

SYNCHRONIZATION OF COUPLED MAP LATTICES

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ABSTRACT. In this paper we address the issue of synchronization of coupled map lattices, introducing concepts of local and global synchronization for a class of systems that extends the coupled map lattices. A criterion for local synchronization is given; numerical experiments are exhibited to illustrate the criteria and also to raise some questions in the end of the text.

Keywords: *coupled map lattice, synchronization, stochastic kernel*

2000 Mathematics Subject Classification: 37C99 (37D99 37L60 82C20)

Original publication:

Proceedings of the Edinburgh Mathematical Society, **66** (1) (2023), 143–163

<https://doi.org/10.1017/S0013091523000081>

1. INTRODUCTION

Coupled map lattices were introduced many years ago as a model of identical (non linear) dynamical systems interacting by means of some coupling (see for example [3, 12, 10] and references therein). The many copies of the same dynamical system $f: X \rightarrow X$ are distributed over the points of some lattice Σ (that can be finite or infinite) and an interaction is used to couple the maps on different points. Among some phenomena that can be studied in this context we have the propagation of some signals through the lattice, solitons, certain kinds of phase transitions when invariant measures are considered (see [11]), the entropy of the map and its dependence on the coupling (see [4]) and the so called synchronization where the orbits of distinct points of the lattice have the same asymptotic behavior (see for example [9]; for an approach using the ideas of extreme value theory, see [7]).

The purpose of this work is to find some general conditions ensuring synchronization when the usual linear coupling is replaced by a general Markov kernel, extending part of what appears in [9] to less symmetrical situations and also to understand what may happen when those conditions are relaxed. The dynamics on the base is assumed to be C^1 -smooth, allowing also the use of the concept of Lyapunov exponent. Some numerical experiments suggest that part of the results about synchronization can remain valid for a Lipschitz map, but the infinitesimal argument used in the C^1 -smooth context can not be reproduced in this more general case.

We also relax some conditions on the lattice Σ , replacing this by a more general set that is assumed to be a Polish space in what we call a coupled map system; a similar idea appeared in [13], where the lattice is replaced by a Cantor set, and a possible time discretization of the Amari neural field equation (see [1], [2]) used in biomathematics leads to a case where the lattice is replaced by the real line. It

is also interesting to notice that some classical analytical objects, like the Hardy-Littlewood operator acting on real functions, can be seen as limits of the action of a certain coupled map system where the lattice is replaced, again, by the real line. In these new situations it is also possible to adapt some of the arguments used to get synchronization.

The paper is organized as follows: in section 2 we fix the notation and the main definitions. In section 3 we have the results about synchronization of C^1 -smooth transformations, for coupled map lattices and coupled map systems. In section 4 we present the proofs. In section 5 we introduce several classes of coupled map lattices and systems to which our results apply. In section 6 we provide some counter-examples which illustrate the need for the hypothesis of the theorems as well as the limitations in their conclusions. Some problems and future directions, part of them based on numerical experiments, are discussed in section 7.

2. DEFINITIONS

A topological space Σ is called a *Polish space* if it is separable and there exists a metric d on Σ which defines the topology of Σ such that (Σ, d) is complete. Every Polish space is a Radon space. In particular every Borel measure on Σ is regular (see [15]). From now on Σ will denote a Polish space and \mathcal{F} will be the Borel σ -algebra on Σ .

A *stochastic kernel* on Σ is any function $K: \Sigma \times \mathcal{F} \rightarrow [0, 1]$ such that

- (1) the function $B \mapsto K(x, B)$, from \mathcal{F} to $[0, 1]$, is a probability measure for any $x \in \Sigma$;
- (2) the function $x \mapsto K(x, B)$, from Σ to $[0, 1]$, is \mathcal{F} -measurable for any $B \in \mathcal{F}$.

The convolution of two kernels K_1 and K_2 on Σ is

$$(K_2 * K_1)(x, B) = \int_{\Sigma} K_1(x, dy) K_2(y, B), \quad x \in \Sigma, B \in \mathcal{F}.$$

The space of kernels on Σ is a convolution semigroup with identity, where the identity is the kernel $K(x, \cdot) = \delta_x$ and δ_x stands for the Dirac measure supported on x . The iterated kernels are defined recursively, setting $K^1 = K$ and for $n \geq 1$,

$$K^{n+1}(x, B) = \int_{\Sigma} K(x, dy) K^n(y, B).$$

A probability measure μ on (Σ, \mathcal{F}) is called *K-stationary* if for all $B \in \mathcal{F}$,

$$\mu(B) = \int K(x, B) \mu(dx).$$

A set $B \in \mathcal{F}$ is said to be *K-invariant* when $K(x, B) = 1$ for all $x \in B$ and $K(x, B) = 0$ for all $x \in X \setminus B$. A *K-stationary* measure μ is called *ergodic* when there is no *K-invariant* set $B \in \mathcal{F}$ such that $0 < \mu(B) < 1$. As usual, ergodic measures are the extremal points in the convex set of *K-stationary* measures.

We say that a stochastic kernel K is *strongly mixing* if it admits a unique stationary probability measure μ with $\text{supp}(\mu) = \Sigma$ and there are constants $C > 0$ and $0 < \rho < 1$ such that for every $\psi \in L^\infty(\Sigma)$, all $x \in \Sigma$ and $n \in \mathbb{N}$,

$$\left| \int_{\Sigma} \psi(y) K^n(x, dy) - \int_{\Sigma} \psi(y) \mu(dy) \right| \leq C \rho^n \|\psi\|_{\infty}.$$

We say that K satisfies the *Doebelin condition* if there is a positive finite measure ρ on (Σ, \mathcal{F}) and some $\varepsilon > 0$ such that for all $x \in \Sigma$ and $B \in \mathcal{F}$,

$$K(x, B) \geq 1 - \varepsilon \quad \Rightarrow \quad \rho(B) \geq \varepsilon.$$

Theorem 2.1. *Let K be a stochastic kernel on (Σ, \mathcal{F}) . If K satisfies the Doebelin condition then there are sets $\Sigma_1, \dots, \Sigma_m$ in \mathcal{F} and probability measures ν_1, \dots, ν_m on Σ such that for all $i, j = 1, \dots, m$,*

- (1) $\Sigma_i \cap \Sigma_j = \emptyset$ when $i \neq j$,
- (2) Σ_i is K -forward invariant, i.e., $K(x, \Sigma_i) = 1$ for $x \in \Sigma_i$,
- (3) ν_i is K -stationary and ergodic with $\nu_i(\Sigma_j) = \delta_{ij}$,
- (4) $\lim_{n \rightarrow +\infty} K^n(x, \Sigma_1 \cup \dots \cup \Sigma_m) = 1$, with geometric uniform speed of convergence, for all $x \in \Sigma$,
- (5) $\nu(\Sigma_1 \cup \dots \cup \Sigma_m) = 1$, for every K -stationary probability ν .

Moreover, for every $1 \leq i \leq m$ there is an integer $p_i \in \mathbb{N}$ and measurable sets $\Sigma_{i,1}, \dots, \Sigma_{i,p_i} \in \mathcal{F}$ such that

- (1) $\{\Sigma_{i,1}, \dots, \Sigma_{i,p_i}\}$ is a partition of Σ_i ,
- (2) $K(x, \Sigma_{i,j+1}) = 1$ for $x \in \Sigma_{i,j}$ and $1 \leq j \leq p_i$, with $\Sigma_{i,p_i+1} = \Sigma_{i,1}$,
- (3) the stochastic kernel K^{p_i} on $\Sigma_{i,j}$ is strongly mixing for all $1 \leq j \leq p_i$.

Proof. See [5, section V-5]. □

In the context of Doebelin condition, we call the sets Σ_i , $1 \leq i \leq m$, the *ergodic components* of K and the integers p_i their *periods*. We call the sets $\Sigma_{i,j}$, $1 \leq j \leq p_i$, the *mixing sub-components* of the ergodic component Σ_i . We shall say that an ergodic component is *aperiodic* when its period is equal to 1.

A stochastic kernel K determines an operator $K : L^\infty(\Sigma) \rightarrow L^\infty(\Sigma)$ defined by

$$(1) \quad (K\psi)(x) := \int_{\Sigma} \psi(y) K(x, dy).$$

This operator satisfies the following.

Proposition 2.2. *For any $\psi \in L^\infty(\Sigma)$,*

- (a) $K\mathbb{1} = \mathbb{1}$, where $\mathbb{1}$ denotes the constant function 1.
- (b) $\int_{\Sigma} K\psi d\mu = \int_{\Sigma} \psi d\mu$,
- (c) $\|K\psi\|_{\infty} \leq \|\psi\|_{\infty}$.

In particular if $H_0 = \{\psi \in L^\infty(\Sigma) : \int_{\Sigma} \psi d\mu = 0\}$ then $L^\infty(\Sigma) = \mathbb{R}\mathbb{1} \oplus H_0$ is a K -invariant decomposition and $\rho(K|_{H_0}) \leq 1$.

Proof. Since $\int_{\Sigma} K(x, dy) = 1$, items (a) and (c) follow. Item (b) is a consequence of μ being a K -stationary probability measure. □

If the kernel K on Σ is strongly mixing we define its *mixing rate* to be the spectral radius of the restriction of K to the invariant subspace H_0 in Proposition 2.2, i.e.,

$$\tau^*(K) := \rho(K|_{H_0}).$$

In general, if K satisfies the Doebelin condition, using the notation of Theorem 2.1, we define its *escape rate* $\beta^*(K)$ to be the spectral radius of the operator $r_{\Lambda} \circ K|_{L^\infty(\Lambda)}$, where $\Lambda = \Sigma \setminus (\Sigma_1 \cup \dots \cup \Sigma_m)$ and $r_{\Lambda} : L^\infty(\Sigma) \rightarrow L^\infty(\Lambda)$ is the restriction operator. Notice that the operator $K_{\Lambda} : L^\infty(\Lambda) \rightarrow L^\infty(\Lambda)$, $K_{\Lambda} = r_{\Lambda} \circ K|_{L^\infty(\Lambda)}$ is sub-stochastic and hence has spectral radius less than 1. We also define the *mixing rate*

of the component Σ_i as $\tau^*(K, \Sigma_i) := \sqrt[p_i]{\tau^*(K^{p_i}, \Sigma_{i,j})}$, where $\tau^*(K^{p_i}, \Sigma_{i,j})$ denotes the mixing rate of the strongly mixing stochastic kernel obtained by restriction of the kernel K^{p_i} to any of the mixing sub-components $\Sigma_{i,j}$, $1 \leq j \leq p_i$.

Let Σ be a discrete space, that we call a *lattice*. Consider a convex and compact set $X \subset \mathbb{R}^d$, a map $f: X \rightarrow X$, the base dynamical system, and a probability kernel $K: \Sigma \times \Sigma \rightarrow [0, 1]$. If Σ is finite with k elements then the stochastic kernel K can be identified with a $k \times k$ stochastic matrix. We recall that a (*row*) *stochastic matrix* on a finite set Σ is any square matrix $A = [a_{ij}] \in \mathbb{R}^{k \times k}$ such that $a_{ij} \geq 0$ for all $i, j \in \Sigma$ and $\sum_{j \in \Sigma} a_{ij} = 1$ for all $i \in \Sigma$. A stochastic matrix A is called *primitive* if there exists $n \geq 1$ for which the power matrix $A^n = [a_{ij}^n]$ has all entries strictly positive, i.e. $a_{ij}^n > 0$ for $i, j \in \Sigma$. Notice that if A is primitive then the kernel K on Σ determined by A is strongly mixing. In particular, Σ is an aperiodic ergodic component. We define the *mixing rate of a primitive matrix* A , denoted by $\tau^*(A)$, as the mixing rate of the strongly mixing kernel defined by A on Σ .

The *coupled map lattice* is the dynamical system $F: X^\Sigma \rightarrow X^\Sigma$ defined by

$$F(\varphi) := K(f \circ \varphi),$$

where X^Σ denotes the space of all functions from Σ to X . Since K is a probability kernel we get that for the constant function $\varphi = c$, the image $K(f \circ \varphi) = Kf(c) = f(c)$. Hence, the set of constant functions is invariant under the dynamics defined by F .

More generally given a Polish space Σ , consider the space $L^\infty(\Sigma, X)$ of all measurable functions $\varphi: \Sigma \rightarrow X$. Notice that since X is compact all functions in $L^\infty(\Sigma, X)$ are bounded. We endow this space with the uniform convergence topology, which is determined by the uniform distance

$$d_\infty(\varphi, \psi) := \|\varphi - \psi\|_\infty.$$

As before we can define a transformation $F: L^\infty(\Sigma, X) \rightarrow L^\infty(\Sigma, X)$ by

$$F(\varphi) := K(f \circ \varphi)$$

that we will refer to as a *coupled map system*.

We say that the coupled map system $F: L^\infty(\Sigma, X) \rightarrow L^\infty(\Sigma, X)$ has *global synchronization over a subset* $\Lambda \subset \Sigma$ if for every $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, and for every $\varphi \in L^\infty(\Sigma, X)$ and all $x, y \in \Lambda$, one has

$$\|F^n(\varphi)(x) - F^n(\varphi)(y)\| < \delta.$$

When $\Lambda = \Sigma$ we simply talk about global synchronization of F . Let $\Delta \subset L^\infty(\Sigma, X)$ denote the compact and convex subset of all constant functions. Notice that F has global synchronization if and only if Δ is a global attractor of F on $L^\infty(\Sigma, X)$.

We say that F has *local synchronization over a subset* $\Lambda \subset \Sigma$ if there exists a neighborhood \mathcal{U} of Δ in $L^\infty(\Sigma, X)$ such that for every $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, and for every $\varphi \in \mathcal{U}$ and all $x, y \in \Lambda$, one has

$$\|F^n(\varphi)(x) - F^n(\varphi)(y)\| < \delta.$$

The map F has local synchronization (over Σ) if and only if Δ is a local attractor of F on $L^\infty(\Sigma, X)$.

Clearly, global synchronization implies local synchronization, but the converse is not true.

3. MAIN RESULTS

Throughout this section we assume that $X \subset \mathbb{R}^d$ is a compact convex set and $f : X \rightarrow X$ is a C^1 -smooth map whose Lipschitz constant we denote by $\text{Lip}(f)$.

We define

$$\ell(f) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \text{Lip}(f^n).$$

This limit exists by Fekete's Lemma. As usual, the derivative of the map f at a point $x \in X$ is denoted by $Df_x : \mathbb{R}^d \rightarrow \mathbb{R}^d$. The *top Lyapunov exponent* of a periodic point $x = f^n(x)$ with period n is defined to be

$$\lambda(f, x) := \frac{1}{n} \log \|Df_x^n\|.$$

By the mean value theorem $\|Df_x\| \leq \text{Lip}(f)$. Hence, for all periodic points $x \in X$ of f one has

$$\lambda(f, x) \leq \ell(f).$$

We define the *oscillation semi-norm* $\|\cdot\|_o$ of a function $\varphi \in L^\infty(\Sigma, \mathbb{R}^d)$ by

$$\|\varphi\|_o := \sup_{x, y \in \Sigma} \|\varphi(x) - \varphi(y)\|.$$

Proposition 3.1. *The oscillation semi-norm satisfies*

- (1) $\|g \circ \varphi\|_o \leq \text{Lip}(g) \|\varphi\|_o$ for any Lipschitz function $g : \mathbb{R}^d \rightarrow \mathbb{R}^d$, and
- (2) $\|\varphi\|_o = 0 \Leftrightarrow \varphi$ is constant.

Proof. Given any Lipschitz function $g : \mathbb{R}^d \rightarrow \mathbb{R}^d$,

$$\begin{aligned} \|g \circ \varphi\|_o &= \sup_{x, y \in \Sigma} \|g(\varphi(x)) - g(\varphi(y))\| \\ &\leq \sup_{x, y \in \Sigma} \text{Lip}(g) \|\varphi(x) - \varphi(y)\| \\ &\leq \text{Lip}(g) \|\varphi\|_o, \end{aligned}$$

and, clearly, $\|\varphi\|_o = 0$ if and only if φ is constant. \square

We define the norm of a stochastic kernel K , regarded as an operator, by

$$\|K\|_o := \sup_{\|\varphi\|_o \neq 0} \frac{\|K\varphi\|_o}{\|\varphi\|_o}.$$

Theorem 3.2. *If $\|K\|_o \text{Lip}(f) < 1$, then F has global synchronization.*

Proof. Given $\varphi \in L^\infty(\Sigma, X)$,

$$\begin{aligned} \|F(\varphi)\|_o &= \|K(f \circ \varphi)\|_o \leq \|K\|_o \|f \circ \varphi\|_o \\ &\leq \|K\|_o \text{Lip}(f) \|\varphi\|_o. \end{aligned}$$

Hence

$$\|F^n(\varphi)\|_o \leq [\|K\|_o \text{Lip}(f)]^n \|\varphi\|_o$$

converges to 0 as $n \rightarrow +\infty$. This proves that F has global synchronization. \square

Theorem 3.3. *Given a finite set Σ and a stochastic primitive matrix A on Σ , if there exists a periodic orbit $x \in X$ with Lyapunov exponent $\lambda = \lambda(f, x)$ such that*

$$\tau^*(A) e^\lambda > 1$$

then F does not have local synchronization over Σ .

Theorem 3.4. *Given a Polish space Σ and a stochastic kernel K on Σ satisfying the Doeblin condition with a unique aperiodic ergodic component Σ_0 , if*

$$\max\{\beta^*(K), \tau^*(K, \Sigma_0)\} e^{\ell(f)} < 1$$

then F has local synchronization over Σ .

The next theorem provides a sufficient condition for local synchronization over the mixing sub-components of an ergodic component.

Theorem 3.5. *Given a Polish space Σ and a stochastic kernel K on Σ satisfying the Doeblin condition with an ergodic component*

$$\Sigma_i = \Sigma_{i,1} \cup \dots \cup \Sigma_{i,p},$$

where the sets $\Sigma_{i,j}$ stand for the mixing sub-components of Σ_i , if

$$\tau^*(K, \Sigma_i) e^{\ell(f)} < 1$$

then F has local synchronization over all mixing sub-components $\Sigma_{i,j}$ of Σ_i .

4. PROOFS

Proof of Theorem 3.3. Let $\Delta = \{u \in X^\Sigma : u_i = u_j \forall i, j \in \Sigma\}$. The transformation F has local synchronization over Σ iff Δ is a local attractor whose basin of attraction is a neighborhood of Δ . To see that F does not have local synchronization over Σ it is enough to find an F -periodic point $u \in \Delta$ with period p such that the derivative DF_u^p has an eigenvalue $\alpha \in \mathbb{C}$ with $|\alpha| > 1$ associated to some (complex) eigenvector which does not lie in the complexification of the tangent space $T_u \Delta$.

Consider the periodic point $x = f^p(x)$ whose existence is hypothesized in this theorem, and let $u = (x, x, \dots, x) \in X^\Sigma$. Then $u \in \Delta$ is a periodic point of F with the same period p . Let $w \in \mathbb{C}^\Sigma$ be an eigenvector of A such that $\sum_{i \in \Sigma} w_i = 0$ and $Aw = \beta w$, where $|\beta| = \tau^*(A)$. Writing $A = [a_{ij}] \in \mathbb{R}^{\Sigma \times \Sigma}$, let $\hat{A} \in (\mathbb{R}^{d \times d})^{\Sigma \times \Sigma}$ be the matrix with entries $\hat{a}_{ij} := a_{ij} I_d$ where I_d is the identity matrix in $\mathbb{R}^{d \times d}$. Let $M_j = Df_{f^j(x)}$ and denote by D_{M_j} the block diagonal matrix

$$D_{M_j} = \begin{bmatrix} M_j & 0 & \cdots & 0 \\ 0 & M_j & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_j \end{bmatrix}.$$

Finally let $v \in \mathbb{C}^d$ be a non-zero vector such that $Df_x^p v = M_{p-1} \cdots M_1 M_0 v = \alpha v$ with $\alpha \in \mathbb{C}$ and $\frac{1}{p} \log |\alpha| = \lambda(x, f)$. The Jacobian matrix of F at each point $F^j(u)$ is the matrix

$$DF_{F^j(u)} = \hat{A} D_{M_j}.$$

Since the matrices \hat{A} and D_{M_j} commute we have

$$DF_u^p = \hat{A}^p D_M$$

where $M = M_{p-1} \cdots M_1 M_0 = Df_x^p$. Now the complex vector $V = (w_i v)_{i \in \Sigma} \in (\mathbb{C}^d)^\Sigma$ satisfies

$$DF_u^p V = \hat{A}^p D_M V = \beta^p \alpha V.$$

Since the vector V does not lie in the complexification $\Delta_{\mathbb{C}} = \{(w, w, \dots, w) : w \in \mathbb{C}\}$ of the tangent space $T_u \Delta$ and

$$|\beta^p \alpha| = \tau^{*p} e^{p\lambda} = (\tau^* e^\lambda)^p > 1$$

the F -invariant set Δ can not be a local attractor. \square

Proof of Theorem 3.4. Assume first that the stochastic kernel K is strongly mixing and consider the space

$$H_0 := \{\varphi \in L^\infty(\Sigma, \mathbb{R}^d) : \int_\Sigma \varphi d\mu = 0\}$$

introduced in Proposition 2.2. Let $\tau^* = \tau^*(K) < 1$ be the spectral radius of $K|_{H_0}$.

In the end we explain how to proceed in the general case. Taking $\kappa > 0$ with $\tau^*(K) e^{\ell(f)} < \kappa < 1$ there exists an integer $m \in \mathbb{N}$ such that $\|K^m\|_0 \text{Lip}(f^m) < \kappa^m$.

Because f is C^1 -smooth then so is F . On the diagonal Δ the derivatives of the linear action of K and of f commute. Consequently for all $\varphi \in \Delta$

$$\|DF_\varphi^n\| \leq \tau^{*n} e^{n\ell} < \kappa^n$$

with $0 < \kappa < 1$. Hence there is a neighborhood \mathcal{U} of Δ where F^n is a contraction. This implies that for all $\varphi \in \mathcal{U}$,

$$\lim_{k \rightarrow +\infty} \|F^{kn} \varphi\|_0 = 0.$$

Thus for all $0 \leq j < n$,

$$\lim_{k \rightarrow +\infty} \|F^{kn+j} \varphi\|_0 = 0$$

and therefore

$$\lim_{k \rightarrow +\infty} \|F^k \varphi\|_0 = 0.$$

The case where K is not strongly mixing but there is only one ergodic acyclic component Σ_0 is the same as the previous one, because the operator $K|_{H_0}$ has spectral radius equal to $\max\{\beta^*(K), \tau^*(K, \Sigma_0)\}$. \square

Proof of Theorem 3.5. Same argument as before working in the mixing sub-component. \square

5. COUPLED MAP CLASSES

In this section we introduce several classes of coupled map systems and lattices to which our results apply. In all examples below, $X \subset \mathbb{R}^d$ is any compact convex set and $f : X \rightarrow X$ is a C^1 -smooth function.

Finite coupled map lattices. Consider a finite set $\Sigma = \{1, \dots, k\}$ and a stochastic $k \times k$ matrix $A = [a_{ij}]$. The associated coupled map lattice is the transformation $F : X^k \rightarrow X^k$ with components

$$F_i(x) = \sum_{j=1}^k a_{ij} f(x_j) \quad \text{where } x = (x_1, \dots, x_k).$$

In this setting Theorem 2.1 is a simple consequence of classical Markov Chain's Theory. Let $\Sigma_1, \dots, \Sigma_m$ be the ergodic components of the stochastic kernel K defined by A on Σ and let $\Lambda = \Sigma \setminus (\Sigma_1 \cup \dots \cup \Sigma_m)$. The escape rate $\beta^*(K)$ and the mixing rate $\tau^*(K, \Sigma_i)$ can be estimated as described below. See [5, section V-2] and also [6, section 5]¹. Recall that $\tau^*(K, \Sigma_i) := \sqrt[p_i]{\tau^*(K^{p_i}, \Sigma_{i,j})}$, where $\tau^*(K^{p_i}, \Sigma_{i,j})$ denotes the mixing rate of the strongly mixing stochastic kernel obtained by restriction of the kernel K^{p_i} to any of the sub-components $\Sigma_{i,j}$. Let $A^{p_i} = [a_{ij}^{p_i}]$. We have that

$$(2) \quad \beta^*(K) = \inf_{n \geq 1} [\beta(K^n)]^{1/n},$$

where

$$\beta(K) = 1 - \min_{\ell \in \Sigma} \sum_{k \in \Sigma_1 \cup \dots \cup \Sigma_m} a_{\ell k} = \max_{\ell \in \Sigma} \sum_{k \in \Lambda} a_{\ell k};$$

and

$$(3) \quad \tau^*(K^{p_i}, \Sigma_{i,j}) = \inf_{n \geq 1} [\tau(K^{np_i}, \Sigma_{i,j})]^{1/n},$$

where

$$\tau(K^{p_i}, \Sigma_{i,j}) = \frac{1}{2} \max_{\ell, \ell' \in \Sigma_{i,j}} \sum_{k \in \Sigma_{i,j}} |a_{\ell k}^{p_i} - a_{\ell' k}^{p_i}| = 1 - \min_{\ell, \ell' \in \Sigma_{i,j}} \sum_{k \in \Sigma_{i,j}} a_{\ell k}^{p_i} \wedge a_{\ell' k}^{p_i}.$$

This class contains, for example, coupled map lattices with translation invariant coupling and arbitrary Lipschitz continuous individual map on the periodic lattice $\mathbb{Z}_L := \{s \in \mathbb{Z} \bmod L\}$ ($L > 1$).

Infinite coupled map lattices. Assume now that Σ is an infinite (countable) lattice, say $\Sigma = \mathbb{N} = \{0, 1, \dots\}$. Consider an infinite (row) stochastic matrix $A = [a_{ij}]$ such that for some $\alpha > 0$ we have $a_{i0} \geq \alpha$ for all $i \in \mathbb{N}$. This assumptions ensures that the stochastic kernel defined by A satisfies the Doeblin condition.

The associated coupled map lattice is the transformation $F : X^{\mathbb{N}} \rightarrow X^{\mathbb{N}}$ with components

$$F_i(x) = \sum_{j \in \mathbb{N}} a_{ij} f(x_j) \quad \text{where } x = (x_j)_{j \in \mathbb{N}}.$$

The previous finite state formulas also hold in this countable case.

¹We notice that all results in section 5 of this reference are abstract and hence hold for general measure spaces (X, m) . This includes the case where X is finite.

Coupled map systems. Assume that Σ is a compact metric space, say $\Sigma = [0, 1]$, and consider a bounded measurable function $k : [0, 1] \times [0, 1] \rightarrow [0, +\infty)$ such that

$$\int_0^1 k(x, y) dy = 1 \quad \text{for all } x \in [0, 1].$$

This function determines a kernel K on $[0, 1]$ defined by

$$K(x, B) = \int_B k(x, y) dy \quad \text{for all } B \subset [0, 1].$$

In this setting the stochastic kernel K satisfies the Doeblin condition.

The associated coupled map system is the transformation $F : L^\infty([0, 1], X) \rightarrow L^\infty([0, 1], X)$

$$F(\varphi)(x) = \int_0^1 k(x, y) f(\varphi(y)) dy \quad \text{where } \varphi \in L^\infty([0, 1], X).$$

Remark 5.1. It is interesting to mention that the uncoupled map, say, the case where $F(\varphi)(x) = f(\varphi(x))$ corresponds to a dynamical system already considered in [8].

Remark 5.2. We notice that the class of coupled map systems may contain the continuum limits of some coupled map lattices as the lattice diameter goes to zero. In [16], the authors study thermodynamic limits of a very similar class of models, also called *coupled map systems*.

The previous given formulas for the escape rate and the mixing rate of the ergodic components Σ_i , $1 \leq i \leq m$, extend to stochastic kernels satisfying the Doeblin condition. See [5, section V-5] and also [6, section 5]². We have that

$$\beta^*(K) = \inf_{n \geq 1} [\beta(K^n)]^{1/n},$$

where

$$\beta(K) = 1 - \inf_{x \in \Sigma} \int_{\Sigma_1 \cup \dots \cup \Sigma_m} k(x, y) dy = \sup_{x \in \Sigma} \int_{\Lambda} k(x, y) dy;$$

and

$$\tau^*(K^{p_i}, \Sigma_{i,j}) = \inf_{n \geq 1} [\tau(K^{np_i}, \Sigma_{i,j})]^{1/n},$$

where

$$\begin{aligned} \tau(K^{p_i}, \Sigma_{i,j}) &= \frac{1}{2} \sup_{x, z \in \Sigma_{i,j}} \int_{\Sigma_{i,j}} |k^{p_i}(x, y) - k^{p_i}(z, y)| dy \\ &= 1 - \inf_{x, z \in \Sigma_{i,j}} \int_{\Sigma_{i,j}} k^{p_i}(x, y) \wedge k^{p_i}(z, y) dy. \end{aligned}$$

²All results in section 5 of this reference are abstract and apply to the current setting.

Example 5.3. The Amari neural field equation used in biomathematics is the integro-differential equation

$$\partial_t u(t, x) = -u(t, x) + \int_{\mathbb{R}} \mathcal{K}(x - y) S(u(t, y)) dy \quad t > 0, x \in \mathbb{R}$$

where u represents the average neural activity, \mathcal{K} is the connectivity kernel (modelling the interaction between neurons at distinct places) and S is the firing rate function. A first order time discretization, considering a time step of size 1, consists in replacing the partial derivative by the difference $u(t + 1, x) - u(t, x)$; we get then the map

$$u(t + 1, x) = F(u(t, x)) = \int_{\mathbb{R}} \mathcal{K}(x - y) S(u(t, y)) dy$$

that is an example of a coupled map system where Σ is the real line (for another approximation of the Amari's model the reader can see [14]).

Example 5.4. Consider an integrable function $\varphi: \mathbb{R} \rightarrow \mathbb{R}$. The Hardy-Littlewood maximal operator maps φ to the function that is defined for any $x \in \mathbb{R}$ as

$$M\varphi(x) = \sup_{r>0} F_r(\varphi)(x)$$

where, for any given $r > 0$, F_r is given by

$$F_r(\varphi)(x) := \frac{1}{|B(x, r)|} \int_{B(x, r)} |\varphi(y)| dy;$$

in this expression dy is the Lebesgue measure on \mathbb{R} , $B(x, r)$ is the open interval $(x - r, x + r)$ and $|B|$ denotes its Lebesgue measure. Notice that F_r corresponds to the coupled map system where $X = \Sigma = \mathbb{R}$, $f(\cdot) = |\cdot|$ and $k(x, y) = \frac{dy}{|B(x, r)|}$ is the normalized Lebesgue measure restricted to the ball $B(x, r)$.

6. COUNTER-EXAMPLES

In this section we provide two counter-examples which illustrate the need for the hypothesis of the theorems as well as the limitations in their conclusions.

In the first example the hypothesis of Theorem 3.4 is satisfied but the coupled lattice map has no global synchronization.

Example 6.1. Let $X = [0, 3]$ and $f: X \rightarrow X$ be a C^1 -smooth map such that $f(1) = 2/3$, $f(8/5) = 2$ and $f(2) = 16/9$. Such an example (see Figure 1) can be constructed such that

$$\text{Lip}(f|_{\{1, \frac{8}{5}, 2\}}) \leq \frac{20}{9} < \text{Lip}(f) < \frac{10}{3}.$$

Indeed, we begin by constructing a piecewise linear interpolation function $h: X \rightarrow X$ with $h(1) = 2/3$, $h(8/5) = 2$, $h(2) = 16/9$ and $\text{Lip}(h) = 20/9$. For any $0 < \delta < 1$ we can then approximate h by a C^1 -function f with the same prescribed values and such that $\text{Lip}(f) < 20/9 + \delta < 10/3$.

Consider the stochastic matrix

$$A = \begin{bmatrix} \frac{7}{10} & \frac{3}{10} & 0 \\ 0 & \frac{7}{10} & \frac{3}{10} \\ 1 & 0 & 0 \end{bmatrix}$$

and the coupled lattice map $F : [0, 3]^3 \rightarrow [0, 3]^3$ defined by

$$F(x, y, z) = \left(\frac{7}{10}f(x) + \frac{3}{10}f(y), \frac{7}{10}f(y) + \frac{3}{10}f(z), f(x) \right).$$

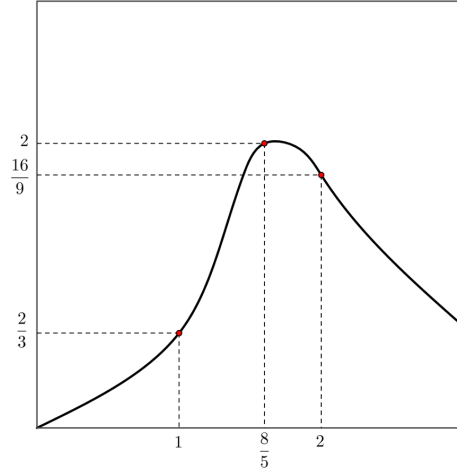


FIGURE 1. The map of example 6.1.

The point $p = (8/5, 1, 2)$ is a fixed point of F . Because this fixed point is off the diagonal $\Delta = \{(x, x, x) : x \in [0, 3]\}$ the map F can not have global synchronization.

The matrix A has eigenvalues 1 and $\frac{1}{10}(2 \pm \sqrt{5}i)$ (with absolute value $3/10$). The stochastic kernel K determined by A has a unique aperiodic ergodic component $\Sigma_0 = \Sigma$. Hence, $\tau^*(K, \Sigma) = 3/10$ and $\beta^*(K) = 0$. Consequently,

$$\max\{\beta^*(K), \tau^*(K, \Sigma)\} \text{Lip}(f) < \frac{3}{10} \frac{10}{3} = 1.$$

The second example illustrates a transition from local synchronization over Σ to local synchronization over an ergodic component.

Example 6.2. Consider the following family of stochastic matrices

$$A_t := \begin{bmatrix} \frac{1-t}{2} + \frac{4t}{9} & \frac{1-t}{2} + \frac{5t}{9} & 0 \\ \frac{1-t}{2} + \frac{t}{6} & \frac{1-t}{2} + \frac{5t}{6} & 0 \\ 0 & \frac{7(1-t)}{18} + \frac{17t}{18} & \frac{11(1-t)}{18} + \frac{t}{18} \end{bmatrix}$$

with $0 \leq t \leq 1$, whose associated family of kernels K_t has a unique aperiodic ergodic component $\Sigma_0 = \{1, 2\}$. Applying the formulas (2)-(3) we get $\beta^*(K_t) = \frac{11-10t}{18}$ and $\tau^*(K_t, \Sigma_0) = \frac{5t}{18}$.

Let $f : [0, 1] \rightarrow [0, 1]$ be the map $f(x) := 4x(1-x)$ which has Lipschitz constant $\text{Lip}(f) = 4$. We have $e^{\ell(f)} = 4$ because 0 is a fixed point with derivative $f'(0) = 4$.

In this example the threshold conditions of theorems 3.4 and 3.5 are respectively

$$\begin{aligned} (\mathcal{R}_1) \quad & \max\{\beta^*(K_t), \tau^*(K_t, \Sigma_0)\} e^{\ell(f)} < 1 \quad \Leftrightarrow \quad 0.65 < t < 0.9 \\ (\mathcal{R}_2) \quad & \tau^*(K_t, \Sigma_0) e^{\ell(f)} < 1 \quad \Leftrightarrow \quad t < 0.9 \end{aligned}$$

An analog of the threshold condition in Theorem 3.3 with Lyapunov exponent $\lambda(f) = \log 2$ w.r.t. Lebesgue measure is

$$(\mathcal{R}_3) \quad \max\{\beta^*(K_t), \tau^*(K_t, \Sigma_0)\} e^{\lambda(f)} < 1 \quad \Leftrightarrow \quad 0.2 < t.$$

See Figure 2.

Using theorems 3.4 and 3.5:

- (1) in region (\mathcal{R}_1) (local) synchronization over Σ occurs;
- (2) in region (\mathcal{R}_2) (local) synchronization over Σ_0 occurs.

For $t < 0.65 = 13/20$, an easy calculation shows that $p_t = (0, 0, \frac{20t-13}{20t-22})$ is a fixed point of the coupled map F_t off the diagonal. This shows that there is no global synchronization for F_t on this range.

We run several numerical experiments, randomly choosing an initial condition in $[0, 1]^3$, computing $n = 200$ iterates and then plotting the last 100 iterates. These experiments indicate that (see figures 3 and 4):

- (3) in region (\mathcal{R}_1) global synchronization over Σ occurs;
- (4) in region (\mathcal{R}_3) local synchronization over Σ occurs;
- (5) outside region (\mathcal{R}_3) synchronization over Σ never occurs.

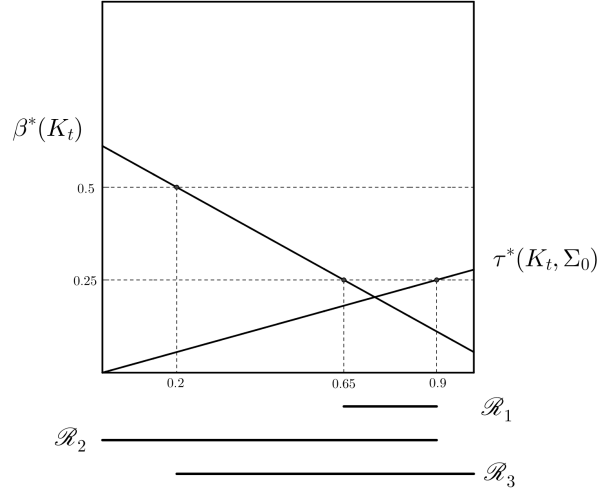


FIGURE 2. Threshold lines.

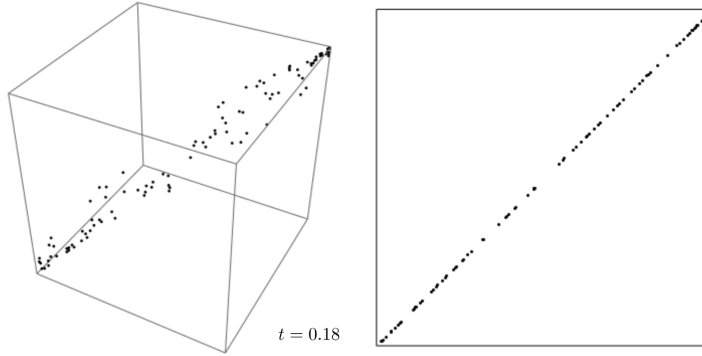


FIGURE 3. In Example 6.2 synchronization occurs, only over Σ_0 , for the values $0 \leq t < 0.2$. The right plot represents the projection of the left one onto the plane \mathbb{R}^{Σ_0} where $\Sigma_0 = \{1, 2\}$.

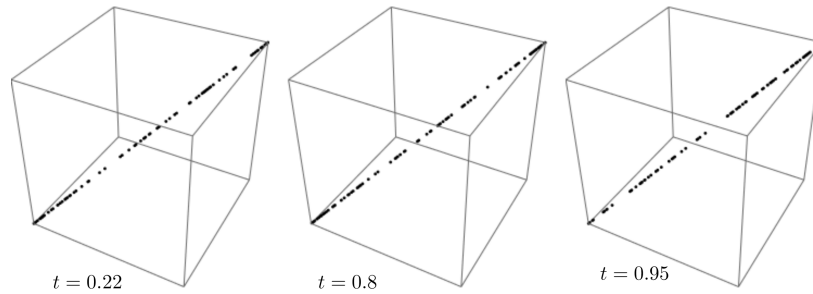


FIGURE 4. In Example 6.2 synchronization over Σ occurs for the values $0.2 < t \leq 1$.

7. CONCLUSIONS

All examples that we have analyzed indicate that theorems 3.4 and 3.5 should hold for any Lipschitz map $f : X \rightarrow X$. We note that the infinitesimal argument used in the C^1 -smooth context can not be reproduced in the Lipschitz case.

This motivates the following problem:

Problem 1. Prove theorems 3.4 and 3.5 for Lipschitz maps $f : X \rightarrow X$, or at least for piecewise smooth maps.

The two (Lipschitz) examples below provide numerical evidence that there is no local synchronization whenever the main hypothesis of theorems 3.4 and 3.5 fail.

Example 7.1. Consider the following family of stochastic matrices

$$A_t := \begin{bmatrix} \frac{1-t}{2} + \frac{4t}{5} & \frac{1-t}{2} + \frac{t}{5} & 0 \\ \frac{1-t}{2} + \frac{t}{5} & \frac{1-t}{2} + \frac{4t}{5} & 0 \\ 0 & \frac{1-t}{4} + \frac{2t}{3} & \frac{3(1-t)}{4} + \frac{t}{3} \end{bmatrix}$$

with $0 \leq t \leq 1$, whose associated family of kernels K_t has a unique aperiodic ergodic component $\Sigma_0 = \{1, 2\}$. Applying the formulas (2)-(3) we have that $\beta^*(K_t) = \frac{9-5t}{12}$ and $\tau^*(K_t, \Sigma_0) = \frac{3t}{5}$.

Let $f : [0, 1] \rightarrow [0, 1]$ be the piecewise linear map

$$f(x) := \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2-2x & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

which has Lipschitz constant $\text{Lip}(f) = 2$. A simple calculation shows that $e^{\ell(f)} = 2$.

In this example the threshold condition

$$\max\{\beta^*(K_t), \tau^*(K_t, \Sigma_0)\} < \frac{1}{2}$$

of Theorem 3.4 is equivalent to $0.6 = \frac{3}{5} < t < \frac{5}{6} \sim 0.833$ (see Figure 5). Experiments we run indicate that in this example local synchronization never occurs whenever the threshold condition fails, i.e., $t < \frac{3}{5}$ or $t > \frac{5}{6}$ (see Figure 6).

In the simulations we randomly chose initial conditions near the diagonal Δ in $[0, 1]^3$ and computed $n = 200$ iterates, plotting the last 100 iterates.

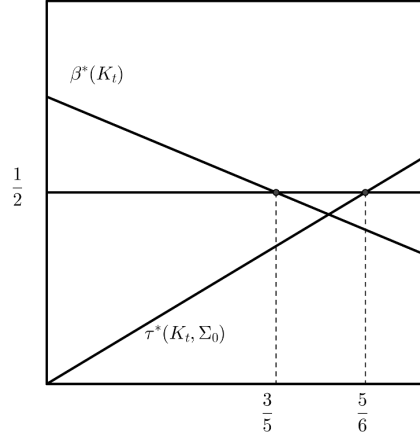


FIGURE 5. Threshold condition: $\max\{\beta^*(K_t), \tau^*(K_t, \Sigma_0)\} < \frac{1}{2}$.

Example 7.2. Consider the following family of stochastic matrices

$$A_t := \begin{bmatrix} 0 & 0 & \frac{1-t}{2} + \frac{7t}{8} & \frac{1-t}{2} + \frac{t}{8} \\ 0 & 0 & \frac{1-t}{2} + \frac{t}{4} & \frac{1-t}{2} + \frac{3t}{4} \\ \frac{1-t}{2} + \frac{3t}{4} & \frac{1-t}{2} + \frac{t}{4} & 0 & 0 \\ \frac{1-t}{2} + \frac{t}{8} & \frac{1-t}{2} + \frac{7t}{8} & 0 & 0 \end{bmatrix}$$

with $0 \leq t \leq 1$. Each associated kernel K_t has a unique ergodic component $\Sigma_1 = \Sigma = \{1, 2, 3, 4\}$ with period 2. The mixing sub-components are $\Sigma_{1,1} = \{1, 2\}$ and $\Sigma_{1,2} = \{3, 4\}$. Since the eigenvalues of A_t are 1, -1 , $\frac{5t}{8}$ and $-\frac{5t}{8}$, we have that $\tau^*(K_t, \Sigma_1) = \frac{5t}{8}$.

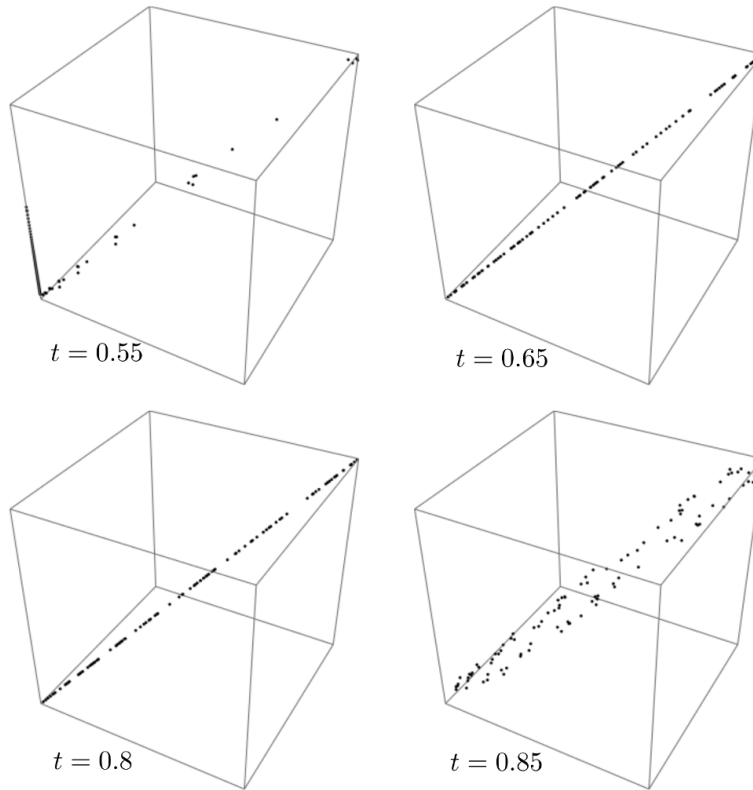


FIGURE 6. In Example 7.1 local synchronization only occurs for the values $0.6 = \frac{3}{5} < t < \frac{5}{6} \sim 0.833$.

Let $f : [0, 1] \rightarrow [0, 1]$ be the piecewise linear map

$$f(x) := \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2 - 2x & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

which has Lipschitz constant $\text{Lip}(f) = 2$. A simple calculation shows that $e^{\ell(f)} = 2$.

In this example the threshold condition

$$\tau^*(K_t, \Sigma_1) < \frac{1}{2}$$

of Theorem 3.5 is equivalent to $t < 0.8$. Experiments we run have shown that in this example local synchronization over the mixing sub-components never occurs whenever the threshold condition fails, i.e., $t > 0.8$.

In the simulations we randomly chose initial conditions near the diagonal Δ in $[0, 1]^4$ and computed $n = 200$ iterates, plotting the last 100 iterates. Figures 7 and 8 illustrate the projections of these orbits in $[0, 1]^4$ to all six coordinate 2-planes.

Example 6.2 (in the smooth setting) and Examples 7.1 and 7.2 (in the Lipschitz case) suggest a weaker concept of synchronization.

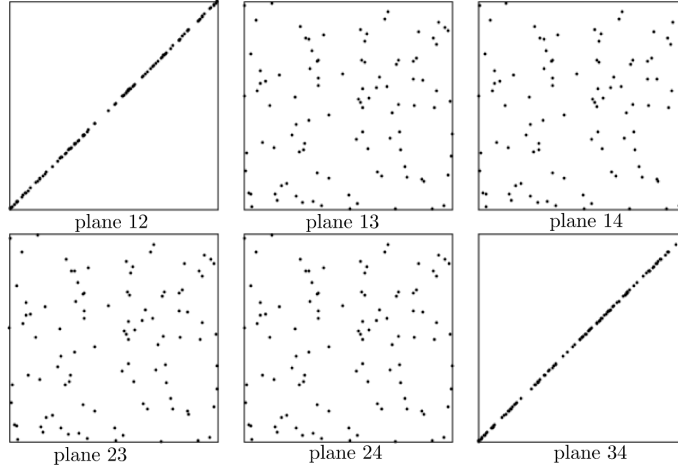


FIGURE 7. In Example 7.2 local synchronization occurs over the mixing sub-components for $t = 0.78$.

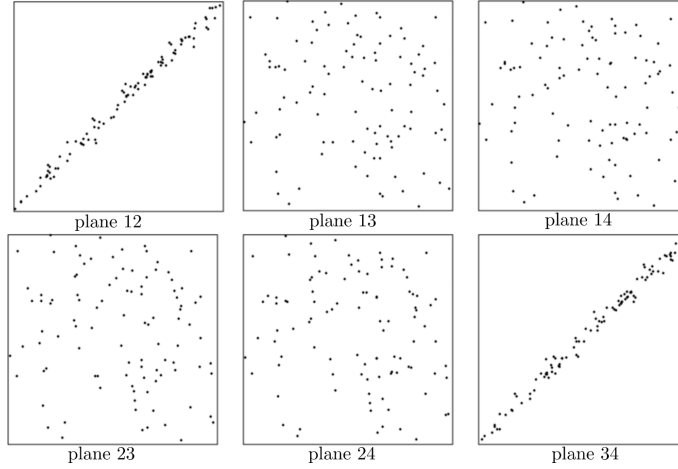


FIGURE 8. In Example 7.2 local synchronization does not occur over the mixing sub-components for $t = 0.82$.

Let $f : X \rightarrow X$ be a piecewise C^1 -diffeomorphism preserving a probability measure μ on the Borel σ -algebra of X . Consider on $L^\infty(\Sigma, X) \subset X^\Sigma$ the product measure μ^Σ . We say that a coupled map system has *almost sure global synchronization* over $\Lambda \subset \Sigma$ if for μ^Σ -almost every $\varphi \in L^\infty(\Sigma, X)$, given $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ and all $x, y \in \Lambda$, one has

$$\|F^n(\varphi)(x) - F^n(\varphi)(y)\| < \delta.$$

In Example 6.2, the simulations indicate that almost sure global synchronization over $\Sigma = \{1, 2, 3\}$ occurs for $t > 0.2$, this in spite of the fact that the existence of a fixed point off the diagonal implies no global synchronization for $t < 0.65$.

This motivates the following problems:

Problem 2. Under the assumptions of Theorem 3.4, denoting by $\lambda(f) = \lambda(f, \mu)$ the top Lyapunov exponent of f w.r.t. μ , if

$$\max\{\beta^*(K), \tau^*(K, \Sigma_0)\} e^{\lambda(f)} < 1$$

can one prove that almost sure global synchronization (over Σ) occurs?

Problem 3. Under the assumptions of Theorem 3.5, denoting by $\lambda(f) = \lambda(f, \mu)$ the top Lyapunov exponent of f w.r.t. μ , if

$$\tau^*(K, \Sigma_i) e^{\ell(f)} < 1$$

can one prove that almost sure global synchronization over all mixing sub-components occur?

In other words, we wonder if almost sure global synchronization occurs when we replace, in the threshold conditions of theorems 3.4 and 3.5, the Lipschitz term $\ell(f)$ by the Lyapunov exponent $\lambda(f)$.

Acknowledgments. The authors would like to thank the referee for his useful comments and suggestions. AB would like to thank the hospitality of Universidade de Lisboa. PD was supported by Fundação para a Ciência e a Tecnologia, through the project UID/MAT/04561/2013. MJT was partially financed by Portuguese Funds through FCT (Fundação para a Ciência e a Tecnologia) within the Projects UIDB/00013/2020 and UIDP/00013/2020. AB, PD and MJT were partially supported by the Project “New trends in Lyapunov exponents” (PTDC/MAT-PUR/29126/ 2017).

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