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On symmetric higher-dimensional automata and bisimilarity

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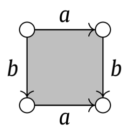
ABSTRACT

It is shown that there exists a hereditary history-preserving bisimulation between a higher-dimensional automaton (HDA) and the symmetric HDA freely generated by it. Consequently, up to hereditary history-preserving bisimilarity, ordinary HDAs and symmetric HDAs are models of concurrency with the same expressive power.

1. Introduction

A higher-dimensional automaton (HDA) is a precubical set with an initial state, a set of final states, and a labeling on 1-cubes such that opposite edges of 2-cubes have the same label [8,14]. Intuitively, an HDA may be seen as an ordinary automaton, or a labeled transition system, equipped with two- and higher-dimensional cubes representing independence or concurrency of actions. If two actions a and b are independent in a state of an HDA, meaning that they may be executed sequentially or simultaneously without any relevant difference, then this is indicated by a square linking the two execution sequences ab and ba:



Similarly, the independence or concurrency of n actions is represented by an n-dimensional cube. It has been shown in [8] that HDAs are a very expressive model of concurrency.

An important category of HDAs is the one of symmetric HDAs, i.e., HDAs with symmetric underlying precubical sets. The construction of HDAs from other models of concurrency often yields symmetric HDAs (see, e.g., [6,8–10]). In a symmetric HDA, the independence of n distinct actions is modeled by one n-dimensional cube for each permutation of the actions instead of possibly just one single n-cube. This is both an advantage and a disadvantage of symmetric HDAs: while it is undoubtedly convenient to model concurrency without the need for ordering actions in a more or less arbitrary way, the redundancy in the representation of independence makes symmetric HDAs inevitably enormous in size.

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Ordinary and symmetric HDAs are related by an adjunction: a symmetric HDA is, in particular, an HDA, and conversely, every HDA $\mathcal{Q}$ freely generates a symmetric HDA $S\mathcal{Q}$. The HDAs $\mathcal{Q}$ and $S\mathcal{Q}$ have isomorphic underlying transition systems, and n transitions are independent in $\mathcal{Q}$ if and only if the corresponding transitions are independent in $S\mathcal{Q}$. This strongly suggests that $\mathcal{Q}$ and $S\mathcal{Q}$ are semantically equivalent HDAs, and the purpose of this paper is to establish that this is indeed the case in a strong sense, namely with respect to hereditary history-preserving bisimulation (as defined in [8]). This result implies as a consequence that, up to hereditary history-preserving bisimilarity, ordinary and symmetric HDAs are equally expressive models of concurrency.

2. Precubical sets and HDAs

This section briefly recalls the definitions of precubical set and higher-dimensional automaton. For more details, explanations, and examples, the reader is referred to, e.g., [2,5,8,11].

Precubical sets

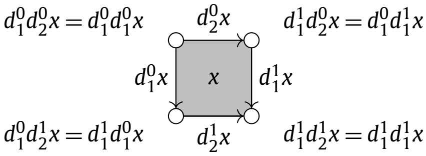
A *precubical set* is a graded set $P = (P_n)_{n\geq 0}$ with *face maps*

$$d_i^k\colon P_n \to P_{n-1} \quad (n > 0,\ k = 0, 1,\ i = 1, \dots, n)$$

satisfying the *cubical identities*

$$d_i^k d_j^l = d_{j-1}^l d_i^k \quad (k, l = 0, 1,\ i < j).$$

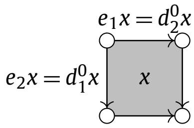
If $x \in P_n$, we say that x is of *degree* or *dimension* n. The elements of degree n are called the *n-cubes* of P. The elements of degree 0 are also called the *vertices* of P, and the 1-cubes are also called the *edges* of P. A face $d_i^k x$ is called a *front face* of x if $k = 0$, and it is called a *back face* of x if $k = 1$. The cubical identities $d_i^k d_j^l = d_{j-1}^l d_i^k$ guarantee that the faces of a cube intersect each other as they should:



The *ith starting edge* of a cube x of degree $n > 0$ is the edge

$$e_i x = d_1^0 \cdots d_{i-1}^0 d_{i+1}^0 \cdots d_n^0 x.$$

In the particular case of a 2-cube, we have $e_i x = d_{3-i}^0 x$:



A *morphism* of precubical sets is a morphism of graded sets that is compatible with the face maps. The category of precubical sets can be seen as the presheaf category of functors $\square^{\mathrm{op}} \to \mathsf{Set}$ where $\square$ is the small subcategory of the category of topological spaces whose objects are the standard n-cubes $[0,1]^n$ $(n \geq 0)$ and whose nonidentity morphisms are composites of the *coface maps* $\delta_i^k\colon [0,1]^n \to [0,1]^{n+1}$ $(k \in \{0,1\},\ n \geq 0,\ i \in \{1, \dots, n+1\})$ given by $\delta_i^k(u_1, \dots, u_n) = (u_1, \dots, u_{i-1}, k, u_i \dots, u_n)$.

Higher-dimensional automata

Throughout this paper, let Σ be an alphabet. A *higher-dimensional automaton (HDA)* over Σ is a tuple

$$\mathcal{Q} = (P, I, F, \lambda)$$

where P is a precubical set, $I \in P_0$ is a vertex, called the *initial state*, $F \subseteq P_0$ is a (possibly empty) set of *final states*, and $\lambda\colon P_1 \to \Sigma$ is a map, called the *labeling function*, such that $\lambda(d_i^0 x) = \lambda(d_i^1 x)$ for all $x \in P_2$ and $i \in \{1, 2\}$ [8]. Higher-dimensional automata form a category, in which a morphism from an HDA $\mathcal{Q} = (P, I, F, \lambda)$ to an HDA $\mathcal{Q}' = (P', I', F', \lambda')$ is a morphism of precubical sets $f\colon P \to P'$ such that $f(I) = I'$, $f(F) \subseteq F'$, and $\lambda'(f(x)) = \lambda(x)$ for all $x \in P_1$.

3. The precubical set of permutations

It is well known that the family of symmetric groups can be given the structure of a *skew-simplicial* or *crossed simplicial group* [3,12]. This implies that it also can be given the structure of a precubical set. In this section, we describe this structure and prove a number of basic facts about it. Recall that the *symmetric group* S_n is the set of permutations of $\{1,\dots,n\}$ with composition as multiplication. Here we understand that $\{1,\dots,0\}=\emptyset$ and that $S_0=\{id_\emptyset\}$.

The maps $\downarrow i$ *and* $\uparrow i$

For an integer i, we define the maps $\downarrow i$ and $\uparrow i$ on integers by

$$m^{\downarrow i}=\begin{cases} m, & m\le i,\\ m-1, & m>i\end{cases} \quad\text{and}\quad m^{\uparrow i}=\begin{cases} m, & m<i,\\ m+1, & m\ge i.\end{cases}$$

Note that $m^{\uparrow i\downarrow i}=m$ and, for $m\neq i$, $m^{\downarrow i\uparrow i}=m$. Note also that for $i<j$,

$$m^{\downarrow j\downarrow i}=m^{\downarrow i\downarrow j-1}=\begin{cases} m, & m\le i,\\ m-1, & i<m\le j,\\ m-2, & m>j.\end{cases}$$

We remark that the maps $\downarrow i$ and $\uparrow i$ are used to define the coface and codegeneracy maps in the simplex category, which plays an important role in the theory of simplicial sets (see, e.g., [7,13]).

The face maps of S

For $n\ge 1$, $\theta\in S_n$, $i\in\{1,\dots,n\}$, and $k\in\{0,1\}$, we define the permutation $d_i^k\theta\in S_{n-1}$ (using one-line notation) by

$$d_i^k\theta=(\theta(1)^{\downarrow i}\ \theta(2)^{\downarrow i}\ \cdots\ \theta(\theta^{-1}(i)-1)^{\downarrow i}\ \theta(\theta^{-1}(i)+1)^{\downarrow i}\ \cdots\ \theta(n)^{\downarrow i}).$$

Thus, $d_i^k\theta(j)=\theta(j^{\uparrow\theta^{-1}(i)})^{\downarrow i}$ and, more explicitly,

$$d_i^k\theta(j)=\begin{cases} \theta(j), & j<\theta^{-1}(i),\ \theta(j)<i,\\ \theta(j)-1, & j<\theta^{-1}(i),\ \theta(j)>i,\\ \theta(j+1), & j\ge\theta^{-1}(i),\ \theta(j+1)<i,\\ \theta(j+1)-1, & j\ge\theta^{-1}(i),\ \theta(j+1)>i.\end{cases}$$

Note that by definition, $d_i^0\theta=d_i^1\theta$. We may therefore simplify the notation by setting

$$d_i\theta=d_i^0\theta=d_i^1\theta.$$

By the next proposition, the face maps turn the graded set S into a precubical set. Although this precubical set has only one vertex and one edge, it is very large: S_n has $n!$ elements. Computationally, each element of degree $n\ge 2$ may be interpreted as representing a simultaneous execution of n copies of the action given by the only element of S_1.

Proposition 3.1. *For* $1\le i<j\le n$, $d_id_j\theta=d_{j-1}d_i\theta$.

Proof. Set

$$r=\begin{cases} i, & \theta^{-1}(i)<\theta^{-1}(j),\\ j, & \theta^{-1}(i)>\theta^{-1}(j)\end{cases} \quad\text{and}\quad s=\begin{cases} j, & \theta^{-1}(i)<\theta^{-1}(j),\\ i, & \theta^{-1}(i)>\theta^{-1}(j).\end{cases}$$

Since $i^{\downarrow j}=i$, we have $i=\theta(\theta^{-1}(i))^{\downarrow j}$ and therefore

$$\begin{aligned} d_id_j\theta &= d_i(\theta(1)^{\downarrow j}\ \cdots\ \theta(\theta^{-1}(j)-1)^{\downarrow j}\ \theta(\theta^{-1}(j)+1)^{\downarrow j}\ \cdots\ \theta(n)^{\downarrow j})\\ &= (\theta(1)^{\downarrow j\downarrow i}\ \cdots\ \theta(\theta^{-1}(r)-1)^{\downarrow j\downarrow i}\ \theta(\theta^{-1}(r)+1)^{\downarrow j\downarrow i}\ \cdots\\ &\quad \cdots\ \theta(\theta^{-1}(s)-1)^{\downarrow j\downarrow i}\ \theta(\theta^{-1}(s)+1)^{\downarrow j\downarrow i}\ \cdots\ \theta(n)^{\downarrow j\downarrow i}).\end{aligned}$$

Since $j^{\downarrow i}=j-1$, we have $j-1=\theta(\theta^{-1}(j))^{\downarrow i}$ and therefore

$$\begin{aligned} d_{j-1}d_i\theta &= d_{j-1}(\theta(1)^{\downarrow i}\ \cdots\ \theta(\theta^{-1}(i)-1)^{\downarrow i}\ \theta(\theta^{-1}(i)+1)^{\downarrow i}\ \cdots\ \theta(n)^{\downarrow i})\\ &= (\theta(1)^{\downarrow i\downarrow j-1}\ \cdots\ \theta(\theta^{-1}(r)-1)^{\downarrow i\downarrow j-1}\ \theta(\theta^{-1}(r)+1)^{\downarrow i\downarrow j-1}\ \cdots\\ &\quad \cdots\ \theta(\theta^{-1}(s)-1)^{\downarrow i\downarrow j-1}\ \theta(\theta^{-1}(s)+1)^{\downarrow i\downarrow j-1}\ \cdots\ \theta(n)^{\downarrow i\downarrow j-1}).\end{aligned}$$

Since $m^{\downarrow j\downarrow i}=m^{\downarrow i\downarrow j-1}$, we have $d_id_j\theta=d_{j-1}d_i\theta$. □

Being a group in each degree, the precubical set S is also an algebraic object. Regarding the compatibility of the precubical and the algebraic structures of S, we have the following proposition:

Proposition 3.2. *Let $n \geq 1$ and $i \in \{1, \dots, n\}$. Then*

(1) $d_i id_{\{1,\dots,n\}} = id_{\{1,\dots,n-1\}}$;
(2) $d_i(\sigma \cdot \theta) = d_i\sigma \cdot d_{\sigma^{-1}(i)}\theta$ for all $\sigma, \theta \in S_n$;
(3) $(d_i\theta)^{-1} = d_{\theta^{-1}(i)}\theta^{-1}$ for all $\theta \in S_n$.

Proof. (1) follows immediately from the definition of $d_i id_{\{1,\dots,n\}}$.

(2) Let $j \in \{1, \dots, n-1\}$. Since $j^{\uparrow\theta^{-1}(\sigma^{-1}(i))} \neq \theta^{-1}(\sigma^{-1}(i))$, we have $\theta(j^{\uparrow\theta^{-1}(\sigma^{-1}(i))}) \neq \sigma^{-1}(i)$ and therefore

$$\theta(j^{\uparrow\theta^{-1}(\sigma^{-1}(i))})^{\downarrow\sigma^{-1}(i)\uparrow\sigma^{-1}(i)} = \theta(j^{\uparrow\theta^{-1}(\sigma^{-1}(i))}).$$

Hence

$$\begin{aligned}(d_i\sigma \cdot d_{\sigma^{-1}(i)}\theta)(j) &= d_i\sigma(d_{\sigma^{-1}(i)}\theta(j))\\ &= d_i\sigma(\theta(j^{\uparrow\theta^{-1}(\sigma^{-1}(i))})^{\downarrow\sigma^{-1}(i)})\\ &= \sigma(\theta(j^{\uparrow\theta^{-1}(\sigma^{-1}(i))})^{\downarrow\sigma^{-1}(i)\uparrow\sigma^{-1}(i)})^{\downarrow i}\\ &= \sigma(\theta(j^{\uparrow\theta^{-1}(\sigma^{-1}(i))}))^{\downarrow i}\\ &= (\sigma \cdot \theta)(j^{\uparrow(\sigma\cdot\theta)^{-1}(i)})^{\downarrow i}\\ &= d_i(\sigma \cdot \theta)(j).\end{aligned}$$

(3) By (1) and (2),

$$d_i\theta \cdot d_{\theta^{-1}(i)}\theta^{-1} = d_i(\theta \cdot \theta^{-1}) = d_i id_{\{1,\dots,n\}} = id_{\{1,\dots,n-1\}}. \quad \square$$

Permutations and face conditions

We will show in Proposition 3.4 below that if two permutations of the same degree have a common face, there exists a permutation that has both of them as faces. The proof requires the following lemma:

Lemma 3.3. *Consider permutations $\sigma, \theta \in S_n$ $(n \geq 1)$, and let $i \in \{1, \dots, n\}$ such that $d_i\sigma = d_i\theta$ and $\sigma^{-1}(i) = \theta^{-1}(i)$. Then $\sigma = \theta$.*

Proof. By definition,

$$d_i\sigma = (\sigma(1)^{\downarrow i} \ \cdots \ \sigma(\sigma^{-1}(i)-1)^{\downarrow i} \ \sigma(\sigma^{-1}(i)+1)^{\downarrow i} \ \cdots \ \sigma(n)^{\downarrow i})$$

and

$$d_i\theta = (\theta(1)^{\downarrow i} \ \cdots \ \theta(\theta^{-1}(i)-1)^{\downarrow i} \ \theta(\theta^{-1}(i)+1)^{\downarrow i} \ \cdots \ \theta(n)^{\downarrow i}).$$

Since $d_i\sigma = d_i\theta$ and $\sigma^{-1}(i) = \theta^{-1}(i)$, we have $\sigma(j)^{\downarrow i} = \theta(j)^{\downarrow i}$ for all $j \neq \sigma^{-1}(i) = \theta^{-1}(i)$. For these j, $\sigma(j) \neq i \neq \theta(j)$ and therefore $\sigma(j) = \sigma(j)^{\downarrow i\uparrow i} = \theta(j)^{\downarrow i\uparrow i} = \theta(j)$. Since $\sigma(\sigma^{-1}(i)) = i = \theta(\theta^{-1}(i))$, we have $\sigma(j) = \theta(j)$ for all $j \in \{1, \dots, n\}$. $\square$

Proposition 3.4. *Consider permutations $\alpha, \beta \in S_n$ $(n \geq 1)$, and let $r \leq s$ be integers such that $d_r\alpha = d_s\beta$. Then there exists a permutation $\theta \in S_{n+1}$ such that $d_r\theta = \beta$ and $d_{s+1}\theta = \alpha$. If $\alpha^{-1}(r) \leq \beta^{-1}(s)$, θ may be chosen such that $\theta^{-1}(r) < \theta^{-1}(s+1)$. If $\alpha^{-1}(r) \geq \beta^{-1}(s)$, θ may be chosen such that $\theta^{-1}(r) > \theta^{-1}(s+1)$.*

Proof. (i) Suppose that $\alpha^{-1}(r) \leq \beta^{-1}(s)$. Set

$$\theta = (\beta(1)^{\uparrow r} \ \cdots \ \beta(\alpha^{-1}(r)-1)^{\uparrow r} \ r \ \beta(\alpha^{-1}(r))^{\uparrow r} \ \cdots \ \beta(n)^{\uparrow r}).$$

Then $d_r\theta = \beta$. Since $r < s+1$,

$$d_rd_{s+1}\theta = d_sd_r\theta = d_s\beta = d_r\alpha.$$

Since $\alpha^{-1}(r) \leq \beta^{-1}(s)$, we have $\alpha^{-1}(r) = \theta^{-1}(r) < \theta^{-1}(s+1)$ and therefore

$$(d_{s+1}\theta)^{-1}(r) = d_{\theta^{-1}(s+1)}\theta^{-1}(r) = \theta^{-1}(r) = \alpha^{-1}(r).$$

By Lemma 3.3, it follows that $d_{s+1}\theta = \alpha$.

(ii) If $\alpha^{-1}(r) \geq \beta^{-1}(s)$, a similar argument shows that

$$\theta = (\alpha(1)^{\uparrow s+1} \ \cdots \ \alpha(\beta^{-1}(s)-1)^{\uparrow s+1} \ \ s+1 \ \ \alpha(\beta^{-1}(s))^{\uparrow s+1} \ \cdots \ \alpha(n)^{\uparrow s+1})$$

has the required properties. $\square$

Cubical identities and permutations

The cubical identities $d_i^k d_j^l = d_{j-1}^l d_i^k$ of precubical sets can be generalized using permutations. This is done after the next lemma.

Lemma 3.5. *Consider a permutation $\theta \in S_n$ $(n \geq 2)$, and let $1 \leq i < j \leq n$. If $d_{\theta(j)}\theta(i) < \theta(j)$, then $d_{\theta(j)}\theta(i) = \theta(i)$ and $d_{\theta(i)}\theta(j-1) = \theta(j) - 1$. Else $d_{\theta(j)}\theta(i) = \theta(i) - 1$ and $d_{\theta(i)}\theta(j-1) = \theta(j)$.*

Proof. We will suppose that $d_{\theta(j)}\theta(i) < \theta(j)$. The other case is analogous. Since $i < j = \theta^{-1}(\theta(j))$, we have $d_{\theta(j)}\theta(i) = \theta(i)$ or $d_{\theta(j)}\theta(i) = \theta(i) - 1$. If we had $d_{\theta(j)}\theta(i) = \theta(i) - 1$, we would have $\theta(i) > \theta(j)$ and thus $d_{\theta(j)}\theta(i) \geq \theta(j)$. Hence $d_{\theta(j)}\theta(i) = \theta(i)$. Since $j - 1 \geq i = \theta^{-1}(\theta(i))$ and $\theta(j-1+1) = \theta(j) > d_{\theta(j)}\theta(i) = \theta(i)$, we have $d_{\theta(i)}\theta(j-1) = \theta(j) - 1$. $\square$

Proposition 3.6. *Let P be a precubical set, and let $n \geq 2$, $1 \leq i < j \leq n$, $k, l \in \{0, 1\}$, $x \in P_n$, and $\theta \in S_n$. Then*

(i) $d^k_{d_{\theta(j)}\theta(i)} d^l_{\theta(j)} x = d^l_{d_{\theta(i)}\theta(j-1)} d^k_{\theta(i)} x$;

(ii) $d^k_{(d_j\theta)^{-1}(i)} d^l_{\theta^{-1}(j)} x = d^l_{(d_i\theta)^{-1}(j-1)} d^k_{\theta^{-1}(i)} x$.

Proof. (i) If $d_{\theta(j)}\theta(i) < \theta(j)$, then, by Lemma 3.5,

$$d^k_{d_{\theta(j)}\theta(i)} d^l_{\theta(j)} x = d^l_{\theta(j)-1} d^k_{d_{\theta(j)}\theta(i)} x = d^l_{d_{\theta(i)}\theta(j-1)} d^k_{\theta(i)} x.$$

If $d_{\theta(j)}\theta(i) \geq \theta(j)$, then, again by Lemma 3.5,

$$d^l_{d_{\theta(i)}\theta(j-1)} d^k_{\theta(i)} x = d^l_{\theta(j)} d^k_{d_{\theta(j)}\theta(i)+1} x = d^k_{d_{\theta(j)}\theta(i)} d^l_{\theta(j)} x.$$

(ii) By Proposition 3.2 and (i), $d^k_{(d_j\theta)^{-1}(i)} d^l_{\theta^{-1}(j)} x = d^k_{d_{\theta^{-1}(j)}\theta^{-1}(i)} d^l_{\theta^{-1}(j)} x = d^l_{d_{\theta^{-1}(i)}\theta^{-1}(j-1)} d^k_{\theta^{-1}(i)} x = d^l_{(d_i\theta)^{-1}(j-1)} d^k_{\theta^{-1}(i)} x$. $\square$

Remark 3.7. By Proposition 3.6, one may consider a second structure of precubical set on S where the face maps are defined by $\partial_i^k \theta = d_{\theta(i)}\theta$. By Proposition 3.2, the map $\theta \mapsto \theta^{-1}$ is an isomorphism between the two precubical sets of permutations.

4. Symmetric precubical sets and HDAs

A symmetric HDA is an HDA with symmetric underlying precubical set. Intuitively, this means that the independence or concurrency of n actions is modeled not by just one n-cube but by one n-cube for each permutation of the actions. Symmetric precubical sets are usually defined as presheaves on a suitable category of cubes (see, e.g., [6,9]). Here we define them equivalently as precubical sets with a crossed action of the precubical set S consisting of the symmetric groups. We also define free symmetric precubical sets and HDAs, which are central to our work in the following sections.

Crossed actions

A *crossed action* of S on a precubical set P is a morphism of graded sets $S \times P \to P$, $(\theta, x) \mapsto \theta \cdot x$ satisfying the following three conditions:

1. For all $n \geq 0$ and $x \in P_n$,

   $$id \cdot x = x.$$

2. For all $n \geq 0$, $\sigma, \theta \in S_n$, and $x \in P_n$,

   $$(\sigma \cdot \theta) \cdot x = \sigma \cdot (\theta \cdot x).$$

3. For all $n \geq 1$, $\theta \in S_n$, $x \in P_n$, $i \in \{1, \ldots, n\}$, and $k \in \{0, 1\}$,

   $$d_i^k(\theta \cdot x) = d_i\theta \cdot d^k_{\theta^{-1}(i)} x.$$

The terminology reflects an analogy with actions of crossed simplicial groups [3]. The multiplication $S \times S \to S$ is a crossed action of S on itself. An example of a precubical set that does not admit a crossed action of S is the *precubical square* consisting of a 2-cube x and its edges and vertices. Indeed, if the precubical square admitted a crossed action of S, one would have $\tau \cdot x = x$ for the transposition $\tau = (2\ 1)$ because x is the only element of degree 2. But then one would also have

$$d_1^0 x = d_1^0(\tau \cdot x) = d_1\tau \cdot d^0_{\tau^{-1}(1)} x = id \cdot d_2^0 x = d_2^0 x,$$

which is not the case.

Symmetric precubical sets

A *symmetric precubical set* is a precubical set P equipped with a crossed action of S on P. For example, S is a symmetric precubical set with respect to the multiplication $S \times S \to S$. On the other hand, as we have seen above, the precubical square cannot be given the structure of a symmetric precubical set.

Symmetric precubical sets form a category, in which the morphisms are morphisms of precubical sets that are compatible with the crossed actions. We remark that the category of symmetric precubical sets is isomorphic to the presheaf category $\mathsf{Set}^{\square_S^{\mathrm{op}}}$ where $\square_S$ is the subcategory of the category of topological spaces whose objects are the standard n-cubes $[0,1]^n$ ($n \geq 0$) and whose morphisms are composites of the coface maps δ_i^k defined in Section 2 and the *permutation maps* $t_\theta \colon [0,1]^n \to [0,1]^n$ ($n \geq 0$, $\theta \in S_n$) given by $t_\theta(u_1, \dots, u_n) = (u_{\theta(1)} \dots, u_{\theta(n)})$.

Recall that the ith starting edge of a cube x of degree $n \geq 1$ of a precubical set is the edge $e_i x = d_1^0 \cdots d_{i-1}^0 d_{i+1}^0 \cdots d_n^0 x$. For the starting edges of cubes in a symmetric precubical set, we have the following formula:

Proposition 4.1. *Let P be a symmetric precubical set, and let $x \in P_n$ and $\theta \in S_n$ ($n \geq 1$). Then for each $i \in \{1, \dots, n\}$, $e_i(\theta \cdot x) = e_{\theta^{-1}(i)} x$.*

Proof. We proceed by induction. For $n = 1$, there is nothing to show. If $n = 2$,

$$e_i(\theta \cdot x) = d^0_{3-i}(\theta \cdot x) = d_{3-i}\theta \cdot d^0_{\theta^{-1}(3-i)} x = id \cdot d^0_{\theta^{-1}(3-i)} x = e_{\theta^{-1}(i)} x.$$

The last two equalities hold because $S_1 = \{id\}$ and $\{1,2\} \setminus \{\theta^{-1}(i)\} = \{\theta^{-1}(3-i)\}$.

Suppose that $n > 2$. Consider first the case $i < n$. By the inductive hypothesis,

$$e_i(\theta \cdot x) = e_i d_n^0(\theta \cdot x) = e_i(d_n\theta \cdot d^0_{\theta^{-1}(n)} x) = e_{(d_n\theta)^{-1}(i)} d^0_{\theta^{-1}(n)} x.$$

Since $i < n$,

$$(d_n\theta)^{-1}(i) = d_{\theta^{-1}(n)}\theta^{-1}(i) = \begin{cases} \theta^{-1}(i), & \theta^{-1}(i) < \theta^{-1}(n), \\ \theta^{-1}(i) - 1, & \theta^{-1}(i) > \theta^{-1}(n). \end{cases}$$

Hence

$$\begin{aligned} e_i(\theta \cdot x) &= e_{(d_n\theta)^{-1}(i)} d^0_{\theta^{-1}(n)} x \\ &= \begin{cases} e_{\theta^{-1}(i)} d^0_{\theta^{-1}(n)} x, & \theta^{-1}(i) < \theta^{-1}(n), \\ e_{\theta^{-1}(i)-1} d^0_{\theta^{-1}(n)} x, & \theta^{-1}(i) > \theta^{-1}(n) \end{cases} \\ &= \begin{cases} d_1^0 \cdots d^0_{\theta^{-1}(i)-1} d^0_{\theta^{-1}(i)+1} \cdots d^0_{n-1} d^0_{\theta^{-1}(n)} x, & \theta^{-1}(i) < \theta^{-1}(n), \\ d_1^0 \cdots d^0_{\theta^{-1}(i)-2} d^0_{\theta^{-1}(i)} \cdots d^0_{n-1} d^0_{\theta^{-1}(n)} x, & \theta^{-1}(i) > \theta^{-1}(n) \end{cases} \\ &= d_1^0 \cdots d^0_{\theta^{-1}(i)-1} d^0_{\theta^{-1}(i)+1} \cdots d_n^0 x \\ &= e_{\theta^{-1}(i)} x. \end{aligned}$$

Suppose now that $i = n$. By the inductive hypothesis,

$$\begin{aligned} e_n(\theta \cdot x) &= e_{n-1} d^0_{n-1}(\theta \cdot x) \\ &= e_{n-1}(d_{n-1}\theta \cdot d^0_{\theta^{-1}(n-1)} x) \\ &= e_{(d_{n-1}\theta)^{-1}(n-1)} d^0_{\theta^{-1}(n-1)} x. \end{aligned}$$

We have

$$\begin{aligned}(d_{n-1}\theta)^{-1}(n-1) &= d_{\theta^{-1}(n-1)}\theta^{-1}(n-1)\\ &= \begin{cases}\theta^{-1}(n), & \theta^{-1}(n) < \theta^{-1}(n-1),\\ \theta^{-1}(n)-1, & \theta^{-1}(n) > \theta^{-1}(n-1).\end{cases}\end{aligned}$$

Hence

$$\begin{aligned}e_n(\theta\cdot x) &= e_{(d_{n-1}\theta)^{-1}(n-1)}d^0_{\theta^{-1}(n-1)}x\\ &= \begin{cases}e_{\theta^{-1}(n)}d^0_{\theta^{-1}(n-1)}x, & \theta^{-1}(n) < \theta^{-1}(n-1),\\ e_{\theta^{-1}(n)-1}d^0_{\theta^{-1}(n-1)}x, & \theta^{-1}(n) > \theta^{-1}(n-1)\end{cases}\\ &= e_{\theta^{-1}(n)}x. \quad \square\end{aligned}$$

Free symmetric precubical sets

Let P be a precubical set. The *free symmetric precubical set* generated by P is the symmetric precubical set SP defined by

- $SP_n = S_n \times P_n$ $(n \geq 0)$;
- $d^k_i(\theta, x) = (d_i\theta, d^k_{\theta^{-1}(i)}x)$ $(n \geq 1, \theta \in S_n, x \in P_n, 1 \leq i \leq n, k \in \{0,1\})$;
- $\sigma\cdot(\theta, x) = (\sigma\cdot\theta, x)$ $(n \geq 0, \sigma, \theta \in S_n, x \in P_n)$.

The fact that SP is a symmetric precubical set follows from Propositions 3.2 and 3.6. Indeed, by Proposition 3.6, for $i < j$,

$$\begin{aligned}d^k_i d^l_j(\theta, x) &= d^k_i(d_j\theta, d^l_{\theta^{-1}(j)}x)\\ &= (d_id_j\theta, d^k_{(d_j\theta)^{-1}(i)}d^l_{\theta^{-1}(j)}x)\\ &= (d_{j-1}d_i\theta, d^l_{(d_i\theta)^{-1}(j-1)}d^k_{\theta^{-1}(i)}x)\\ &= d^l_{j-1}(d_i\theta, d^k_{\theta^{-1}(i)}x)\\ &= d^l_{j-1}d^k_i(\theta, x).\end{aligned}$$

Moreover, by Proposition 3.2,

$$\begin{aligned}d^k_i(\sigma\cdot(\theta, x)) &= d^k_i(\sigma\cdot\theta, x)\\ &= (d_i(\sigma\cdot\theta), d^k_{(\sigma\cdot\theta)^{-1}(i)}x)\\ &= (d_i\sigma\cdot d_{\sigma^{-1}(i)}\theta, d^k_{\theta^{-1}(\sigma^{-1}(i))}x)\\ &= d_i\sigma\cdot(d_{\sigma^{-1}(i)}\theta, d^k_{\theta^{-1}(\sigma^{-1}(i))}x)\\ &= d_i\sigma\cdot d^k_{\sigma^{-1}(i)}(\theta, x).\end{aligned}$$

The free symmetric precubical set is functorial. Given a morphism of precubical set $f\colon P \to Q$, $Sf\colon SP \to SQ$ is the graded map defined by $(\theta, x) \mapsto (\theta, f(x))$. The free symmetric precubical set SP is freely generated by the precubical set P in the same sense as a free abelian group is freely generated by a basis: it has the universal property that every morphism of symmetric precubical sets $SP \to Z$ is determined by what it is doing on the basis precubical set P. More formally, we have the following proposition:

Proposition 4.2. *The functor S from the category of precubical sets to the category of symmetric precubical sets is left adjoint to the forgetful functor.*

Proof. Let P be a precubical set, and let Z be a symmetric precubical set. The adjunct of a morphism of precubical sets $f\colon P \to Z$ is the morphism of symmetric precubical sets $\hat{f}\colon SP \to Z$ given by $\hat{f}(\theta, x) = \theta\cdot f(x)$. The adjunct of a morphism of symmetric precubical sets $g\colon SP \to Z$ is the morphism of precubical sets $\breve{g}\colon P \to Z$ given by $\breve{g}(x) = g(id, x)$. $\square$

The starting edges of cubes in a free symmetric precubical set are related as follows to the starting edges of cubes in the generating precubical set:

Proposition 4.3. *Let P be a precubical set, and let $(\theta, x) \in SP_n$ $(n \geq 1)$. Then for each $i \in \{1, \dots, n\}$, $e_i(\theta, x) = (id, e_{\theta^{-1}(i)}x)$.*

Proof. By Proposition 4.1,

$$e_i(\theta, x) = e_i(\theta \cdot (id, x)) = e_{\theta^{-1}(i)}(id, x).$$

A simple inductive argument shows that $d_{i_1}^{k_1} \cdots d_{i_r}^{k_r}(id, x) = (id, d_{i_1}^{k_1} \cdots d_{i_r}^{k_r} x)$. In particular, $e_j(id, x) = (id, e_j x)$. Thus, $e_i(\theta, x) = (id, e_{\theta^{-1}(i)} x)$. $\square$

Symmetric HDAs

A *symmetric HDA* is an HDA $\mathcal{Q} = (P, I, F, \lambda)$ equipped with a crossed action of S on P. Symmetric HDAs form a category, in which the morphisms are morphisms of HDAs that also are morphisms of symmetric precubical sets. The *free symmetric HDA* generated by an HDA $\mathcal{Q} = (P, I, F, \lambda)$ is the symmetric HDA $S\mathcal{Q} = (SP, (id, I), S_0 \times F, \mu)$ where $\mu(id, x) = \lambda(x)$ $(x \in P_1)$ and the crossed action is the one of SP. The difference between the HDAs $\mathcal{Q}$ and $S\mathcal{Q}$ lies in the higher-dimensional structure: for each n-cube x in $\mathcal{Q}$, there exist $n!$ n-cubes in $S\mathcal{Q}$, all representing the independence of the same n actions as x. The assignment $\mathcal{Q} \mapsto S\mathcal{Q}$ defines a functor from the category of HDAs to the category of symmetric HDAs, which is left adjoint to the forgetful functor.

5. Cube paths

Throughout this section, let $\mathcal{Q} = (P, I, F, \lambda)$ denote an HDA. A *cube path* in $\mathcal{Q}$ is a sequence of cubes and face maps

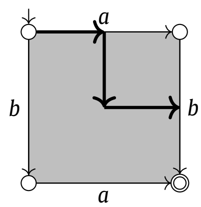
$$\pi = x_0 \xrightarrow{d_{i_1}^{k_1}} x_1 \xrightarrow{d_{i_2}^{k_2}} x_2 \xrightarrow{d_{i_3}^{k_3}} \cdots \xrightarrow{d_{i_m}^{k_m}} x_m$$

such that $x_0 = I$ and the following conditions hold for all $j \in \{1, \dots, m\}$ (cf. [8]):

- $2 \sum\limits_{p=1}^{j} k_p \leq j$
- $x_j \in P_{j - 2 \sum\limits_{p=1}^{j} k_p}$
- $i_j \in \{1, \dots, j - k_j - 2 \sum\limits_{p=1}^{j-k_j} k_p\}$
- $d_{i_j}^{k_j} x_{j-k_j} = x_{j-1+k_j}$

By the last condition, after the initial state, each cube in a cube path either is a face of its predecessor or has its predecessor as a face. The role of the first three conditions is to guarantee that the last condition makes sense. For the second condition to be meaningful, the first has to hold. Note that the first condition implies that $k_1 = 0$ and hence that $j - k_j \in \{1, \dots, m\}$ for all j. Given the second condition, it is clear that the third has to be fulfilled. The second condition can be understood as follows: A cube path starts in the initial state, which has degree 0. At each subsequent cube, the degree either increases or decreases by 1. So if at x_j, the degree has decreased r times and increased $j - r$ times, x_j must have degree $j - 2r$. Since the degree decreases at x_p exactly when $k_p = 1$, we have $r = \sum\limits_{p=1}^{j} k_p$.

Given a cube path $\pi = x_0 \xrightarrow{d_{i_1}^{k_1}} x_1 \xrightarrow{d_{i_2}^{k_2}} x_2 \xrightarrow{d_{i_3}^{k_3}} \cdots \xrightarrow{d_{i_m}^{k_m}} x_m$, we refer to m as the *length* of π and write $end(\pi)$ for x_m. We write $\pi \to \pi'$ if π extends to a cube path π'. A cube path in $\mathcal{Q}$ represents a partial execution of the concurrent system modeled by $\mathcal{Q}$. An example of a cube path is indicated by the thick arrows in the following very simple HDA:



If the 2-cube in this HDA is x, with $d_2^0 x$ the upper horizontal edge and $d_1^1 x$ the right vertical edge, the depicted cube path is

$$d_1^0 d_2^0 x \xrightarrow{d_1^0} d_2^0 x \xrightarrow{d_2^0} x \xrightarrow{d_1^1} d_1^1 x.$$

The *split trace* of a cube path $\pi = x_0 \xrightarrow{d_{i_1}^{k_1}} x_1 \xrightarrow{d_{i_2}^{k_2}} x_2 \xrightarrow{d_{i_3}^{k_3}} \cdots \xrightarrow{d_{i_m}^{k_m}} x_m$ is the sequence

$$\textit{split-trace}(\pi) = ((\lambda(e_{i_1}x_{1-k_1}), k_1), \dots, (\lambda(e_{i_m}x_{m-k_m}), k_m))$$

(see [8]). The split trace is the sequence of actions starting (with second component 0) and terminating (with second component 1) along the cube path. The split trace of the cube path in the example above is the sequence

$$((a,0),(b,0),(a,1)).$$

This cube path describes thus an (incomplete) execution of the system modeled by the HDA where first a starts, then b starts, and finally a terminates.

Definition 5.1 *([8]).* Two cube paths

$$\pi = x_0 \xrightarrow{d_{i_1}^{k_1}} x_1 \xrightarrow{d_{i_2}^{k_2}} \cdots \xrightarrow{d_{i_m}^{k_m}} x_m \quad \text{and} \quad \gamma = y_0 \xrightarrow{d_{r_1}^{q_1}} y_1 \xrightarrow{d_{r_2}^{q_2}} \cdots \xrightarrow{d_{r_m}^{q_m}} y_m$$

of the same length $m \geq 2$ are said to be *ℓ-adjacent* $(1 \leq \ell < m)$, denoted $\pi \overset{\ell}{\longleftrightarrow} \gamma$, if $x_j = y_j$ for all $j \neq \ell$, $k_j = q_j$ and $i_j = r_j$ for all $j \neq \ell, \ell+1$, and one of the following conditions holds:

(i) $k_\ell = q_{\ell+1} = k_{\ell+1} = q_\ell = 0$ and $i_\ell = r_{\ell+1} < i_{\ell+1} = r_\ell + 1$
(ii) $q_\ell = k_{\ell+1} = q_{\ell+1} = k_\ell = 0$ and $r_\ell = i_{\ell+1} < r_{\ell+1} = i_\ell + 1$
(iii) $k_\ell = q_{\ell+1} = 0$, $k_{\ell+1} = q_\ell = 1$, and $i_\ell = r_{\ell+1} < i_{\ell+1} = r_\ell + 1$
(iv) $q_\ell = k_{\ell+1} = 0$, $q_{\ell+1} = k_\ell = 1$, and $r_\ell = i_{\ell+1} < r_{\ell+1} = i_\ell + 1$
(v) $k_\ell = q_{\ell+1} = 0$, $k_{\ell+1} = q_\ell = 1$, and $i_\ell = r_{\ell+1} + 1 > i_{\ell+1} = r_\ell$
(vi) $q_\ell = k_{\ell+1} = 0$, $q_{\ell+1} = k_\ell = 1$, and $r_\ell = i_{\ell+1} + 1 > r_{\ell+1} = i_\ell$
(vii) $k_\ell = q_{\ell+1} = k_{\ell+1} = q_\ell = 1$ and $i_\ell = r_{\ell+1} + 1 > i_{\ell+1} = r_\ell$
(viii) $q_\ell = k_{\ell+1} = q_{\ell+1} = k_\ell = 1$ and $r_\ell = i_{\ell+1} + 1 > r_{\ell+1} = i_\ell$

For example, the cube path considered above and the cube path

$$d_1^0 d_2^0 x \xrightarrow{d_1^0} d_2^0 x \xrightarrow{d_1^1} d_1^1 d_2^0 x \xrightarrow{d_1^0} d_1^1 x$$

with split trace $((a,0),(a,1),(b,0))$ are 2-adjacent, satisfying condition (v). Note that ℓ-adjacency is defined in [8] in a symmetric way using four conditions corresponding to our conditions (i), (vii), (iii), and (v). Applying these conditions to both paths, one obtains the eight conditions above.

Cube paths in $S\mathcal{Q}$ and the map ϕ

Consider a cube path

$$\pi = (\theta_0, x_0) \xrightarrow{d_{i_1}^{k_1}} (\theta_1, x_1) \xrightarrow{d_{i_2}^{k_2}} \cdots \xrightarrow{d_{i_m}^{k_m}} (\theta_m, x_m)$$

in $S\mathcal{Q}$. Then, depending on the value of k_j, either (θ_j, x_j) is a face of (θ_{j-1}, x_{j-1}) or vice versa. Therefore also either x_j is a face of x_{j-1} or vice versa, which suggests that π induces a cube path in $\mathcal{Q}$. This indeed turns out to be the case: we may define the *cube path induced by π* by setting

$$\phi(\pi) = x_0 \xrightarrow{d^{k_1}_{\theta^{-1}_{1-k_1}(i_1)}} x_1 \xrightarrow{d^{k_2}_{\theta^{-1}_{2-k_2}(i_2)}} \cdots \xrightarrow{d^{k_m}_{\theta^{-1}_{m-k_m}(i_m)}} x_m.$$

Let us check that $\phi(\pi)$ satisfies the conditions to be a cube path in $\mathcal{Q}$. Since $(\theta_0, x_0) = (id, I)$, we have $x_0 = I$. Since for all $j \in \{1, \dots, m\}$, $2\sum\limits_{p=1}^{j} k_p \leq j$, $(\theta_j, x_j) \in S_{j-2\sum_{p=1}^{j} k_p} \times P_{j-2\sum_{p=1}^{j} k_p}$, $i_j \in \{1, \dots, j-k_j-2\sum\limits_{p=1}^{j-k_j} k_p\}$, and

$$(d_{i_j}\theta_{j-k_j}, d^{k_j}_{\theta^{-1}_{j-k_j}(i_j)} x_{j-k_j}) = d^{k_j}_{i_j}(\theta_{j-k_j}, x_{j-k_j}) = (\theta_{j-1+k_j}, x_{j-1+k_j}),$$

we also have for all $j \in \{1, \dots, m\}$, $x_j \in P_{j-2\sum_{p=1}^{j} k_p}$, $\theta_{j-k_j} \in S_{j-k_j-2\sum_{p=1}^{j-k_j} k_p}$, $\theta^{-1}_{j-k_j}(i_j) \in \{1, \dots, j-k_j-2\sum\limits_{p=1}^{j-k_j} k_p\}$, and $d^{k_j}_{\theta^{-1}_{j-k_j}(i_j)} x_{j-k_j} = x_{j-1+k_j}$. Hence $\phi(\pi)$ is indeed a cube path in $\mathcal{Q}$.

The map ϕ will play a fundamental role in the proof of our main result in the next section. The remainder of this section is devoted to the behavior of ϕ with respect to adjacency.

Proposition 5.2. *Consider cube paths*

$$\pi = (\theta_0, x_0) \xrightarrow{d^{k_1}_{i_1}} (\theta_1, x_1) \xrightarrow{d^{k_2}_{i_2}} \cdots \xrightarrow{d^{k_m}_{i_m}} (\theta_m, x_m)$$

and

$$\gamma = (\sigma_0, y_0) \xrightarrow{d^{q_1}_{r_1}} (\sigma_1, y_1) \xrightarrow{d^{q_2}_{r_2}} \cdots \xrightarrow{d^{q_m}_{r_m}} (\sigma_m, y_m)$$

in $S\mathcal{Q}$ such that $\pi \overset{\ell}{\longleftrightarrow} \gamma$. Then $\phi(\pi) \overset{\ell}{\longleftrightarrow} \phi(\gamma)$.

Proof. By our hypothesis, $(\theta_j, x_j) = (\sigma_j, y_j)$ for all $j \neq \ell$, $k_j = q_j$ and $i_j = r_j$ for all $j \neq \ell, \ell+1$, and one of the conditions of Definition 5.1 holds. In all cases, $x_j = y_j$ for all $j \neq \ell$ and $k_j = q_j$ and $\theta^{-1}_{j-k_j}(i_j) = \sigma^{-1}_{j-q_j}(r_j)$ for all $j \neq \ell, \ell+1$. We will suppose that condition 5.1(i) holds, i.e.,

$$k_\ell = q_{\ell+1} = k_{\ell+1} = q_\ell = 0 \quad \text{and} \quad i_\ell = r_{\ell+1} < i_{\ell+1} = r_\ell + 1.$$

The arguments in the remaining situations are analogous. We have

$$(\theta_\ell, x_\ell) = d^0_{i_{\ell+1}}(\theta_{\ell+1}, x_{\ell+1}) = (d_{i_{\ell+1}}\theta_{\ell+1}, d^0_{\theta^{-1}_{\ell+1}(i_{\ell+1})} x_{\ell+1})$$

and therefore

$$\begin{aligned}\theta_\ell^{-1}(i_\ell) &= (d_{i_{\ell+1}}\theta_{\ell+1})^{-1}(i_\ell) = d_{\theta^{-1}_{\ell+1}(i_{\ell+1})}\theta^{-1}_{\ell+1}(i_\ell)\\ &= \begin{cases} \theta^{-1}_{\ell+1}(i_\ell) = \sigma^{-1}_{\ell+1}(r_{\ell+1}), & \theta^{-1}_{\ell+1}(i_\ell) < \theta^{-1}_{\ell+1}(i_{\ell+1}),\\ \theta^{-1}_{\ell+1}(i_\ell) - 1 = \sigma^{-1}_{\ell+1}(r_{\ell+1}) - 1, & \theta^{-1}_{\ell+1}(i_\ell) > \theta^{-1}_{\ell+1}(i_{\ell+1}).\end{cases}\end{aligned}$$

Since $q_{\ell+1} = 0$, $(\sigma_\ell, y_\ell) = d^0_{r_{\ell+1}}(\sigma_{\ell+1}, y_{\ell+1}) = (d_{r_{\ell+1}}\sigma_{\ell+1}, d^0_{\sigma^{-1}_{\ell+1}(r_{\ell+1})} y_{\ell+1}) = (d_{i_\ell}\theta_{\ell+1}, d^0_{\theta^{-1}_{\ell+1}(i_\ell)} x_{\ell+1})$ and therefore

$$\begin{aligned}\sigma_\ell^{-1}(r_\ell) &= (d_{i_\ell}\theta_{\ell+1})^{-1}(i_{\ell+1} - 1) = d_{\theta^{-1}_{\ell+1}(i_\ell)}\theta^{-1}_{\ell+1}(i_{\ell+1} - 1)\\ &= \begin{cases} \theta^{-1}_{\ell+1}(i_{\ell+1}), & \theta^{-1}_{\ell+1}(i_{\ell+1}) < \theta^{-1}_{\ell+1}(i_\ell),\\ \theta^{-1}_{\ell+1}(i_{\ell+1}) - 1, & \theta^{-1}_{\ell+1}(i_{\ell+1}) > \theta^{-1}_{\ell+1}(i_\ell).\end{cases}\end{aligned}$$

If $\sigma^{-1}_{\ell+1}(r_{\ell+1}) = \theta^{-1}_{\ell+1}(i_\ell) < \theta^{-1}_{\ell+1}(i_{\ell+1})$, we obtain

$$\theta_\ell^{-1}(i_\ell) = \sigma^{-1}_{\ell+1}(r_{\ell+1}) < \theta^{-1}_{\ell+1}(i_{\ell+1}) = \sigma_\ell^{-1}(r_\ell) + 1,$$

which shows that $\phi(\pi)$ and $\phi(\gamma)$ satisfy condition 5.1(i). If $\sigma^{-1}_{\ell+1}(r_{\ell+1}) = \theta^{-1}_{\ell+1}(i_\ell) > \theta^{-1}_{\ell+1}(i_{\ell+1})$, we obtain

$$\sigma_\ell^{-1}(r_\ell) = \theta^{-1}_{\ell+1}(i_{\ell+1}) < \sigma^{-1}_{\ell+1}(r_{\ell+1}) = \theta_\ell^{-1}(i_\ell) + 1,$$

which shows that $\phi(\pi)$ and $\phi(\gamma)$ satisfy condition 5.1(ii). Consequently, $\phi(\pi) \overset{\ell}{\longleftrightarrow} \phi(\gamma)$. □

As a partial converse to Proposition 5.2, we have the following result:

Proposition 5.3. *Consider cube paths*

$$\pi = (\theta_0, x_0) \xrightarrow{d^{k_1}_{i_1}} (\theta_1, x_1) \xrightarrow{d^{k_2}_{i_2}} \cdots \xrightarrow{d^{k_m}_{i_m}} (\theta_m, x_m)$$

and

$$\gamma = (\sigma_0, y_0) \xrightarrow{d^{q_1}_{r_1}} (\sigma_1, y_1) \xrightarrow{d^{q_2}_{r_2}} \cdots \xrightarrow{d^{q_m}_{r_m}} (\sigma_m, y_m)$$

in $S\mathcal{Q}$ such that $(\theta_j, x_j) = (\sigma_j, y_j)$ for all $j \neq \ell$, $k_j = q_j$ and $i_j = r_j$ for all $j \neq \ell, \ell+1$, and $\phi(\pi) \overset{\ell}{\longleftrightarrow} \phi(\gamma)$. Then $\pi \overset{\ell}{\longleftrightarrow} \gamma$.

Proof. Since the cube paths $\phi(\pi)$ and $\phi(\gamma)$ are ℓ-adjacent, they satisfy one of the conditions of Definition 5.1. We will suppose that condition (i) holds. The arguments in the remaining situations are analogous. So our hypothesis is that $k_\ell = q_{\ell+1} = k_{\ell+1} = q_\ell = 0$ and $\theta_\ell^{-1}(i_\ell) = \sigma^{-1}_{\ell+1}(r_{\ell+1}) < \theta^{-1}_{\ell+1}(i_{\ell+1}) = \sigma_\ell^{-1}(r_\ell) + 1$. We have

$$r_\ell = \sigma_\ell(\sigma_\ell^{-1}(r_\ell)) = d_{r_{\ell+1}}\theta_{\ell+1}(\theta_{\ell+1}^{-1}(i_{\ell+1})-1)$$
$$= \begin{cases} \theta_{\ell+1}(\theta_{\ell+1}^{-1}(i_{\ell+1})) = i_{\ell+1}, & i_{\ell+1} < r_{\ell+1}, \\ i_{\ell+1}-1, & i_{\ell+1} > r_{\ell+1} \end{cases}$$

and

$$i_\ell = \theta_\ell(\theta_\ell^{-1}(i_\ell)) = d_{i_{\ell+1}}\theta_{\ell+1}(\theta_{\ell+1}^{-1}(r_{\ell+1}))$$
$$= \begin{cases} \theta_{\ell+1}(\theta_{\ell+1}^{-1}(r_{\ell+1})) = r_{\ell+1}, & r_{\ell+1} < i_{\ell+1}, \\ r_{\ell+1}-1, & r_{\ell+1} > i_{\ell+1}. \end{cases}$$

Hence either $r_\ell = i_{\ell+1} < r_{\ell+1} = i_\ell + 1$ or $i_\ell = r_{\ell+1} < i_{\ell+1} = r_\ell + 1$, which shows that π and γ satisfy either condition 5.1(ii) or condition 5.1(i). Thus $\pi \overset{\ell}{\leftrightarrow} \gamma$. $\square$

The last result of this section is Proposition 5.5, which establishes an adjacency lifting property of ϕ. For the proof, we will need the following lemma:

Lemma 5.4. *Consider cube paths $\pi = (\theta_0, x_0)\xrightarrow{d_{i_1}^{k_1}}(\theta_1, x_1)\xrightarrow{d_{i_2}^{k_2}}\cdots\xrightarrow{d_{i_m}^{k_m}}(\theta_m, x_m)$ in $S\mathcal{Q}$ and $\rho = y_0\xrightarrow{d_{r_1}^{q_1}}y_1\xrightarrow{d_{r_2}^{q_2}}\cdots\xrightarrow{d_{r_m}^{q_m}}y_m$ in $\mathcal{Q}$ such that $\phi(\pi) \overset{\ell}{\leftrightarrow} \rho$. Then*

(i) for all $j \in \{1, \dots, m\}$, $2\sum\limits_{p=1}^{j} q_p \le j$;
(ii) for $j \ne \ell$, $(\theta_j, y_j) \in SP_{j-2\sum_{p=1}^{j} q_p}$;
(iii) for $j \ne \ell, \ell+1$, $i_j \in \{1, \dots, j - q_j - 2\sum\limits_{p=1}^{j-q_j} q_p\}$;
(iv) for $j \ne \ell, \ell+1$, $d_{i_j}^{q_j}(\theta_{j-q_j}, y_{j-q_j}) = (\theta_{j-1+q_j}, y_{j-1+q_j})$.

Proof. Statement (i) follows because ρ is a cube path. Since $\phi(\pi) \overset{\ell}{\leftrightarrow} \rho$, we have $k_j = q_j$ for all $j \ne \ell, \ell+1$ and $x_j = y_j$ and $\sum\limits_{p=1}^{j} k_p = \sum\limits_{p=1}^{j} q_p$ for all $j \ne \ell$. Since π is a cube path, this implies (ii). Since $j - q_j = j - k_j \ne \ell$ and $j - 1 + q_j = j - 1 + k_j \ne \ell$ for all $j \ne \ell, \ell+1$, (iii) and (iv) follow as well. $\square$

Proposition 5.5. *Let $\pi = (\theta_0, x_0)\xrightarrow{d_{i_1}^{k_1}}(\theta_1, x_1)\xrightarrow{d_{i_2}^{k_2}}\cdots\xrightarrow{d_{i_m}^{k_m}}(\theta_m, x_m)$ be a cube path in $S\mathcal{Q}$, and let $\rho = y_0\xrightarrow{d_{r_1}^{q_1}}y_1\xrightarrow{d_{r_2}^{q_2}}\cdots\xrightarrow{d_{r_m}^{q_m}}y_m$ be a cube path in $\mathcal{Q}$ such that $\phi(\pi) \overset{\ell}{\leftrightarrow} \rho$. Then there exists a cube path γ in $S\mathcal{Q}$ such that $\phi(\gamma) = \rho$ and $\pi \overset{\ell}{\leftrightarrow} \gamma$.*

Proof. Since $\phi(\pi) \overset{\ell}{\leftrightarrow} \rho$, we have $x_j = y_j$ for all $j \ne \ell$ and $k_j = q_j$ for all $j \ne \ell, \ell+1$. By Proposition 5.3, it is therefore enough to construct a cube path

$$\gamma = (\sigma_0, y_0)\xrightarrow{d_{s_1}^{q_1}}(\sigma_1, y_1)\xrightarrow{d_{s_2}^{q_2}}\cdots\xrightarrow{d_{s_m}^{q_m}}(\sigma_m, y_m)$$

such that $\sigma_j = \theta_j$ for all $j \ne \ell$, $s_j = i_j$ for all $j \ne \ell, \ell+1$, and $\phi(\gamma) = \rho$. In order to start the construction of γ, we set $\sigma_j = \theta_j$ for $j \ne \ell$ and $s_j = i_j$ for $j \ne \ell, \ell+1$, as required. It is then clear that $(\sigma_0, y_0) = (id, I)$. Moreover, since $j - q_j = j - k_j \ne \ell$ for $j \ne \ell, \ell+1$, the fact that $\phi(\pi) \overset{\ell}{\leftrightarrow} \rho$ implies that $\sigma_{j-q_j}^{-1}(s_j) = \theta_{j-k_j}^{-1}(i_j) = r_j$ for $j \ne \ell, \ell+1$. Therefore, by Lemma 5.4, to finish the construction of γ, it remains to define the permutation $\sigma_\ell \in S_{\ell-2\sum_{p=1}^{\ell} q_p}$ and to check that with $s_\ell = \sigma_{\ell-q_\ell}(r_\ell)$ and $s_{\ell+1} = \sigma_{\ell+1-q_{\ell+1}}(r_{\ell+1})$, one has $d_{s_\ell}^{q_\ell}(\sigma_{\ell-q_\ell}, y_{\ell-q_\ell}) = (\sigma_{\ell-1+q_\ell}, y_{\ell-1+q_\ell})$ and $d_{s_{\ell+1}}^{q_{\ell+1}}(\sigma_{\ell+1-q_{\ell+1}}, y_{\ell+1-q_{\ell+1}}) = (\sigma_{\ell+q_{\ell+1}}, y_{\ell+q_{\ell+1}})$.

By our hypothesis, $\phi(\pi)$ and ρ satisfy one of the conditions of Definition 5.1. We will only consider conditions 5.1(i), (iii), and (iv). In each of the remaining situations, the arguments are analogous to those used in one of these three cases.

Suppose first that $\phi(\pi)$ and ρ satisfy condition 5.1(i). Then $k_\ell = q_{\ell+1} = k_{\ell+1} = q_\ell = 0$ and $\theta_\ell^{-1}(i_\ell) = r_{\ell+1} < \theta_{\ell+1}^{-1}(i_{\ell+1}) = r_\ell + 1$. In this situation, we set $\sigma_\ell = d_{\theta_{\ell+1}(r_{\ell+1})}\theta_{\ell+1} = d_{\sigma_{\ell+1}(r_{\ell+1})}\sigma_{\ell+1}$. Since $\sigma_{\ell+1} \in S_{\ell+1-2\sum_{p=1}^{\ell+1} q_p}$ and $r_{\ell+1} \in$

$\{1,\dots,\ell+1-2\sum_{p=1}^{\ell+1} q_p\}$, σ_ℓ is a well-defined element of $S_{\ell-2\sum_{p=1}^{\ell} q_p}$. We compute $d^0_{s_{\ell+1}}(\sigma_{\ell+1}, y_{\ell+1}) = d^0_{\sigma_{\ell+1}(r_{\ell+1})}(\sigma_{\ell+1}, y_{\ell+1}) = (d_{\sigma_{\ell+1}(r_{\ell+1})}\sigma_{\ell+1}, d^0_{r_{\ell+1}} y_{\ell+1}) = (\sigma_\ell, y_\ell)$ and, using Proposition 3.6,

$$
\begin{aligned}
d^0_{s_\ell}(\sigma_\ell, y_\ell) &= (d_{s_\ell}\sigma_\ell, d^0_{\sigma_\ell^{-1}(s_\ell)} y_\ell) = (d_{\sigma_\ell(r_\ell)}\sigma_\ell, d^0_{\sigma_\ell^{-1}(\sigma_\ell(r_\ell))} y_\ell) \\
&= (d_{d_{\theta_{\ell+1}(r_{\ell+1})}\theta_{\ell+1}(r_\ell)} d_{\theta_{\ell+1}(r_{\ell+1})}\theta_{\ell+1}, d^0_{r_\ell} y_\ell) \\
&= (d_{d_{\theta_{\ell+1}(r_{\ell+1})}\theta_{\ell+1}(r_\ell+1-1)} d_{\theta_{\ell+1}(r_{\ell+1})}\theta_{\ell+1}, y_{\ell-1}) \\
&= (d_{d_{\theta_{\ell+1}(r_\ell+1)}\theta_{\ell+1}(r_{\ell+1})} d_{\theta_{\ell+1}(r_\ell+1)}\theta_{\ell+1}, y_{\ell-1}) \\
&= (d_{d_{i_{\ell+1}}\theta_{\ell+1}(r_{\ell+1})} d_{i_{\ell+1}}\theta_{\ell+1}, y_{\ell-1}) \\
&= (d_{\theta_\ell(r_{\ell+1})}\theta_\ell, y_{\ell-1}) = (d_{i_\ell}\theta_\ell, y_{\ell-1}) = (\theta_{\ell-1}, y_{\ell-1}) = (\sigma_{\ell-1}, y_{\ell-1}).
\end{aligned}
$$

Suppose now that $\phi(\pi)$ and ρ satisfy condition 5.1(iii). Then $k_\ell = q_{\ell+1} = 0$, $k_{\ell+1} = q_\ell = 1$, and $\theta_\ell^{-1}(i_\ell) = r_{\ell+1} < \theta_\ell^{-1}(i_{\ell+1}) = r_\ell + 1$. Set $\sigma_\ell = d_{\theta_{\ell+1}(r_{\ell+1})}\theta_{\ell+1}$. As before, σ_ℓ is a well-defined element of $S_{\ell-2\sum_{p=1}^{\ell} q_p}$. Since $r_\ell \geq r_{\ell+1} = \theta_\ell^{-1}(i_\ell)$, we have

$$
s_\ell = \theta_{\ell-1}(r_\ell) = d_{i_\ell}\theta_\ell(r_\ell) = \begin{cases} \theta_\ell(r_\ell+1) = i_{\ell+1}, & i_{\ell+1} < i_\ell, \\ \theta_\ell(r_\ell+1) - 1 = i_{\ell+1} - 1, & i_{\ell+1} > i_\ell. \end{cases}
$$

Since $r_{\ell+1} < \theta_\ell^{-1}(i_{\ell+1})$, we have

$$
s_{\ell+1} = \theta_{\ell+1}(r_{\ell+1}) = d_{i_{\ell+1}}\theta_\ell(r_{\ell+1}) = \begin{cases} \theta_\ell(r_{\ell+1}) = i_\ell, & i_\ell < i_{\ell+1}, \\ \theta_\ell(r_{\ell+1}) - 1 = i_\ell - 1, & i_\ell > i_{\ell+1}. \end{cases}
$$

Thus either $s_\ell = i_{\ell+1} < i_\ell = s_{\ell+1} + 1$ or $i_\ell = s_{\ell+1} < i_{\ell+1} = s_\ell + 1$. In the first situation,

$$
d_{s_\ell}\sigma_{\ell-1} = d_{i_{\ell+1}}\theta_{\ell-1} = d_{i_{\ell+1}} d_{i_\ell}\theta_\ell = d_{i_\ell-1} d_{i_{\ell+1}}\theta_\ell = d_{s_{\ell+1}}\theta_{\ell+1} = d_{\theta_{\ell+1}(r_{\ell+1})}\theta_{\ell+1} = \sigma_\ell.
$$

In the second situation,

$$
d_{s_\ell}\sigma_{\ell-1} = d_{i_{\ell+1}-1}\theta_{\ell-1} = d_{i_{\ell+1}-1} d_{i_\ell}\theta_\ell = d_{i_\ell} d_{i_{\ell+1}}\theta_\ell = d_{s_{\ell+1}}\theta_{\ell+1} = d_{\theta_{\ell+1}(r_{\ell+1})}\theta_{\ell+1} = \sigma_\ell.
$$

Hence $d^1_{s_\ell}(\sigma_{\ell-1}, y_{\ell-1}) = (d_{s_\ell}\sigma_{\ell-1}, d^1_{\sigma_{\ell-1}^{-1}(s_\ell)} y_{\ell-1}) = (\sigma_\ell, d^1_{r_\ell} y_{\ell-1}) = (\sigma_\ell, y_\ell)$ and $d^0_{s_{\ell+1}}(\sigma_{\ell+1}, y_{\ell+1}) = d^0_{\theta_{\ell+1}(r_{\ell+1})}(\theta_{\ell+1}, y_{\ell+1}) = (d_{\theta_{\ell+1}(r_{\ell+1})}\theta_{\ell+1}, d^0_{r_{\ell+1}} y_{\ell+1}) = (\sigma_\ell, y_\ell)$.

Suppose finally that $\phi(\pi)$ and ρ satisfy condition 5.1(iv). Then $k_\ell = q_{\ell+1} = 1$, $k_{\ell+1} = q_\ell = 0$, and $r_\ell = \theta_{\ell+1}^{-1}(i_{\ell+1}) < r_{\ell+1} = \theta_{\ell-1}^{-1}(i_\ell) + 1$. We also assume that $i_\ell \leq i_{\ell+1}$ and leave the analogous case $i_\ell > i_{\ell+1}$ to the reader. Since $d_{i_\ell}\theta_{\ell-1} = \theta_\ell = d_{i_{\ell+1}}\theta_{\ell+1}$ and $\theta_{\ell-1}^{-1}(i_\ell) \geq \theta_{\ell+1}^{-1}(i_{\ell+1})$, Proposition 3.4 implies that there exists a permutation $\sigma_\ell \in S_{\ell-2\sum_{p=1}^{\ell-1} k_p} = S_{\ell-2\sum_{p=1}^{\ell} q_p}$ such that $d_{i_\ell}\sigma_\ell = \theta_{\ell+1}$, $d_{i_{\ell+1}+1}\sigma_\ell = \theta_{\ell-1}$, and $\sigma_\ell^{-1}(i_\ell) > \sigma_\ell^{-1}(i_{\ell+1}+1)$. We have

$$
r_\ell = \theta_{\ell+1}^{-1}(i_{\ell+1}) = (d_{i_\ell}\sigma_\ell)^{-1}(i_{\ell+1}) = d_{\sigma_\ell^{-1}(i_\ell)}\sigma_\ell^{-1}(i_{\ell+1}) = \sigma_\ell^{-1}(i_{\ell+1}+1)
$$

and

$$
r_{\ell+1} = \theta_{\ell-1}^{-1}(i_\ell) + 1 = (d_{i_{\ell+1}+1}\sigma_\ell)^{-1}(i_\ell) + 1 = d_{\sigma_\ell^{-1}(i_{\ell+1}+1)}\sigma_\ell^{-1}(i_\ell) + 1 = \sigma_\ell^{-1}(i_\ell).
$$

Hence $s_\ell = \sigma_\ell(r_\ell) = i_{\ell+1} + 1$ and $s_{\ell+1} = \sigma_\ell(r_{\ell+1}) = i_\ell$. We therefore have $d^0_{s_\ell}(\sigma_\ell, y_\ell) = d^0_{i_{\ell+1}+1}(\sigma_\ell, y_\ell) = (d_{i_{\ell+1}+1}\sigma_\ell, d^0_{\sigma_\ell^{-1}(i_{\ell+1}+1)} y_\ell) = (\theta_{\ell-1}, d^0_{r_\ell} y_\ell) = (\sigma_{\ell-1}, y_{\ell-1})$ and $d^1_{s_{\ell+1}}(\sigma_\ell, y_\ell) = d^1_{i_\ell}(\sigma_\ell, y_\ell) = (d_{i_\ell}\sigma_\ell, d^1_{\sigma_\ell^{-1}(i_\ell)} y_\ell) = (\theta_{\ell+1}, d^1_{r_{\ell+1}} y_\ell) = (\sigma_{\ell+1}, y_{\ell+1})$. □

6. Hereditary history-preserving bisimilarity

A *hereditary history-preserving bisimulation* between two HDAs is a relation R between their cube paths such that the following conditions hold:

(1) The cube paths of length 0 are related.
(2) If $\pi R \rho$, then $\textit{split-trace}(\pi) = \textit{split-trace}(\rho)$.
(3) If $\pi R \rho$ and $\pi \stackrel{\ell}{\leftrightarrow} \pi'$, then $\exists \rho'$ with $\rho \stackrel{\ell}{\leftrightarrow} \rho'$ and $\pi' R \rho'$.
(4) If $\pi R \rho$ and $\rho \stackrel{\ell}{\leftrightarrow} \rho'$, then $\exists \pi'$ with $\pi \stackrel{\ell}{\leftrightarrow} \pi'$ and $\pi' R \rho'$.
(5) If $\pi R \rho$ and $\pi \to \pi'$, then $\exists \rho'$ with $\rho \to \rho'$ and $\pi' R \rho'$.
(6) If $\pi R \rho$ and $\rho \to \rho'$, then $\exists \pi'$ with $\pi \to \pi'$ and $\pi' R \rho'$.
(7) If $\pi R \rho$, then $\textit{end}(\pi)$ is a final state if and only if $\textit{end}(\rho)$ is a final state.
(8) If $\pi R \rho$ and $\pi' \to \pi$, then $\exists \rho'$ with $\rho' \to \rho$ and $\pi' R \rho'$.
(9) If $\pi R \rho$ and $\rho' \to \rho$, then $\exists \pi'$ with $\pi' \to \pi$ and $\pi' R \rho'$.

Two HDAs are called *hhp-bisimilar* if there exists a hereditary history-preserving bisimulation between them. Hereditary history-preserving bisimilarity has been introduced in [8], along with some weaker concepts of bisimilarity for HDAs. Further notions of bisimilarity for HDAs can be found in [1,4].

The following theorem is the main result of this paper:

Theorem 6.1. *Let $\mathcal{Q}$ be an HDA. Then $\mathcal{Q}$ and $S\mathcal{Q}$ are hhp-bisimilar.*

Proof. Consider the relation R on cube paths of $S\mathcal{Q}$ and $\mathcal{Q}$ defined by

$$\pi R \rho \Leftrightarrow \rho = \phi(\pi),$$

where $\phi(\pi)$ is the cube path defined in Section 5. We show that R is a hereditary history-preserving bisimulation. Properties (1), (5), (7), (8), and (9) are obvious. Property (3) follows from Proposition 5.2. Property (4) follows from Proposition 5.5. It remains to establish properties (2) and (6).

(2) Let λ and μ denote the labeling functions of $\mathcal{Q}$ and $S\mathcal{Q}$, respectively. By Proposition 4.3, we have

$$\begin{aligned} \textit{split-trace}(\pi) &= ((\mu(e_{i_j}(\theta_{j-k_j}, x_{j-k_j}), k_j))_{j=1,\dots,m} \\ &= ((\mu(id, e_{\theta^{-1}_{j-k_j}(i_j)} x_{j-k_j}), k_j))_{j=1,\dots,m} \\ &= ((\lambda(e_{\theta^{-1}_{j-k_j}(i_j)} x_{j-k_j}), k_j))_{j=1,\dots,m} \\ &= \textit{split-trace}(\phi(\pi)). \end{aligned}$$

(6) Consider a cube path π in $S\mathcal{Q}$, and suppose that $\phi(\pi) \to \rho'$. We may suppose that $\rho' = \phi(\pi) \xrightarrow{d^k_r} y$; the general case follows by induction. Let $\textit{end}(\pi) = (\theta, x)$, and suppose that $(\theta, x) \in S_n \times P_n$. Then $\textit{end}(\phi(\pi)) = x \in P_n$. Suppose first that $k = 1$. Then $y \in P_{n-1}$, $r \in \{1, \dots, n\}$, and $y = d^1_r x$. Since

$$d^1_{\theta(r)}(\theta, x) = (d_{\theta(r)}\theta, d^1_{\theta^{-1}(\theta(r))} x) = (d_{\theta(r)}\theta, y),$$

we may extend π to $\pi' = \pi \xrightarrow{d^1_{\theta(r)}} (d^1_{\theta(r)}\theta, y)$. We then have $\phi(\pi') = \phi(\pi) \xrightarrow{d^1_r} y = \rho'$, i.e., $\pi' R \rho'$.

Suppose now that $k = 0$. Then $y \in P_{n+1}$, $r \in \{1, \dots, n+1\}$, and $x = d^0_r y$. Define $\sigma \in S_{n+1}$ by

$$\sigma = (\theta(1)^{\uparrow 1} \ \theta(2)^{\uparrow 1} \ \cdots \ \theta(r-1)^{\uparrow 1} \ 1 \ \theta(r)^{\uparrow 1} \ \cdots \ \theta(n)^{\uparrow 1}).$$

Then $\sigma^{-1}(1) = r$ and

$$d_1\sigma = (\theta(1)^{\uparrow 1 \downarrow 1} \ \theta(2)^{\uparrow 1 \downarrow 1} \ \cdots \ \theta(r-1)^{\uparrow 1 \downarrow 1} \ \theta(r)^{\uparrow 1 \downarrow 1} \ \cdots \ \theta(n)^{\uparrow 1 \downarrow 1}) = \theta.$$

Hence

$$d^0_1(\sigma, y) = (d_1\sigma, d^0_{\sigma^{-1}(1)} y) = (\theta, d^0_r y) = (\theta, x).$$

We may therefore extend π to $\pi' = \pi \xrightarrow{d^0_1} (\sigma, y)$. Since $\sigma^{-1}(1) = r$, we have $\phi(\pi') = \phi(\pi) \xrightarrow{d^0_r} x = \rho'$, i.e., $\pi' R \rho'$. $\square$

If one views symmetric HDAs as HDAs of a particular type, it is natural to define two symmetric HDAs to be hhp-bisimilar if they are hhp-bisimilar as HDAs. From this point of view, symmetric HDAs are a priori at most as expressive as ordinary HDAs. Since, by Theorem 6.1, every HDA is hhp-bisimilar to a symmetric one, symmetric HDAs are actually as expressive as ordinary HDAs. The fact that symmetric HDAs are at least as expressive as ordinary HDAs can also be inferred from the following corollary of Theorem 6.1:

Corollary 6.2. *Two HDAs $\mathcal{Q}$ and $\mathcal{Q}'$ are hhp-bisimilar if and only if $S\mathcal{Q}$ and $S\mathcal{Q}'$ are hhp-bisimilar.*

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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