, this seemed to be scarcely promoted in the observed lessons. Generally, the tasks tended to be implemented in a segmented way, i.e., when an interpretation of fraction was approached, only tasks on that interpretation were selected. João began by working in quotient interpretation, then moved to part-whole and measure interpretations, and finally to the operator interpretation. The articulation of these interpretations of fractions would have promoted a consolidation and integration of knowledge. Indeed, students needed explicit help on learning to perform these articulations.

4. Discussion and conclusions

The results of the observation of teacher João’s classes suggest different fragilities in both mathematical and didactical knowledge: concretely, in the domain of different fractional interpretations and in the knowledge of didactic strategies that make those same interpretations meaningful to students.

Within the quotient interpretation, the teacher made sequences of the values of the fraction (numerator and denominator) to produce equivalent fractions, consequently promoting a mechanized learning — according to Hiebert and Lefevre (1986), procedural knowledge can be developed from either meaningful learning or mechanized learning, while it is impossible to directly generate conceptual knowledge from mechanized learning. Instead, the teacher could have applied the direct proportional reasoning that naturally emerges in a quotient situation (e.g., twice chocolate bars and twice children means that each child still gets the same) (Nunes & Bryant, 2007, 2011; Streefland, 1991).

Concerning the ordering of fractions in the quotient interpretation, it was observed a reduction of this interpretation to the part-whole one, leaving it undesirably unexplored. According to Nunes and Bryant (2007), the quotient interpretation foments the understanding of the ordering and equivalence of fractions, as it calls for the use of correspondences as the scheme of action: children establish correspondences between portions and recipients. Indeed, children are quite good at establishing correspondences to produce equal shares — thus thinking about a direct relation between the quantities — whereas they experience much difficulty in partitioning continuous quantities — which leads to thinking about an inverse relation between the quantities involved in the problem (Nunes & Bryant, 2007, 2011). The former type of reasoning arises in the quotient interpretation and the latter in the part-whole one (Nunes & Bryant, 2007, 2011; Streefland, 1991). Teacher João seemed to ignore all these questions, given his little exploration of the quotient interpretation in the classroom.

These teaching fragilities regarding the quotient interpretation might be particularly noteworthy: the quotient situation is known as the most appropriate for the appliance of children’s informal knowledge about fractions (Mamede, 2018; Mamede & Nunes, 2008).

The results also suggest weaknesses concerning the measure interpretation (representation of fractions on the number line): within a task that most properly would have benefitted from its use, the teacher converted instead the fractions involved in the task to decimals, representing the latter on the number line. Such a tendency of approach prevents the important and useful capability of conceptualising a fraction as a point on a line.

Finally, the approach to the subjects was, in general, too segmented: the teacher rarely interpolated tasks involving different interpretations. This suggests that either the teacher did not recognise the importance of articulating interpretations when building on the concept of fractions, or the teacher felt uncomfortable on doing this articulation, or perhaps, both. The articulation of different interpretations of fractions would have promoted a consolidation and integration of knowledge, and would have revealed stronger knowledge of the teacher on the domain of pedagogical content knowledge regarding the teaching of fractions.

The above-mentioned teaching fragilities on interpretations of fractions — either by inappropriate approach or absence of approach — naturally prevent students' comprehensive and valuable knowledge on fractions. Such limitation is felt both on understanding each of the interpretations and on understanding their interrelationship. Eventually, it does not promote children's: a) mathematical thought (particularly regarding the development of number sense); b) development of mental structures that foster intellectual growth; c) knowledge to widely connect fractions, whenever useful, to everyday situations, thus preventing their ability to manage situations in the real world.

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