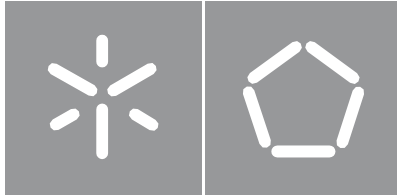




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A critical analysis on friction force
models in dynamical systems



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RESUMO

A critical analysis on friction force models in dynamical systems

As sociedades atuais têm enfrentado bastantes desafios no que diz respeito à economia de recursos naturais, reciclagem e reutilização destes mesmos recursos, assim como na procura por novas fontes de energia mais sustentáveis. O contacto e movimento relativo entre as superfícies de componentes mecânicos são responsáveis pela dissipação de percentagens consideráveis de energia sob a forma de ruído, calor, entre outras, contribuindo para um desgaste acelerado e falha prematura destes mesmos componentes.

A correta modelação e previsão do comportamento dinâmico de componentes e mecanismos dos mais variados tipos de máquinas e equipamentos recorrendo a ferramentas computacionais é crucial para o correto controlo do seu funcionamento e prolongamento da sua vida útil, possibilitando a economia de recursos naturais e energéticos que de outra forma seriam gastos na manufatura de componentes de substituição.

Deste modo, este trabalho centra-se, principalmente, na análise crítica de modelos de força de atrito teóricos, comparando o desempenho computacional destes na previsão do comportamento dinâmico de sistemas mecânicos de complexidade crescente na presença de fenómenos dissipativos relacionados com as forças de atrito que se desenvolvem entre os diferentes tipos de contacto.

Depois da análise dos conceitos fundamentais e teóricos associados aos fenómenos de atrito são analisadas as formulações dos modelos de atrito estáticos e dinâmicos para contactos secos, comumente aplicados para a simulação destes fenómenos dissipativos em problemas que se inserem no quadro da dinâmica de sistemas multicorpo.

O projeto de uma bancada simples e versátil para a determinação de dados relativos a forças de atrito é apresentado com o objetivo de calibrar e prever os parâmetros numéricos associados aos diferentes modelos de atrito existentes na literatura e que melhor permitem representar as propriedades estruturais, físicas e cinemáticas de um determinado sistema mecânico.

O estudo da influência dos modelos de força de atrito apresentado neste trabalho culmina na análise de um sistema multicorpo, onde é feita uma análise qualitativa e quantitativa dos efeitos destes fenómenos dissipativos no comportamento dinâmico da suspensão de um carro do tipo Formula Student.

Palavras-chave: Bancada Experimental, Dinâmica de Sistemas Multicorpo, Modelos de Atrito, Simulação Computacional

ABSTRACT

A critical analysis on friction force models in dynamical systems

Current societies have been facing considerable challenges regarding the management of natural resources, recycling, and the search for new and more sustainable energy sources. The contact and relative movement between the surfaces of mechanical components are responsible for the dissipation of significant amounts of energy, which is wasted in the form of noise, heat, among others, contributing to accelerated wear and premature failure of these components.

Proper modeling and prediction of the dynamic behavior of components and mechanisms in various types of machines and equipment utilizing computational resources is crucial for controlling their operation and extending their lifespan, enabling the conservation of natural and energy resources that would otherwise be spent on manufacturing replacement components.

Thus, this work focuses primarily on the critical analysis of theoretical friction force models, comparing their computational performance in predicting the dynamic behavior of mechanical systems of increasing complexity in the presence of dissipative phenomena related to the friction forces developed between different contact types.

After analyzing the fundamental and theoretical concepts associated with friction phenomena, static and dynamic friction models for dry contacts which are commonly applied for simulating these dissipative phenomena in problems within the framework of multibody systems dynamics are examined in detail.

The design of a simple and versatile experimental apparatus for determining data related to friction forces is presented with the aim of calibrating and predicting the numerical parameters associated with different friction models found in the literature, which best represent the structural, physical, and kinematic properties of a specific mechanical system.

The study of the influence of friction force models presented in this work culminates in the analysis of a multibody system, where a qualitative and quantitative analysis of the effects of these dissipative phenomena on the dynamic behavior of the suspension system of a Formula Student car is carried out.

Keywords: Computational Analysis, Experimental Apparatus, Friction Models, Multibody Dynamics

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LIST OF ABBREVIATIONS

Abbreviation	Description
CAD	Computer-Aided Design
CAE	Computer-Aided Engineering
CG	Center of Gravity
CNC	Computer Numerical Control
DC	Direct Current
DoF	Degree(s) of Freedom
FS	Formula Student
FSAE	Formula SAE
FSUM	Formula Student of University of Minho
ga	Genetic algorithm
MBD	Multibody Dynamics
MF	Magic Formula Tire Model
NI	National Instruments
SA	Shock Absorber
SAE	Society of Automobile Engineers
s-d	Spring-damper element within multibody dynamics framework
TTC	Tire Test Consortium

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LIST OF SYMBOLS

Chapter 2

Symbol	SI Unit	Description
n_{DoF}	-	Number of Degrees of Freedom
n_b	-	Number of Bodies
$\mathbf{q}_i$	-	Generalized coordinate vector for i -index body
$\mathbf{r}_i$	-	Translational coordinates of i -index body in global coordinates
$\mathbf{p}_i$	-	Euler parameters vector of i -index body
ϕ	rad	Angle of rotation
$\mathbf{u}_e$	-	Unit vector
$\mathbf{v}_i$	-	Generalized velocities vector for i -index body
$\dot{\mathbf{r}}_i$	-	Vector of translational velocities
$\ddot{\mathbf{r}}_i$	-	Vector of translational accelerations
$\boldsymbol{\omega}_i$	-	Vector of angular velocities
$\dot{\boldsymbol{\omega}}_i$	-	Vector of angular accelerations
$\Phi(q, t)$	-	Set of holonomic constraints
t	s	Time variable
$\dot{\Phi}$	-	Velocity constraint equations in matrix form
$\mathbf{D}$	-	Jacobian matrix
$\mathbf{U}$	-	Right-hand side of the velocities constraint equations
$\ddot{\Phi}$	-	Acceleration constraint equations
$\ddot{\mathbf{v}}_i$	-	Generalized accelerations vector
$\boldsymbol{\gamma}$	-	Right-hand side of the acceleration constraint equations
$\mathbf{M}$	-	Mass matrix
$\mathbf{g}$	-	Generalized forces and gyroscopic moments
$\boldsymbol{\lambda}$	-	Vector of Lagrange multipliers
α	-	Velocity feedback control parameter of Baumgarte stabilization method

β	-	Position feedback control parameter of Baumgarte stabilization method
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Chapter 3

Symbol	SI Unit	Description
α	rad	Inclination angle
f	N	External force
f_g	N	Gravity force

Chapter 4

Symbol	SI Unit	Description
f_i	N	Intrinsic frictional force
f_a	N	Force required to shear adhered junctions
f_d	N	Force associated with deformation energy
v_t	m/s	Relative tangential velocity
A	m ²	Total real area of contact
W	N	Normal load
f	N	External force
f_g	N	Gravity force
f_n	N	Normal force
f_f	N	Friction force
m	kg	Mass of the block
g	m/s ²	Gravity acceleration
θ	rad	Inclination angle
μ	-	Friction coefficient
μ_k	-	Kinetic friction coefficient
μ_s	-	Static friction coeffic
f_e	N	Resultant of external forces applied
f_c	N	Coulomb friction force magnitude
M_r	N·m	Rolling resistance moment

Chapter 5

Symbol	SI Unit	Description
μ_k	-	Kinetic friction coefficient
μ_s	-	Static friction coefficient
f_s	N	Static friction force magnitude
f_C	N	Coulomb friction force magnitude
v_t	m/s	Relative tangential velocity
f_v	N	Viscous friction force
δ_v	-	Non-linear velocity dependent variable
$f(v_t)$	-	Arbitrary velocity dependent friction function
v_s	m/s	Stribeck velocity
γ	-	Constant factor defined to control the shape of the Stribeck curve
δ	-	Factor dependent on the geometry of contacting surfaces
λ	-	Specific curve fitting parameter
α	-	Specific curve fitting parameter
t	s	Time
k	N/m	Constant slope of the elastic range of friction force-displacement relationship
s	m	Displacement
Δt	s	Small time interval
f_{fu}	N	Friction force behavior for the upper curve acceleration phase
f_{fl}	N	Friction force behavior for the lower curve deceleration phase
a	-	Parameter to adjust the acceleration phase slope
b	-	Parameter to adjust the deceleration phase slope

Chapter 6

Symbol	SI Unit	Description
μ_k	-	Kinetic friction coefficient
μ_s	-	Static friction coefficient
f_s	N	Static friction force magnitude
f_C	N	Coulomb friction force magnitude
v_t	m/s	Relative tangential velocity
f_n	N	Normal force
k_l	N/m	Linear slope of the transition from positive to negative kinetic friction force
e	-	Exponential function
v_0	m/s	Threshold velocity
v_1	m/s	Threshold velocity
α	-	Fitting parameter associated with surface roughness or bonding affinity
β	-	Fitting parameter associated with surface roughness or bonding affinity
f_r	-	Ratio of static and kinetic friction forces
ξ	-	Positive parameter associated with the negative slope of the Stribeck effect
k_t	N/m	Tangential stiffness of the static contact
x	m	Displacement
f_v	N·s/m	Viscous friction force
γ	-	Temporal parameter related to the rising slope of the static friction
t_2	s	Dwell-time
τ_1	s	Frictional lag time constant
v_s	m/s	Stribeck characteristic velocity
$f_{s,a}$	N	Stribeck friction force at the end of the previous sliding phase
$f_{s,\infty}$	N	Stribeck friction force for an external force applied at an infinitely small rate
μ_{Hv}	-	Hollars viscous friction coefficient

μ_v	-	Viscous friction coefficient
p	-	Decaying parameter of the transition from static to kinetic friction
$k_{\tanh}$	N/m	Constant rising slope from null friction force to stiction level
k_s	N/m	Contact stiffness
$\dot{v}_t$	m/s ²	Tangential acceleration
f_0	N	Friction force correspondent to the initial value of the sticking phase
f_{d+}	N	Kinetic friction force related to the acceleration phase
f_{d-}	N	Kinetic friction force related to the deceleration phase
$f(v_t)$	N	Breakaway force
$\Delta f(v_t)$	N	Range of breakaway variation
v_A	m/s	Adjusting parameter representing an average value of the Stribeck velocity
$f_R(x_t, v_t)$	N	Stochastic function of the tangential displacement and velocity
$g(v_t, \dot{v}_t)$	-	Nonlinear function of tangential velocity and acceleration
τ	s	Time delay
v_{Sp}	m/s	Stribeck peak velocity
c_v	N·s/m	Damping friction coefficient

Chapter 7

Symbol	SI Unit	Description
f_f	N	Friction force
f_c	N	Coulomb friction force
σ_0	N/m	Bristle stiffness
α	-	Parameter defining the shape of the stress-strain curve
z	m	Bristle deflection
σ_i	N/m	Stiffness of an individual bristle
b_i	m	Reference position of the initial bond of the bristles

x_i	m	Relative position of the bristles
δ_i	m	Strain limit of a bristle
Δ	-	Bristle range
v_t	m/s	Relative tangential velocity
z_0	m	Pre-determined value of bristle deflection
a	-	Static and kinetic friction force ratio
f_s	N	Static friction force
σ_1	N·s/m	Bristle damping coefficient
s	-	Space variable/Sticking state function
t	s	Time variable
x_s	m	Absolute relative displacement
$\frac{dz}{dt}$	m/s	Bristle deflection time rate
σ_2	N·s/m	Viscous friction coefficient
d	m	Separation distance from the interacting surfaces planes
h	m	Relative height to a reference plane
δ	m	Asperity deformation
d_∞	m	Separation distance between reference planes when tending to infinite
μ_s	-	Static friction coefficient
σ	Pa	Stress induced in the asperities
$\bar{\delta}$	m	Average asperity deformation
$F_h(z)$	N	Nonlinear hysteretic function
F_b	N	Hysteretic behavior force in the beginning of the transition curve
$F_d(z)$	N	Point-symmetrical strictly crescent function
$S(v_t)$	N	Constant velocity friction force in the sliding phase
Ψ	-	Curve fitting parameter
w	m	Plastic displacement

$F_{f\text{ss}}(v_t)$	N	Friction force in the steady-state
i	-	Integer exponent coefficient
$\alpha(z, v_t)$	-	Piecewise function to capture stiction
z_{ba}	m	Breakaway bristle deflection
$z_{\text{ss}}(v_t)$	m	Steady-state bristle deflection
z_{max}	m	Maximum bristle deflection
C_i	-	Attraction parameter
k_i	N/m	Stiffness of each elementary spring used to model the asperities behavior
s_i	-	Stribeck velocity function
v_i	-	Velocity scale parameter
$\frac{dz_{\text{st}}}{dt}$	m/s	Static bristle deflection rate
$\frac{dz_{\text{sl}}}{dt}$	m/s	Dynamic/sliding bristle deflection rate
$\text{dir}_0(v_t, v_0)$	-	Piecewise function of the tangential velocity and threshold velocity
v_0	m/s	Threshold velocity
s_{dw}	-	Dwell-time state variable
τ_{dw}	s	Bristle dwell-time constant
z_{smax}	m	Maximum static bristle deflection
z_{kmax}	m	Maximum kinetic bristle deflection
t_0	m/s	Initial time instant of contact

Chapter 8

Symbol	SI Unit	Description
f_{ext}	N	External force
f_g	N	Gravity force
f_N	N	Normal force
f_f	N	Friction force
$\ddot{x}$	m/s ²	Acceleration
m	kg	Mass
μ_s	-	Static friction coefficient
μ_k	-	Kinetic friction coefficient
v_s	m/s	Stribeck velocity
σ_0	N/m	Bristle stiffness constant
σ_1	N·s/m	Bristle damping constant
σ_2	N·s/m	Viscous friction coefficient
f_s	N	Static friction force
f_c	N	Coulomb friction force
x	-	Non-normalized parameter
x_{norm}	-	Normalized parameter
LB_{norm}	-	Lower bound of normalized parameter
UB_{norm}	-	Upper bound of normalized parameter
LB	-	Lower bound of non-normalized parameter
UB	-	Upper bound of non-normalized parameter
x_{lit}	-	Friction model specific parameter value from literature
x_{opt}	-	Optimized friction force model specific parameter

Chapter 9

Symbol	SI Unit	Description
$\mathbf{p}_i$	-	Vector of Euler parameters
e_0, e_1, e_2, e_3	-	Euler parameters
$\mathbf{u}_e$	-	Unit vector
$\mathbf{A}$	-	Transformation matrix
ϕ	rad	Rotation angle
$\xi\eta\zeta$	-	Local reference frame
xyz	-	Global reference frame
zxz	-	Euler angle convention
D, C, B	-	Individual rotation matrices
$\text{tr } \mathbf{A}$	-	Trace of transformation matrix $\mathbf{A}$
f_z	N	Normal/vertical tire force
κ	-	Slip
α	rad	Slip angle
γ	rad	Camber angle
f_x	N	Longitudinal tire force
f_y	N	Lateral tire force
M_x	N·m	Overturning moment
M_y	N·m	Rolling resistance moment
M_z	N·m	Aligning torque
C_z	N/m	Tire radial stiffness
ρ_z	m	Vertical tire compression
ζ, η	m	Elliptical tire cross-section shape parameters
a	m	Unloaded tire radius
b	m	Tire width
d_z	N·s/m	Tire damping coefficient
$\dot{\rho}_z$	m/s	Vertical tire compression rate

c_{crit}	N·s/m	Critical damping
m_t	kg	Tire mass
$\mathbf{a}_i$	-	Longitudinal unit vector
$\mathbf{m}_j$	-	Unit vector
$\mathbf{n}_j$	-	Unit vector
$\mathbf{d}$	-	Vector defined between P_i and P_j points
Φ	-	Set of constraint equations
$\mathbf{g}$	-	Vector of external generalized forces
$\mathbf{M}$	-	Mass matrix
$\dot{\mathbf{v}}$	-	Acceleration vector
$\mathbf{D}^T$	-	Transpose of the Jacobian matrix
λ	-	Vector of Lagrange multipliers
$\tilde{\mathbf{a}}_i, \mathbf{m}_j, \tilde{\mathbf{s}}_i^P, \tilde{\mathbf{s}}_j^P, \mathbf{d}$	-	Skew-symmetric matrices of the respective vectors
$\lambda_1, \lambda_2, \lambda_3, \lambda_4$	-	Reaction forces
$\mathbf{f}_f(\lambda_i)$	-	Generalized friction forces vector obtained from reaction forces
P_{C1}	-	Point contact 1
P_{C2}	-	Point contact 2
d	-	Norm of vector $\mathbf{d}$
P_{ii}	-	Auxiliar point in i -index body
P_{jj}	-	Auxiliar point in j -index body
d_{ii}	m	Immutable dimension on i -index body
d_{jj}	m	Immutable dimension on j -index body
v_t	m/s	Tangential relative velocity
f_{NCP}	N	Normal force in contact point
f_{fCP}	N	Friction force in contact point
n_{fCP}	N·m	Friction torque in contact point

k_s	N/m	Spring stiffness
f_{pl}	N	Preload
k_{bs}	N/m	Bumstop stiffness
δ	m	Spring deformation
c_d	N·s/m	Damping coefficient
$\dot{\delta}$	m/s	Time rate of spring deformation
α, β	-	Baumgarte stabilization method parameters
$\mathbf{k}_s$	-	Vector of spring stiffnesses
$\{\delta_\xi \quad \delta_\eta \quad \delta_\zeta\}$	-	Vector of springs deformations
$\mathbf{c}_d$	-	Vector of damping constants
$\{\dot{\delta}_\xi \quad \dot{\delta}_\eta \quad \dot{\delta}_\zeta\}$	-	Vector of time-rate of spring deformation
$\{\xi_i \quad \eta_i \quad \zeta_i\}^T$	-	Direction of actuation of spring-damper forces in local coordinates

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1. INTRODUCTION

The rapid advancement in computational power over recent decades has transformed the design and validation processes for mechanical systems. The growing market competitiveness demands optimized product development while minimizing costs, resources, and time. These constraints render traditional trial-and-error procedures impractical due to their inefficiencies. Additionally, the increasing complexity of mechanical systems has exacerbated the challenges of performing purely analytical analyses. Consequently, Computer-Aided Engineering (CAE) has emerged as an essential tool in modern engineering practices, facilitating enhanced reliability and efficiency across the product development cycle.

CAE tools have become indispensable for their ability to perform simulations and analyses that encompass multidisciplinary perspectives, enabling engineers to achieve higher performance levels in system design. This shift from experimental or trial-based methods to computational approaches reflects the demand for integrated and realistic simulations that consider all factors significantly impacting the performance of a product. These methodologies have become crucial across all engineering domains, particularly in fields like multibody system dynamics and its application to areas such as automotive, robotics, aeronautical and aerospace, which are characterized by their substantial economic and technological relevance.

As the field advances, it increasingly necessitates the incorporation of complex phenomena such as friction, which can significantly influence dynamic behavior. Frictional forces play a pivotal role in determining system performance, yet accurately modeling these forces remains a challenge due to their nonlinear and complex nature. Understanding the influence of frictional phenomena is therefore crucial for achieving realistic and reliable simulations in dynamical systems.

This work aims to explore and evaluate various friction force models employed in the simulation of mechanical systems. By critically analyzing their applicability, limitations, and impact on system dynamics, it is possible to enhance the fidelity of dynamic simulations and address the growing demands of modern engineering challenges.

1.1 Motivation

The primary aim of this study is to investigate the impact of friction modeling on the dynamic behavior of mechanical systems. Friction is commonly described as a dissipative force that acts as a resistance between two contacting surfaces that exhibit a relative tangential motion between them.

The increasing evolution and development of modern societies highly relies in industrial activities that can only operate when transportation and power generation sectors exist alongside. These sectors involve the movement of people and a wide variety of materials, utilizing diverse machines and mechanical systems composed of numerous moving components with interacting surfaces. The smooth, reliable, and durable operation of such machines is critically dependent on the effective control of friction at these interacting surfaces (Holmberg & Erdemir, 2017).

The dissipative nature of friction forces indicates that these forces lead to a loss of energy that in the majority of cases is not recoverable. Figure 1.1 shows the final energy consumption data by sector in the European Union in 2022 from Eurostat (2024).

Final Energy Consumption by Sector (EU, 2022)

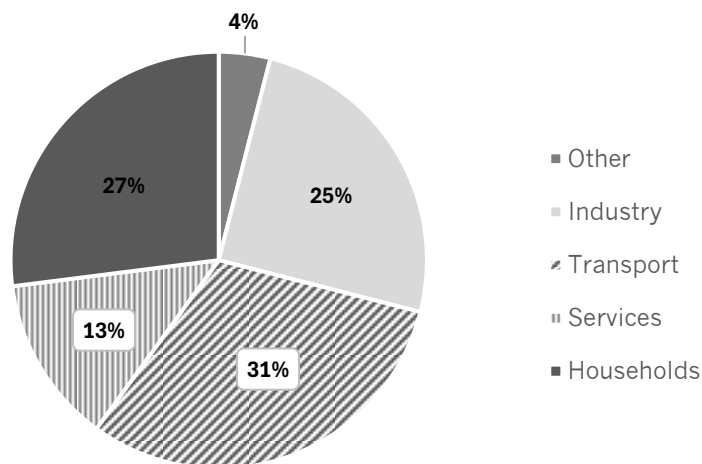


Figure 1.1 – Final energy consumption by sector in the European Union in 2022 (Adapted from Eurostat, 2024)

Holmberg & Erdemir (2017) estimated from the energy consumed in a paper mill and in the mining industry that the overall energy consumption to overcome friction in industry is roughly 20% of the total supplied energy. In the transportation sector, the effects of friction in energy consumption were calculated with great detail for the passenger vehicles, trucks and buses, reaching a value of around 32% of total energy usage just to overcome friction. Although there was not enough available data about the aviation and marine sectors, it was estimated that the share of energy wasted to overcome friction is about 10%

and 20%, respectively, which leads to an estimate 30% of total supplied energy consumption to overcome friction in the transportation sector (Holmberg & Erdemir, 2017).

In what concerns the services and households category, there is a wide range of energy conversion and utilization technologies, such as HVAC systems, lighting, household appliances and business equipment. The mechanical systems which need energy and thus friction can impact their performance are mainly ventilation systems, fans, pumps, etc., where an estimate of 10% of the total energy is used to overcome friction (Holmberg & Erdemir, 2017). In average, considering the economic sectors explored, a total of 20% of the global energy use is utilized to overcome friction. Table 1.1 summarizes the percentage of total energy that is used to overcome friction in the economic sectors considered.

Table 1.1 – Energy expended to overcome frictional forces (Adapted from (Holmberg & Erdemir, 2017)).

Economic sector	Percentage of total energy use to overcome friction [%]
Industry	20
Transportation	30
Services and Households	10
Average	20

Currently, significant efforts are focused on developing more energy-efficient vehicles and machines, driven not only by economic considerations, but also by the need to meet CO₂ emission reduction targets set by the Kyoto Protocol. In certain countries, these efforts are further motivated by the desire to avoid substantial government-imposed financial penalties (Holmberg & Erdemir, 2017).

Although friction is commonly seen as an undesirable phenomenon, there are several examples of its utility role in everyday life. Some examples are related to the friction forces generated in the contact patch of automobile tires with the ground, the high friction forces needed in cars disk brakes, the high friction forces associated with clutch-type transmissions, the friction between the soles of shoes and the ground, or the friction force that allows a robot gripper to pick and place objects. On the other hand, there are several applications where friction is undesirable and should be minimized such as bearings and other machine elements, since its dissipative effects cause temperature rise and other phenomena that can lead to excessive wear and ultimately to the premature and/or unexpected failure of mechanical components and systems.

The latest technological advancements that can reduce friction are associated with the use of new synthetic lubricants, which contain anti-wear and anti-friction additives, that perform better than traditional mineral oil lubricants and new materials and new surface treatments that reduce friction between contacting surfaces, as well as the introduction of super-hard and self-lubricating particles within the material's structure using additive manufacturing techniques (Holmberg & Erdemir, 2017).

A key technology that allows to better comprehend, model and predict frictional behavior between contacting bodies is the use of computational methods. The utilization of multiscale integrated material modelling and simulation based on sophisticated computer codes, which can integrate various methods such as finite element analysis and multibody dynamics, permits the prediction and correction of the dynamical behavior of mechanical systems considering friction phenomena. By simulating the behavior of a system in a virtual environment, it is possible to reduce monetary expenses both in designing, building and testing prototypes, as well as during the system's useful life, since the final design should be optimized in terms of energy consumption to take into account friction forces, and by knowing how many more cycles the component/system has left until failure, considering friction phenomena. Computational modeling of friction phenomena has paramount importance in the design of control systems, because using a suitable friction force model, the system's controller can compensate for the effects of friction without requiring high gain control loops (Canudas De Wit and Lischinsky, 1995).

To summarize, the presented work is motivated by identifying friction phenomena and the influence that friction forces have in the dynamic behavior of mechanical systems through computational and experimental approaches in dry contacts.

1.2 Scope and Objectives

The main scope of this work centers on the critical analysis and evaluation of friction force models in dynamic mechanical systems. Friction, a nonlinear and multifaceted phenomenon, plays a pivotal role in influencing the behavior of mechanical systems.

Accurately modeling and understanding friction is essential for developing simulations that reflect real-world dynamics, enabling engineers to optimize system performance, durability, and efficiency. This work delves into the theoretical foundations, practical implementations, and experimental validations of various friction models, aiming to bridge the gap between theoretical insights and their practical applications.

In order to explore the different dimensions in which the critical analysis of friction force models is divided, it is of paramount importance to establish a division between the tasks required to analyze the impact of the dissipative effects of friction in the dynamic behavior of mechanical systems in a systematic manner. The subsequent topics indicate the objectives that must be fulfilled:

- present a brief review of the most important contributions throughout the centuries to the scientific knowledge of friction;
- present and explore in detail the different phenomena associated with friction;
- review the most utilized dry friction force models in literature in the context of multibody dynamics simulations;
- develop an intuitive and simple interface to test different static and dynamic friction force models within the context of a simple one degree of freedom benchmark problem;
- design and manufacture a simple and versatile experimental apparatus to obtain friction force data;
- develop an optimization routine in MATLAB to predict and calibrate friction force model parameters;
- develop a rigid multibody model of the 2023 Formula Student of University of Minho (FSUM) car prototype, comprising the modeling of tire-ground contact utilizing the Magic Formula (MF) tire model calibrated with experimental data and subject the obtained model in a variety of dynamic test to assess the influence of incorporating friction phenomena in the behavior of the suspension subsystem.

Throughout the different chapters and sections of this work is possible to verify the accomplishments of the aforementioned topics.

1.3 State of the Art of Friction Force Models

Friction is a force that opposes the relative motion between two surfaces in contact. It encompasses various phenomena, including dry friction, which occurs between unlubricated surfaces, and lubricated friction, which involves surfaces with a lubricating layer. Each of these frictional interactions is characterized by distinct mechanisms and behaviors that influence the resistance to motion.

The importance of studying friction phenomena was recognized several centuries ago, being the first works developed by Da Vinci, followed by Amontons and Coulomb (Dowson, 1998). The Coulomb friction model is a foundational model for describing friction and serves as the basis for many subsequent

friction models. It is a relatively simple approach, requiring only a single parameter, the coefficient of friction, to characterize the frictional force. However, the Coulomb model has significant limitations, particularly its inability to capture many complex frictional phenomena. One of its main problems is the discontinuity in the friction force at null relative velocity between the contacting bodies. This discontinuity poses a major challenge in dynamic simulations, as it introduces numerical instabilities.

Further research into frictional behavior has suggested that friction forces are greater when an object is at rest (Morin, 1833; Rabinowicz, 1951). This led to the differentiation of two distinct friction coefficients: one for static friction and another for kinetic friction. These variations in frictional force contribute to the occurrence of the stick-slip phenomenon (Rabinowicz, 1957, 1961).

The continuous progress made to better understand and characterize friction led to the identification of several complex phenomena such as viscous and Stribeck effects, pre-sliding displacement, micro-slip and frictional lag (Andersson et al., 2007; Olsson et al., 1998). Several friction models have been proposed in literature. They can be divided into multiple categories based on the capability of modeling certain phenomena, complexity or simply by chronological order of appearance. The most common criterion used to differentiate friction models consists of dynamic and static models (Andersson et al., 2007a; Marques et al., 2016; Olsson et al., 1998; Pennestri et al., 2016). Usually, static friction models are simpler in terms of mathematical formulations and do not have the ability to capture complex phenomena, being sufficiently accurate in describing the behavior of friction forces at steady state. Dynamic friction models are capable of modeling complex phenomena due to the use of extra state variables, which although increases the accuracy of the friction force results, normally produces more computational effort and less efficient simulation times.

Different works are found in literature regarding the different phenomena associated with friction and review works concatenating the state of the art static and dynamic friction force models. Olsson et al. (1998) presented a review of the recognized complex friction phenomena and the friction models of interest for automatic control. Regarding friction phenomena, static friction (stiction) is identified as being the friction force level reached before a system initiates macroscopic motion, being the breakaway force the level of force required to leave the static state. The microscopic motion verified between the fully static state and the breakaway force level is described as pre-sliding displacement (Olsson et al., 1998). The frictional lag phenomenon is associated with the hysteresis relationship verified between friction force and velocity where for decreasing velocities the friction force is lower than for increasing velocities. This highly non-linear phenomenon is one of the main reasons that dynamic friction forces are required to describe the physical behavior of contacting surfaces (Olsson et al., 1998). The static friction force models

presented are mainly based on the Coulomb friction model considering viscous and Stribeck effects as well as stiction. The only static regularized approaches presented for friction modeling are the Karnopp and the Armstrong friction models. For the dynamic category, Dahl, Bristle Model, Reset Integrator, Bliman and Sorine and LuGre models are presented in a comprehensive way. The LuGre friction force model is further analyzed by comparing the numeric results with experimental data obtained from a laboratory test rig.

Andersson et al. (2007) presented various friction models tailored for different sliding contact conditions, including dry, boundary, and mixed lubrication scenarios. The models analyzed include the Coulomb friction model, which is widely used but often inadequate for accurately representing friction behavior; the viscous friction model, which simplifies simulations but may not reflect real conditions; and the Stribeck friction model, which accounts for lubrication effects and varying friction with sliding speed.

Additionally, the paper introduces combined models, such as the Coulomb and viscous friction model, which aim to mitigate the limitations of individual models. The Dankowicz and Dahl models are also discussed, focusing on their applicability in control engineering and their representation of friction as a function of displacement. The authors emphasize the importance of considering micro-slip phenomena and the stochastic nature of friction, suggesting that friction can be modeled as a stochastic process influenced by the interactions of surface asperities. Regarding the experimental apparatus, the authors utilized a one-degree-of-freedom (1 DoF) dynamic system to study the behavior of different friction models through numerical simulations conducted in MATLAB.

Marques et al. (2016) provides a very complete review and comparison of various friction force models for multibody mechanical systems in a dry sliding regime. The authors categorize friction models into two primary groups: static models, which describe steady-state friction behaviors, and dynamic models, which incorporate additional state variables to capture time-dependent friction effects. Complex phenomena such as stick-slip, viscous friction, and the Stribeck effect are also described in a comprehensive manner within the friction force models that consider them. A comparative analysis of the computational efficiency of the different friction force models was conducted for the ode45 and ode15s solvers of MATLAB.

The basic Coulomb model serves as a foundation, describing friction as a force proportional to the normal force that acts in the opposite direction of the relative motion between contacting bodies. Due to its limitations, variants such as models with viscous and Stribeck components were developed to address speed-dependent and transitional friction phenomena. Dynamic models, like the LuGre model,

add complexity by using state variables to simulate effects such as pre-sliding displacement and frictional lag.

In terms of experimental setups, the authors use a single-degree-of-freedom (1-DOF) mass-spring system as a benchmark for testing these models. The apparatus consists of a mass on a conveyor belt connected to a spring, allowing measurement of frictional forces as the belt moves at a constant speed. This setup enables the simulation of stick-slip motions, frictional hysteresis, and dynamic responses. The results show that models like LuGre and Elasto-Plastic provide closer adherence to real-world behavior by capturing pre-sliding displacement and frictional lag, in contrast to simpler static models, which fail to capture these details. This comparative study emphasizes the influence of selecting an appropriate friction model, as this choice significantly affects the accuracy and dynamic response predictions for mechanical systems with frictional contacts.

Another very important aspect investigated by Marques et al. (2016) is the influence that the choice of the numeric value of the parameters considered by friction force models have in the global response of the system. A systematic sensitivity analysis is presented for the parameters of the dynamic models of Dahl, Reset-Integrator, LuGre and Gonthier where the dynamic behavior of the system is studied by analyzing the relationship between the friction force and both the relative velocity and displacement of the mass-spring model. Pennestri et al. (2016) developed a review and comparison of dry friction force models very similar to the one realized by Marques et al. (2016).

Marques et al., (2019) developed a review work of the friction force effects on the dynamical behavior of multibody systems with rolling contacts. To assess the capabilities of the different analyzed models, the paper presents two experimental applications. In the first, a 2D roller-crank mechanism examines continuous contact frictional behavior, and in the second, a journal-bearing system explores impact scenarios. These setups demonstrate that models such as the LuGre model effectively capture complex frictional responses, although they are sensitive to changes in normal load, particularly under impact conditions. This sensitivity affects their reliability, suggesting simpler models like the Brown and McPhee may be preferable in contexts requiring stable results with lower computational complexity.

In this work, a detailed analysis of the different complex phenomena associated with friction is made. Moreover, a wide variety of friction force models of both static and dynamic categories are presented, where the formulations are analyzed in detail together with their parameters. The friction force models that are considered to be more suitable for multibody dynamics applications are then applied to a real-world complex multibody model to assess their influence in the dynamical response of the system.

1.4 Organization of the Dissertation

The main objective of this work is to critically analyze a broad variety of friction force models commonly utilized in the multibody dynamics simulation of mechanical systems, emphasizing their theoretical development, computational implementation, and practical applications. By integrating experimental validation, numerical benchmarks, and case studies, the research aims to bridge the gap between abstract modeling and real-world performance, ultimately enabling more precise and efficient simulations tailored to modern engineering challenges.

The first chapter indicates the motivation and main objectives of this work, where it is underscored the importance of considering the influence of the dissipative effects associated with friction to predict the dynamic behavior and design better products in terms of energy efficiency and durability. It is also introduced the state-of-the-art in what concerns the review of dry friction force models for multibody applications.

In Chapter Two, a literature review of the multibody dynamics formulations is described with a special focus in the formulation of the equations of motion and the Baumgarte constraint violation stabilization method. A review of the main experimental apparatuses described in literature regarding the acquisition of friction force data and correspondent phenomena identification is also made. The chapter ends with a collection of typical friction force magnitude values verified in automotive suspension systems in both numeric and experimental tests presented in literature by various authors.

Chapter Three is dedicated to the historical development of friction studies, beginning with practical applications by ancient civilizations like the Egyptians and progressing to foundational scientific contributions by Leonardo da Vinci, Guillaume Amontons, Leonhard Euler, and Charles-Augustin de Coulomb. It is highlighted the evolution from rudimentary experimental apparatuses to systematic laws and mathematical models that form the basis of modern tribology. The advancements in understanding friction over time have enabled increasingly precise descriptions of surface interactions, laying the groundwork for contemporary approaches to modeling and simulation in mechanical systems.

Chapter Four explores friction from its fundamental concepts to the distinctions between static and kinetic friction, and the factors influencing relative motion. Key models, such as Coulomb friction, were explored alongside phenomena like sliding and rolling friction, emphasizing their underlying mechanisms and challenges in computational modeling. Influencing variables, from contact geometry to lubrication and vibrations, were also briefly outlined.

Chapter Five provides an in-depth analysis of key friction phenomena, including stiction, breakaway force, dwell-time, viscous effects, the Stribeck effect, stick-slip motion, pre-sliding displacement, micro-slip, frictional lag, and non-reversibility. These phenomena collectively define the complex and nonlinear nature of friction and its influence on mechanical systems. The discussion emphasized their role in mathematical friction models and the challenges of integrating them into simulations.

Chapter Six is dedicated to the evaluation of 14 static friction force models, comparing their computational performance, physical realism, and ability to capture phenomena like stiction and the Stribeck curve. A benchmark test programmed in MATLAB along with an interactive interface highlighted performance difference, emphasizing the importance of selecting appropriate models and solvers for specific applications.

In Chapter Seven a total of 11 dynamic friction models were analyzed, emphasizing their ability to simulate complex phenomena like hysteresis, frictional lag, and pre-sliding displacement through the introduction of a state variable, while also highlighting the problems associated with extensive parameter calibration and higher computational costs compared to static models. The dynamic models were also compared using the same benchmark problem, showcasing the trade-offs between accuracy and computational feasibility in dynamic simulations.

Chapter Eight details the development of a custom experimental apparatus for friction force data acquisition, highlighting its design, manufacturing, and testing processes. A tracking approach was used to determine the kinematic properties of the slider. The chapter also describes the optimization procedure of the LuGre friction model parameters using experimental data, demonstrating the inadequacy of general-use parameters and emphasizing the importance of tailored parameter determination for accurate multibody simulations and control applications.

Chapter Nine is dedicated to the analysis of the impact of friction forces on the dynamic behavior of an automotive suspension system. Within this case study, a multibody model of the FSUM car was created in MUBODYNA. Friction forces were modeled in key contact points of the shock absorber, and the effects of these forces on the suspension's compression and rebounding phases were evaluated. The chapter compared different friction models, highlighting their computational efficiency and showing that friction forces, while small in magnitude, can affect suspension system performance.

Chapter Ten is dedicated to the global conclusions and discussion of the work developed throughout the thesis and to the suggestion of future research works within the topics addressed in the different chapters.

1.5 Contributions of this Work

The main contributions of this work are not directly related to the critical analysis of friction force models, although those tasks were successfully developed within this work, since many authors such as Marques et al. (2016) and Pennestri et al. (2016) have previously presented comprehensive and extensive works regarding the review of multiple dry friction force models.

Instead, this work contributes with a bridge between the theoretical aspects of friction force modeling and the experimental and numerical assessment of friction phenomena influence in the dynamic behavior of real-world mechanical systems.

The proposal of a simple and versatile experimental apparatus was designed and manufactured in order to obtain friction force data, allowing the calibration and prediction of friction force model parameters to better represent and simulate the kinematic and dynamic properties of complex multibody systems.

Furthermore, a new methodology was developed regarding the calculation of friction forces and their effect on the dynamic behavior of an automotive suspension system within the framework of multibody dynamics formulations based on the kinematic joints reaction forces. Also, a proposal of a methodology to simplify the tire-ground contact for non-rolling suspension testing in four-poster test platforms was proposed in this work.

2. LITERATURE REVIEW

2.1 Multibody Dynamics

A general multibody dynamics (MBD) system can be defined as a set of bodies that describe large translational and rotational displacements that are constrained by kinematic joints and acted by generalized forces (Flores, 2015; Nikravesh, 1988a). Figure 2.1 shows an abstract schematic representation of a general multibody system.

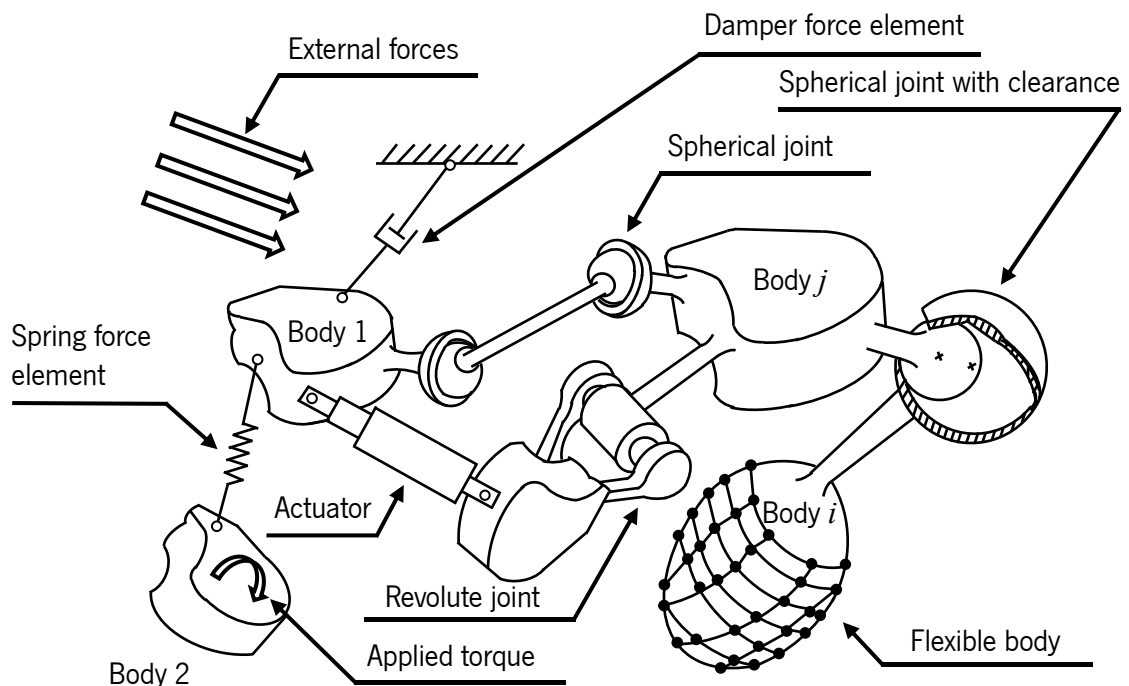


Figure 2.1 – Abstract representation of a generalized multibody system.

In a multibody system, the bodies represent mechanical components that can be modeled as either rigid or flexible. For a rigid body, degrees of freedom are limited to translational and rotational motions, as it is assumed that no deformation occurs within the body. Conversely, a flexible body incorporates additional degrees of freedom to account for deformation, which increases the complexity of the system. The inclusion of flexible bodies in dynamic simulations incurs a substantial computational cost and should therefore be reserved for cases where deformation significantly impacts system behavior. In most applications, assuming rigid bodies offers a reasonable and computationally efficient approximation for modeling.

In the three-dimensional space, the motion of a free rigid body can be fully described by using six generalized coordinates associated with the six degrees of freedom, three translations and three rotations,

that are known as Cartesian coordinates (Flores, 2015). This form of representation of a body in the three-dimensional space is called body-coordinate formulation or simply absolute coordinate and it is the method employed in this work. The formulation requires the construction of a large system of equations, as an equilibrium equation is required for each coordinate, along with the inclusion of existing constraint equations. However, this approach enables a straightforward and systematic implementation in computational algorithms (Flores, 2015).

The number of degrees of freedom of a system is defined as the number of independent generalized coordinates necessary to uniquely specify its configuration. Understanding the degrees of freedom is essential for accurately modeling and analyzing multibody systems. The mathematical expression that allows to determine the number of DoF of a multibody system is the Grübler-Kutzbach criterion defined according to Equation (2.1)

$$n_{DoF} = 6n_b - m \quad (2.1)$$

where n_b is the number of bodies that compose the multibody system and m is the number of independent constraints.

In a multibody system, the degrees of freedom are directly influenced by the kinematic constraints imposed, particularly those associated with kinematic joints. Each joint type permits specific relative motions between connected bodies while restricting others (Flores, 2015).

The angular position of a rigid body can be described through three successive rotations using methods such as Euler or Bryant angles. However, representing the orientation of a rigid body with three coordinates can lead to singularities in certain configurations, causing numerical instabilities, thus imposing computational challenges. To avoid these singularities, this work adopts Euler parameters (Nikravesh, 1988a) to define angular orientation. This approach leverages Euler's theorem on finite rotation, which states that any rotation in three-dimensional space can be represented as a rotation about a specific axis by a defined angle. Consequently, the generalized coordinate vector for a given body i comprises seven coordinates given as

$$\mathbf{q}_i = \{\mathbf{r}_i \quad \mathbf{p}_i\}^T \quad (2.2)$$

where $\mathbf{r}_i$ represents the vector of the translational coordinates of body i in global coordinates, defined as

$$\mathbf{r}_i = \{x_i \quad y_i \quad z_i\}^T \quad (2.3)$$

and $\mathbf{p}_i$ represents the vector of Euler parameters for body i , which can be written in the following form

$$\mathbf{p}_i = \{e_0 \quad e_1 \quad e_2 \quad e_3\}^T = \left\{ \cos \frac{\phi}{2} \quad \mathbf{u}_e \quad \cos \frac{\phi}{2} \right\}^T \quad (2.4)$$

where ϕ represents the angle through which the body must rotate about a unique axis, given by the unit vector $\mathbf{u}_e$ to align its reference frame with the global reference frame. Although using Euler parameters requires four coordinates to describe the orientation of the body, the system retains only three independent coordinates due to the constraint that $\mathbf{p}_i$ remains a unit vector.

Regarding generalized velocities and accelerations, the problem of singular positions does not exist. Thus, the generalized velocities of a rigid body can be written as

$$\mathbf{v}_i = \{\dot{\mathbf{r}}_i \quad \boldsymbol{\omega}_i\}^T \quad (2.5)$$

in which $\dot{\mathbf{r}}_i$ is the vector of the translational velocities and $\boldsymbol{\omega}_i$ is the vector of angular velocities of body i . By differentiating Equation (2.5), the generalized accelerations vector is obtained being defined as

$$\dot{\mathbf{v}}_i = \{\ddot{\mathbf{r}}_i \quad \dot{\boldsymbol{\omega}}_i\}^T \quad (2.6)$$

where $\ddot{\mathbf{r}}_i$ is the vector of the translational accelerations and $\dot{\boldsymbol{\omega}}_i$ is the vector of angular accelerations.

The degree of accuracy wanted for a certain dynamic analysis dictates the type of joints modeled to simulate the connections between the different bodies of a multibody system that constrain their relative motion. Two main categories of joints can be distinguished in the framework of multibody dynamics: ideal joints and imperfect joints. The ideal joints are simply kinematic relations defined between the connected bodies that resort on algebraic equations that constrain the motion through vectors, coordinates of points, etc. The imperfect joints are usually based on contact-impact events and can model more complex phenomena such as clearance, friction and wear. Despite the enhanced realistic behavior of imperfect joints, their computational efficiency is normally lower compared to the simpler kinematic joints.

Ideal joints are mathematically defined by kinematic constraint equations, where the number of independent equations represents the number of constrained degrees of freedom. Kinematic constraints are classified as holonomic or nonholonomic, depending on whether they can be integrated to form a position constraint or not, respectively (Flores, 2015). Additionally, constraints are categorized as

rheonomic if they explicitly depend on time or scleronomic if not (Flores, 2015). Therefore, in a constrained multibody system, the set of holonomic constraint equations can be expressed as follows

$$\Phi \equiv \Phi(\mathbf{q}, t) = \mathbf{0} \quad (2.7)$$

where $\mathbf{q}$ denotes the vector of the generalized coordinates of the system and t is the time variable. The velocities constraint equations are obtained from differentiating Equation (2.7) in order to time and are represented in matrix form as

$$\dot{\Phi} \equiv \mathbf{D}\mathbf{v} - \mathbf{v} = \mathbf{0} \quad (2.8)$$

in which $\mathbf{D}$ is the Jacobian matrix of the constraint equations, $\mathbf{v}$ represents the vector of the generalized velocities of the system and $\mathbf{v}$ represents the right-hand side of the velocities constraint equations. The second derivative of Equation (2.7) with respect to time yields the constraint equations at acceleration level defined as

$$\ddot{\Phi} \equiv \mathbf{D}\dot{\mathbf{v}} - \boldsymbol{\gamma} = \mathbf{0} \quad (2.9)$$

where $\dot{\mathbf{v}}$ is the vector of generalized accelerations and $\boldsymbol{\gamma}$ is the right-hand side vector of the acceleration constraint equations.

In this work, the equations of motion of the multibody system are derived according to Newton-Euler's methodology. The Newton-Euler equations establish a relation between the motion of the center of mass of the rigid bodies and the external and reaction forces and moments of the kinematic constraints, which are formulated as

$$\mathbf{M}\dot{\mathbf{v}} + \mathbf{D}^T\boldsymbol{\lambda} = \mathbf{g} \quad (2.10)$$

where $\mathbf{M}$ represents the mass matrix of the system which encompasses both the mass and the inertia tensor of the system's bodies and the vector $\mathbf{g}$ denotes the vector of the external generalized forces and gyroscopic moments. The term $\mathbf{D}^T\boldsymbol{\lambda}$ includes the reaction forces and moments due to the kinematic joints in the global coordinate system, in which $\boldsymbol{\lambda}$ denotes the vector of the Lagrange multipliers, where its size is equal to the number of holonomic constraints, with each element representing the reaction force or moment associated with a specific kinematic constraint.

In dynamic analysis, a unique solution is obtained when the algebraic constraint equations at the acceleration level are considered simultaneously with the differential equations of motion (Flores, 2015). The matrix form of the dynamic equations of motion of a multibody system can be obtained by introducing

Equation (2.9) in Equation (2.10) which results in the following system of differential algebraic equations (DAE)

$$\begin{bmatrix} \mathbf{M} & \mathbf{D}^T \\ \mathbf{D} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{v}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{g} \\ \boldsymbol{\gamma} \end{Bmatrix} \quad (2.11)$$

that is then solved for accelerations vector, $\dot{\mathbf{v}}$, and Lagrange multipliers vector, $\boldsymbol{\lambda}$. At each integration time step, the acceleration vector, $\dot{\mathbf{v}}$, along with the velocity vector, $\mathbf{V}$, is integrated to compute the system's velocities and positions for the subsequent time step. This process is iteratively repeated until the final analysis time is reached (Flores, 2015).

Solving Equation (2.11) determines the accelerations of the system at each time step, but does not ensure that the position and velocity constraints are satisfied, since the constraints at both the position and velocity levels are not explicitly included in the equations of motion. Consequently, errors introduced through numerical integration can lead to violations of both position and velocity constraint equations. To address constraint violations, several methods are available, categorized into three main approaches: constraint stabilization techniques, coordinate partitioning methods, and direct correction formulations, which are well detailed and analyzed in literature (Marques et al., 2017).

In this work, the Baumgarte stabilization technique (Baumgarte, 1972) was introduced in the dynamic equations of motion in order to keep the violation of the constraints under acceptable values. It should be noted that the constraint stabilization approaches do not fully eliminate the violations but allows to keep their values under control. In this constraint stabilization approach, the acceleration constraint equations are modified to include two additional terms that provide feedback control to correct violations of the position and velocity constraints. The system of equations of Equation (2.11) can then be rewritten as

$$\begin{bmatrix} \mathbf{M} & \mathbf{D}^T \\ \mathbf{D} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{v}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{g} \\ \boldsymbol{\gamma} - 2\alpha\dot{\boldsymbol{\Phi}} - \beta^2\boldsymbol{\Phi} \end{Bmatrix} \quad (2.12)$$

in which α and β are positive values that represent the feedback control parameters of the velocity and position constrain violations, respectively.

The numerical resolution of Equation (2.12) should be carefully addressed. The multibody dynamics equations of motion are differential and algebraic equations (DAE), but it is useful to convert them to ordinary differential equations (ODE), since in the later form it is possible to solve the equations through numerical integration algorithms that are rather simple to implement (Flores, 2015). The most

widely used numerical integration methods in the literature include the Euler method, Runge-Kutta methods, and Adams predictor-corrector methods (Flores, 2015), which are not the most accurate nor efficient due to their fixed integration time-step nature. However, MATLAB and other programming languages have ODE solvers already integrated that can utilize variable time-step integrators, which results in improved accuracy and efficiency of the solution of highly non-linear problems.

2.2 Experimental Apparatus to Study Friction Phenomena

In the middle centuries of the previous millennium, many scientists were interested in understanding the dissipative phenomena associated with parts that exhibited relative motion between them, which led to wear and premature failure of mechanical components.

Leonardo da Vinci constructed simple wooden devices with different weights to investigate friction forces, driven by his interest in optimizing mechanical systems like gears and pulleys. Guillaume Amontons developed various sliding apparatuses using blocks, plates and ropes to measure frictional resistance, aiming to address mechanical inefficiencies in machines. Charles-Augustin Coulomb designed more sophisticated apparatuses to analyze friction, motivated by the need to improve naval and structural engineering. These tools were built to systematically measure and quantify friction for solving real-world mechanical challenges.

Experimental apparatuses have been pivotal to understanding friction phenomena, providing controlled environments to observe and quantify frictional effects. From hysteretic behaviors to stick-slip dynamics and non-reversible friction characteristics, these setups have allowed researchers to validate theoretical models and refine them.

There are apparatuses built specifically to study certain isolated friction phenomena. The work by Awrejcewicz & Olejnik (2007), for example, consists of a two-degree-of-freedom mechanical system with a feedback-reinforced laboratory rig, where a block moving on a belt coupled with springs and sensors allows to measure both friction force and displacement in order to explore stick-slip dynamics and develop a friction model accounting for the behavior at both positive and negative tangential velocities. Wojewoda et al. (2008) built a setup designed to study hysteretic behaviors in dry friction, focusing on pre-sliding displacement and non-reversible friction characteristics. The apparatus captures phenomena such as the Stribeck effect and varying breakaway forces in order to investigate hysteresis during transitions from static to dynamic friction and validate friction models through experimental results compared with numerical simulations.

The phenomenon of non-reversibility is another complex phenomenon associated with friction, that due to its complexity and stochastic nature, has led to the construction of experimental apparatus dedicated to capture this specific phenomenon. Wiercigroch et al. (1999) built a Coulomb oscillator equipped with a pneumatic actuator to control the normal force, allowing precise measurement of friction dynamics and non-reversibility of friction force during acceleration and deceleration phases. This apparatus allowed to investigate non-reversible friction characteristics and validate theoretical friction models through dynamic experimentation. Guo et al. (2008) conducted similar studies utilizing a reciprocating motion test rig with a vibration table and laser displacement sensors to measure frictional forces during sinusoidal and rectangular wave motions with the goal of modeling and identifying non-reversible friction characteristics, including hysteresis in pre-sliding and sliding regimes, and to analyze the effects of relative velocity and acceleration in the overall dynamic response of the system.

Friction phenomena identification and modeling are important to develop energy efficient and high precision mechanical and electromechanical systems. Thus, it is of paramount importance to also build tailored made experimental rigs to test real-world systems in order to correctly predict and compensate for the dissipative effects of friction forces. The work of Kebairi et al. (2015) describes an apparatus for accurately measuring the influence of friction in modeling actuator dynamics, incorporating static and LuGre friction models, and evaluating their performance under various operating conditions. The experimental rig includes a DC motor, a two-stage spur gear system, a helical spring, and position sensors to measure friction torque and actuator displacement, simulating real-world engine conditions that affect the operation of a Bosch GPAS electromechanical actuator used in Diesel engines to control air intake.

The incorporation of friction phenomena in automation and mechatronic systems, such as pneumatic and hydraulic cylinders or robots, is crucial for ensuring precision and reliability in dynamic control systems. Friction, as a dissipative phenomenon, significantly influences system behavior, especially at low speeds or during transitions between static and dynamic states. Accurate identification and modeling of friction allow control systems to mitigate its negative effects, enhancing performance in terms of stability, accuracy, and energy efficiency.

Felix & Silveira (2018) built a low-cost rig using a Raspberry Pi controller and pneumatic instrumentation to measure friction forces in real time. The setup calculates friction based on pressures in actuator chambers and piston dynamics to analyze the static and dynamic friction behaviors of pneumatic actuators, enabling precise parameter identification for friction compensation in industrial positioning systems. Wakasawa et al. (2018) performed tests in a pneumatic cylinder driven by a stepper motor, equipped with load cells, and pressure and velocity sensors to isolate and measure friction

contributions from rod and piston seals under varying lubrication conditions. Feng et al. (2019) performed an identification procedure of nonlinear friction parameters using an improved Stribeck model and implemented friction compensation, improving trajectory accuracy and mitigating low-velocity issues in electro-hydraulic systems. The real-world application consisted of a robotic excavator arm equipped with hydraulic cylinders, pressure sensors, and displacement sensors to measure friction forces during controlled movements at constant velocities. Qian et al. (2022) studied the impact of piston speed, pressure, and pressure differences on friction in pneumatic cylinders utilizing a friction test rig featuring two pneumatic cylinders mounted face-to-face with their chamber pressures balanced to eliminate pressure effects on force measurements, enabling precise and isolated evaluation of friction forces.

The experimental test rig built and tested in the context of this work focuses specifically on the parameter estimation of numerical friction force models, allowing a better understanding of the influence of parameter selection in the dynamical behavior of mechanical systems.

2.3 Influence of Friction on Vehicle Suspension Systems

Friction is modeled in suspension systems through various formulations such as finite element analysis, vibrations and multibody dynamics. In highly complex and realistic simulations, multiple formulations can be employed in a single model, by combining finite element analysis and multibody dynamics, for example.

The influence of friction in the dynamical behavior of suspension systems can be explored by two different perspectives: comfort and NVH (Noise Vibration and Harshness) requirements, usually of capital importance for regular passenger vehicles or dynamic performance, for sports and racing cars. Primarily, friction forces affect the vertical dynamics of a vehicle. The increased sales of electric vehicles in recent years brought particular interest in studying the influence of friction in suspension systems, because the substitution of the internal combustion engine, that produces higher noise and vibration levels compared to an electric or hybrid powertrain, leads to an increased perception of the noise generated by suspension components, reducing the overall NVH quality of a certain vehicle.

The topic of friction influence analysis in shock absorber's performance is not a novelty in literature. Several works explore the critical role of friction in shock absorbers and suspension systems, focusing on its effect on vehicle dynamics and ride comfort.

Lizarraga et al. (2008) provide insights into the dual role of friction in suspension systems and its influence on both comfort and energy dissipation. Their research addresses limitations in existing friction

models by proposing a new formulation that combines the Stribeck effect with viscous friction, improving the understanding of shock absorber behavior. A specialized hydraulic testing device was developed to isolate and measure friction, minimizing hydraulic forces' influence. Experimental data confirmed the presence of the Stribeck effect, highlighting stick-slip phenomena and asymmetrical friction characteristics during rebound and compression phases.

Benini et al. (2017) explored the impact of friction in suspension systems for racing cars, finding that friction significantly affects the vehicle's vertical dynamics, especially at low frequencies near the car's natural pitch frequency. Using a Formula 4 car tested on a Four Poster Rig, the study highlights that friction, particularly in the front suspension, acts as an additional damping element, influencing vehicle behavior during dynamic testing. Friction's role in vertical response and transient maneuvers indicates that ignoring it in simulations can result in an inaccurate assessment of the overall vehicle dynamic behavior.

Körner & Mayer (2020) focus on the friction characteristics of dampers in relation to driving comfort and vehicle dynamics. The study reveals that frictional forces at damper contact surfaces, between seals made from rubber and other materials, significantly influence the damper's response to excitation and the initial breakaway force at the start of movement. The research demonstrated that internal pressure and relative velocity impact friction. Unlike pure Coulomb friction, damper friction exhibits elasto-hydrodynamic characteristics, influenced by movement direction and surface conditioning. Although frictional forces are generally lower than hydraulic damping forces, the authors stated that understanding them is critical for damper optimization and comfort enhancement.

Herzog & Augsburg (2021) presented a two-part study on modeling friction in shock absorbers. The first part develops a dynamic friction model within a finite element framework, focusing on unwanted static friction generated by seals and guides. Using a hybrid model, the researchers integrated physical and dynamic friction models, applying the LuGre model to simulate contact friction. This model, validated through high accuracy simulations in Ansys Mechanical, accurately represents shock absorber friction dynamics, enhancing shock absorber design efficiency by reducing physical testing needs. The second part (Herzog & Augsburg, 2021) validates the friction model using an experimental setup with two specialized test rigs, which measure friction at key contact points in shock absorbers, such as the rod guide assembly and piston/tube interfaces. The data obtained confirmed the simulation model's accuracy in predicting friction across various components, though some deviations were noted in the floating piston/tube interface due to the complexity of seal deformation modeling. The validated model effectively aids shock absorber design and optimization.

Lv et al. (2021) examined the kinematics and compliance (K&C) characteristics of double wishbone air suspensions (DWAS), addressing limitations in traditional models that often overlook friction and joint clearances. This research incorporates a nonlinear dynamic model using ADAMS/VIEW, integrating flexible components and friction forces to enhance suspension simulation accuracy. Experimental validation using a K&C test rig demonstrated that including friction and joint clearances in the model significantly improved predictions of suspension performance, accurately replicating force-related and displacement-related behaviors observed in real-world conditions. The findings underscore the importance of these factors in suspension design, since they have substantial impacts on kinematics, though they affect compliance less.

Deubel et al. (2024) investigated the importance of accurately modeling friction in shock absorbers, noting that friction characteristics are often treated as quasi-static, despite being influenced by dynamic conditions. Using modified valve-free shock absorbers, the researchers observed dynamic friction properties, which revealed that dynamic friction forces significantly exceed quasi-static friction forces, especially at lower velocities (up to 60 mm/s) and under substantial side forces. The study found that side force and velocity should be considered in friction analysis, as conventional models underestimate total shock absorber forces by neglecting dynamic friction as a key component. This understanding can lead to improvements in active suspension control and vehicle simulations by allowing better parametrization of shock absorber models.

A common problem across the various studies on the friction role in the dynamic behavior of suspension systems is the general way friction is studied within these systems, where real world road events are not considered nor a clear quantification of the friction effect is made (Deubel & Prokop, 2024). Another notable issue is the limited availability of studies that provide accurate, reliable data on friction levels within vehicle suspension systems. This scarcity applies both to overall suspension friction and to friction in specific components. Furthermore, there is a lack of comprehensive research on how various parameters, such as temperature and amplitude, may influence suspension friction (Deubel & Prokop, 2024).

The limited number of studies in this area may be attributed to several challenges: (i) the need for extensive measurements to achieve accurate and reproducible friction data, (ii) the significant effort required to model friction with precision, both in terms of capturing all the phenomena associated with friction, as well as the required computational power, and (iii) the inherent difficulties in practically validating how variations in friction affect vehicle dynamics (Deubel & Prokop, 2024).

Table 2.1 shows a summary of the quantification of friction force influence on suspension systems.

Table 2.1 – Quantification of friction force in suspension systems.

Authors	Data Source	Friction Force Values
Kölsch (1994)	Information obtained from Deubel & Prokop (2024) paper.	25 N
Manger (1995)	Information obtained from Deubel & Prokop (2024) paper.	30 – 45 N
Lizarraga et al. (2008)	Experimental test of a shock absorber in a dedicated test rig and numerical simulation using the Coulomb friction model modified to account for the Stribeck effect.	Numerical Simulation Compression: 32 N Rebounding: 23 N Experimental Data Compression: 22 N Rebounding: 15 N
Fujimoto et al. (2018)	MATLAB/Simulink numerical data comparison with experimental data	20 – 150 N, depending on the type of shock absorber and damping level
Wegener (2020)	List of front and rear axles data for various vehicles. No information about test conditions.	75 – 350 N
Körner & Mayer (2020)	Experimental data obtained from dedicated testbench.	Asymmetrical behavior of the shock absorber, ranging from 20 to 70 N for compression and 10 to 40 N for rebounding, depending on excitation frequency and pressure.
Herzog & Augsburg (2021)	FEA analysis in Ansys and experimental data from dedicated test rig.	Values ranging from 35 N for compression to 30 N for rebounding.
Deubel & Prokop (2024)	Analytical formulation based on kinematic constraint forces and experimental data from full-scale four-poster rig	62 – 134 N for different combinations of displacement and wheel lateral forces.

In this work, a model of the Formula Student of University of Minho will be used to assess the influence of friction forces in the dynamic behavior of suspension systems and its influence on the overall vehicle dynamics.

3. A BRIEF HISTORY ABOUT FRICTION

Friction represents a challenging phenomenon within the dynamic behavior of mechanical systems since ancient times. Early civilizations, such as the Egyptians, developed ingenious methods to manage friction while moving massive stones and statues.

The Renaissance marked a turning point with Leonardo da Vinci's pioneering experiments, laying the foundation for the study of friction. Following the first scientific methodology to study friction presented by Da Vinci, Guillaume Amontons formalized the first laws of friction, followed by the critical refinements of Leonhard Euler and Charles-Augustin de Coulomb.

This chapter traces the evolution of friction's understanding, from the practical applications of each historic era to the outcome scientific knowledge of the last century.

3.1 Friction in the Early Human Civilizations

It is known that throughout the ages the importance of friction and resistance of motion has no doubt been recognized. Evidence from early civilizations, such as rock art and engravings, indicate an awareness of this phenomena's influence on everyday tasks. The Early Human Civilizations, more specifically the Sumerian and Egyptian civilizations, have developed tools in which the friction phenomena were beneficial and of great use such as the so-called drills that employed alternating rotary motion and were developed to produce fire and holes (Dowson, 1998). Figure 3.1. shows the specific example of an Egyptian individual using a bow drill.



Figure 3.1 – Representation of an Egyptian individual using a bow drill (Dowson, 1998).

Alongside the creation and use of artefacts based on the friction phenomenon, the foremost innovation in the Mesopotamian and Egyptian civilizations linked to friction is related with the transportation of huge stone blocks and carved figures utilized in the construction of the Pyramids and palaces. Figure 3.2 exhibits the transportation of an Egyptian colossus from the tomb of Tehuti-Hetep, El-Bersheh dated from c.1800 b.c. One of the Egyptian methods of moving stone statues was to utilize sledges where there were not any types of rollers or levers, and the movement was provided by the pulling force that a large number of men exerted using ropes, which indicates the enormous degree of physical labour demanded (Dowson, 1998). Since the sledge method of transportation created a significant amount of friction in the contact between the sledge and the ground, it is noticeable in the area delimited by the ellipse in Figure 3.2 that a worker is responsible for pouring some type of substance with lubricant properties that allowed for a smoother and less effort sliding movement of the sledge, thus significantly reducing the frictional resistance.

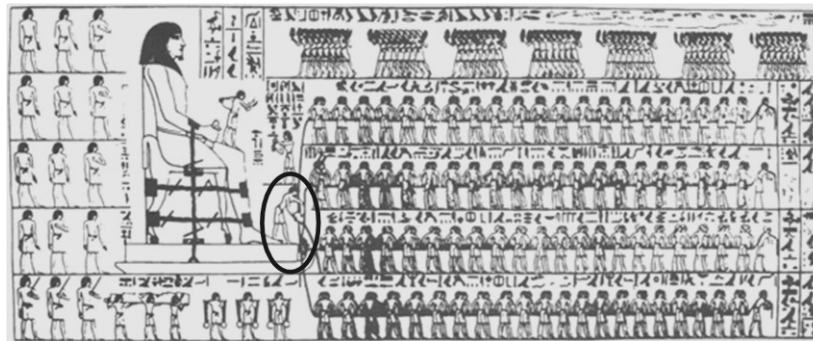


Figure 3.2 – Transportation of an Egyptian colossus from the tomb of Tehuti-Hetep, El-Bersheh (Dowson, 1998).

3.2 Friction Studies by Leonardo da Vinci

The Renaissance Era saw the emergence of several researchers who had a great influence on the development of scientific knowledge. One of the most recognized personalities of the Renaissance era is Leonardo da Vinci (1452 – 1519), regarded universally as a brilliant architect, astronomer, artist, philosopher, and visionary engineer (Pitenis et al., 2014). Da Vinci's results and conclusions quality, obtained and formulated through various experiments, can be attributed to the scientific method that this scientist adopted in his friction studies, which provided a clear demonstration of the impact of the scientific revolution on the Renaissance scene.

With the purpose of better understanding why Leonardo da Vinci showed so much interest in understanding and describing the phenomenon of friction, it is important to analyze some of his inventions and research interests. In the many notebooks written by da Vinci, a large number of sketches regarding

machine elements and mechanisms, such as balances and levers, pulleys and wheels, and inclined planes and screws can be found. The fact that da Vinci was an architect, sculptor and military engineer is an important explanation for the reasons that led this multidisciplinary individual to study friction.

Since Leonardo's work often involved the transport and positioning of heavy weights, there was a certain desire and obsession to understand the rules which governed the friction phenomenon. Analyzing the systems used to transport and position objects, being pulleys one of the most important types of machine elements at the time, some could be simplified by disregarding friction, a practice Leonardo also employed in certain instances. Nevertheless, Leonardo da Vinci recognized the pivotal role of friction in reducing the efficiency and precision of these devices, which led to the establishment of the difference between rolling and sliding friction and the beneficial effects of lubricants (Hutchings, 2016; MacCurdy, 1955).

The *Codex Atlanticus* and the *Codex Arundel* are the notebooks that contain most of da Vinci's studies and experiments regarding friction and offer a remarkable view of the first recorded qualitative application of a scientific method to tribological problems (Hutchings, 2016; Pitenis et al., 2014). Figure 3.3 shows the sketches included in the aforementioned notebooks and demonstrate various experimental apparatus that can be built to measure the force of friction between objects.

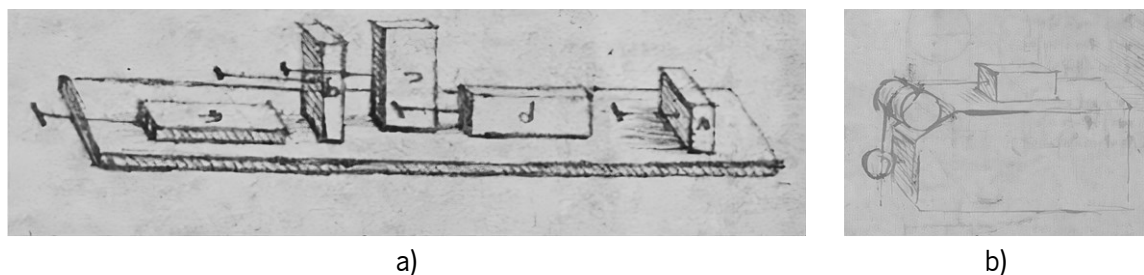


Figure 3.3 – Leonardo Da Vinci's friction experimental apparatus: (a) Sliding blocks in a horizontal surface disposed in various configurations (MacCurdy, 1955); (b) Experimental apparatus used to measure friction using a weight connected to a sliding block (Pitenis et al., 2014).

Figure 3.3 presents two illustrations of da Vinci's friction studies, being the first one related to the influence of apparent contact area upon frictional resistance. Concerning the effect of the apparent area of contact and the normal force upon friction, Leonardo stated that "The friction made by the same weight will be of equal resistance at the beginning of its movement although the contact may be of different breadths and lengths" (Dowson, 1998). This sentence can be considered as the embryogenesis of the well-known friction law that specifies that the apparent area of contact does not influence the friction response (Flores et al., 2023).

It is important to note that da Vinci does not refer to friction as a force. This can be supported by the fact that at the time, mathematicians and engineers were still not capable of defining adequately the phenomenon of force, although it was present in many commonly observed phenomena. The indefiniteness of the concept of force persisted for 200 years until Isaac Newton studied it. Leonardo da Vinci discussed the nature of force in some detail, but the use of the term “forza” appears to be related to the concept of maintaining a constant velocity of an object against some resistance. In fact, it carries a closer resemblance to the contemporary idea of energy or work rather than force (Dowson, 1998).

3.3 Friction Studies by Guillaume Amontons

The period immediately before the Industrial Revolution was characterized by accelerated progress in friction studies, mainly associated with the fact that many new machines were being created and engineers and scientists started investigating ways of improving the performance of those machines by reducing the unwanted effects of friction phenomena. In some cases, as happens today, that performance was often limited due to friction forces that caused overheating and excessive wear of the contact surfaces, leading to wasted power and a reduction of the component's lifetime.

At the end of the 17th century, Guillaume Amontons was a pioneer in studying friction, presenting experiments and results interpretations of his works in a classical paper dated of 1699 (Dowson, 1998). Amontons was motivated by the recognition of friction as a phenomenon that affected the machine's performance and should be reduced to improve energy efficiency and improve the lifetime and reliability of parts.

Guillaume Amontons developed a set of experimental apparatus to support his studies on friction, presenting a practical approach to the friction investigation. Figure 3.4 shows one of those apparatus that is composed by an AA surface test specimen, a BB test block specimen, a CCC spring type element and a spring connected to a scale.

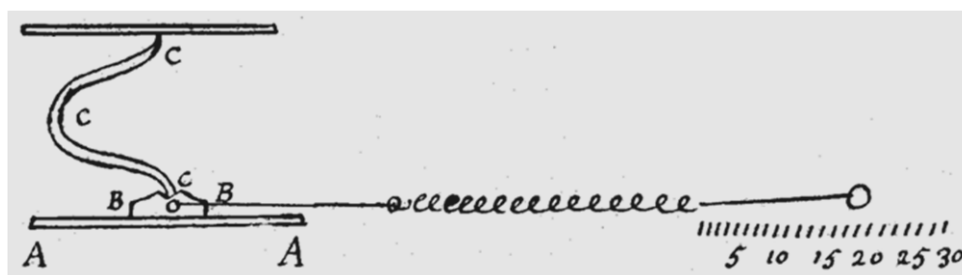


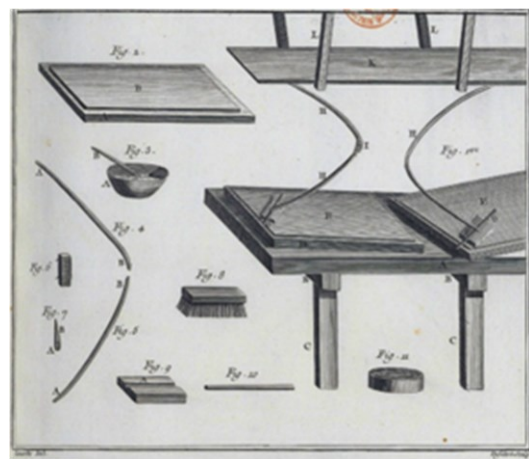
Figure 3.4 – Experimental apparatus developed by Guillaume Amontons to measure friction. A specimen BB is sliding along the test surface AA while being pressed downwards by a CCC flexible body (*flèche*). The spring connected to the scale is used to measure the friction force (Hutchings, 2021).

The working principle of the apparatus presented in Figure 3.4 is based on the downward force provided by the elastic spring element CCC, which promoted the contact between the slider block BB and the surface test specimen AA. The spring displacement was utilized to measure the horizontal force required to overcome friction and initiate the sliding movement of the test block BB. The specimens used in these experiments were made of copper, iron, lead and wood combined in various sets. It is important to note that in each experiment the surfaces were coated with pork fat, which allows to conclude that Amontons was aware of the practical effect of lubricants in reducing friction.

A more critical analysis of the apparatus of Figure 3.4 allows to notice the unusual and complex system to exert the normal force on top of the BB block. The constructive solution can be explained by analyzing an earlier work developed by Amontons in 1699 where the author describes the design of a novel heat engine based on the thermal expansion and contraction of air (Amontons, 1699). The estimation of the power produced by the heat engine was made by comparing it to the effort produced by men and horses and to determine the rate of mechanical working by a human worker, Amontons measured the forces and displacements involved in the manual polishing of plate glass. This process of glass polishing consisted of moving a polishing block repeatedly forwards and backwards over the surface of a glass sheet in the presence of sand and emery. Figure 3.5 shows a group of workers polishing glass in a workshop on the left side and on the right side is presented a schematization of the different components of the glass polishing workbench.



a)



b)

Figure 3.5 – a) Seventeenth century workshop with workers polishing glass; b) Components of the workbench used to polish glass (Hutchings, 2021).

The workbench of Figure 3.5 was developed so that the worker would not get tired of pressing the block down nor of having to continuously accelerate and decelerate a heavy mass, employing all his

physical effort only on sliding the block horizontally. Analyzing both Figure 3.4 and Figure 3.5 it is possible to observe that Amontons was inspired by the glass polishing workbenches to conceptualized and build the experimental apparatus utilized to conduct his friction studies (Hutchings, 2021).

Based on experimental works, Amontons studied the frictional characteristics of greased surfaces. The conditions of the experiments can nowadays be described as boundary lubrication, although the enunciated laws are frequently, but inaccurately, described as the laws of “dry” friction by some present-day authors (Dowson, 1998). The main conclusions described by Amontons can be stated as:

1. The resistance caused by rubbing only increases or decreases proportionally to greater or minor load and not according to the greater or lesser extent of the surfaces;
2. The resistance caused by rubbing is more or less the same for wood, lead, iron and copper in any combination if the surfaces are coated with pork fat;
3. The resistance is more or less equal to one-third of the load applied.

At this stage, it is important to mention the two laws of friction, which can be expressed as:

1. The force of friction is directly proportional to the weight.
2. The force of friction is independent of the apparent area of contact.

The aforementioned laws are usually attributed to Amontons, but Leonardo da Vinci statements, despite being more restrictive, represent the first statements in scientific history of laws governing the phenomenon of friction, especially in what regards the effect of the area of contact (Dowson, 1998).

Another experiment built and studied by Amontons can be observed in Figure 3.6, consisting in a set of weightless blocks A that were intercalated with blocks represented by the B letter. It was stated that the force required to move simultaneously the weightless blocks A is dependent upon the number of sliding faces between the blocks A and B when a weight C is applied to the whole system. The preceding statement may appear contradictory to Amontons’ second law of friction, which states that resistance is independent of the apparent area of contact, but Amontons explains that the resistance at each sliding surface is proportional to the weight C and that the total resistance is equal to the force associated to the weight C multiplied by the number of sliding surfaces (Dowson, 1998).

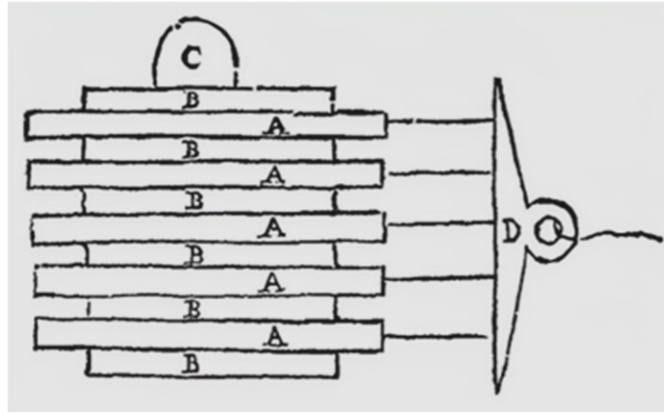


Figure 3.6 – Guillaume Amontons's experiment using a set of specimen blocks A rubbing against a set of specimen blocks B while being pressed down by the weight body C (Serrano et al., 2015).

Amontons also recognized the role of asperities, definition given to the microscopic elements that define the roughness of a certain surface, in friction, although the first studies and physical basis of the laws originated in the behavior of small protuberances should be attributed to Phillipe de la Hire (Dowson, 1998; Hutchings, 2021). Amontons described that the asperities could be compared to a spring element where the force needed to lift them from point A to B can be proportionally compared to the load-displacement relation that a spring normally exhibits due to Hooke's law. Figure 3.7 shows the spring behavior of asperities according to Amontons.

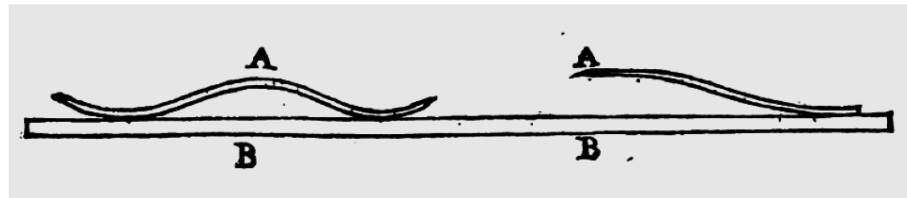


Figure 3.7 – Amontons' theory on friction being described by the spring behaviour of asperities. (Desplanques, 2014).

The lift force represents the elastic force required to strain those asperities and is directly and proportionally related to the friction offered by a surface due to its natural characteristic roughness (Desplanques, 2014; Dowson, 1998). Although somehow abstract, the concept of asperity deformation was developed and improved throughout the centuries by several researchers, leading more recently to friction force models to be based on this deformation phenomenon. LuGre model, for example, is one of the most common friction models associated to this phenomenon (Canudas De Wit & Lischinsky, 1995a; Dowson, 1998).

3.4 Friction Studies by Leonhard Euler

The first mathematical studies of friction were attributed to Leonhard Euler in the 18th century. Euler (1750b) treated the aspects concerning the equilibrium state and uniform motion of bodies, developing mathematical formulations to describe these phenomena.

The first mathematical approach to friction proposed by Euler was the interlocking asperity theory, where the surfaces of contacting bodies were represented by a set of triangular-shaped asperities in a sawtooth pattern, which interlock perfectly and are inclined at an angle, α , in relation to the horizontal plane (Besson, 2013). Figure 3.8 shows a schematic representation of the sawtooth pattern constituted by small triangular-shaped asperities that originate the roughness of a surface.

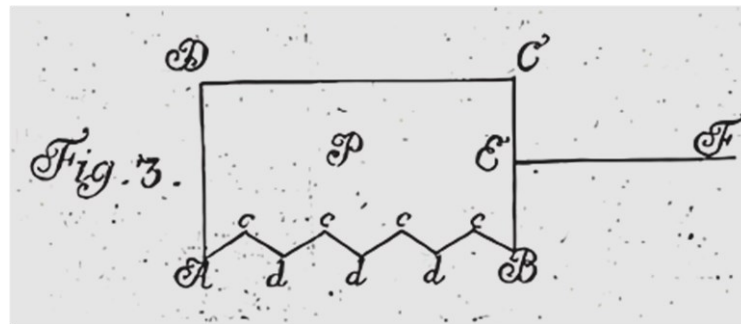


Figure 3.8 – Schematic representation of surface roughness as a set of triangular-shaped asperities in a saw tooth pattern by Euler. Each side of the triangular shape exhibits an inclination, α , in relation to the horizontal plane (Besson, 2013).

The block placed in an inclined plane, represented in Figure 3.9, is in equilibrium state when the external applied force, f , required to move the block up the plane and the frictional force acting on the block, f_f , exhibit the following equality relation

$$f = f_f \Leftrightarrow f_g \sin \alpha = \mu f_g \cos \alpha \quad (3.1)$$

Utilizing the tangent trigonometric function, it is possible to rewrite Equation (3.1) in a more simple and convenient form

$$\mu = \frac{\sin \alpha}{\cos \alpha} \Leftrightarrow \mu = \tan \alpha \quad (3.2)$$

which leads to the conclusion that Euler demonstrated the dependency relation of the friction coefficient, μ , with the tangent of the inclined plane angle, α . Moreover, it can also be attributed to Euler the introduction of the greek letter, μ , into the friction study area to represent the coefficient of friction.

Another important contribution of Euler to the scientific knowledge of friction is the distinction made between static friction and kinetic friction. The distinction between static and kinetic friction was also made using the inclined plane apparatus represented in Figure 3.9.

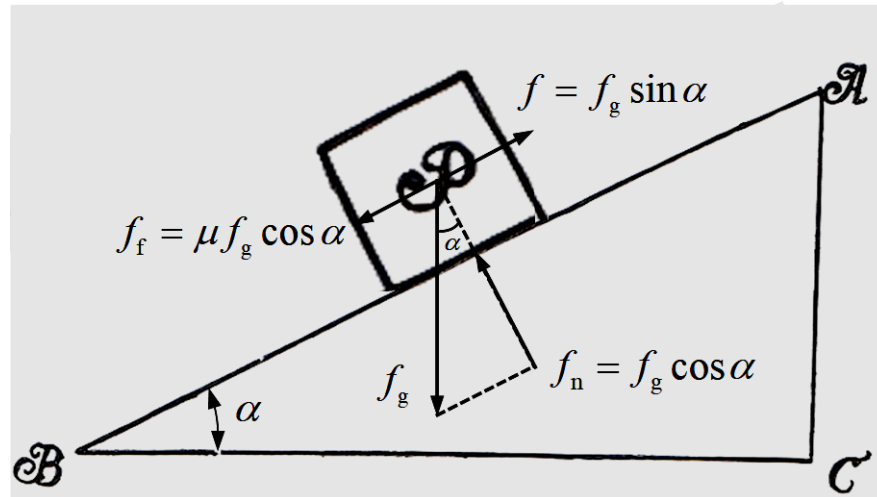


Figure 3.9 – Adapted Euler's inclined plane schematic (Dowson, 1998).

Although Euler did not present a justification that allowed to explain the loss of mechanical energy due to friction, since that problem did not belong to the scientific paradigm of the 18th century, demonstrations were made and led to the conclusion that if the plane was inclined until reaching a limit slope where the static equilibrium was verified, then a very small angle increase would result in a disturbance of the equilibrium state of the block and it would exhibit a fast linear motion down the plane. This observation contradicted Euler's first hypothesis that a small angle increase would result in the block's motion with a very small velocity, concluding that the kinetic friction, associated with the motion of a body, must assume a smaller value in comparison to the static friction (Dowson, 1998).

3.5 Friction Studies by Charles August Coulomb

Charles August Coulomb (1736 – 1806) is undoubtedly a pioneer researcher in the scientific area of friction. As described by Dowson (1998), Coulomb's studies on friction are a good example of high-quality scientific work prompted by practical problems.

The base for Coulomb's friction works relies on the developments by Amontons, although Coulomb demonstrated through several experiments that some of his conclusions contradicted the ones formulated by Amontons. Coulomb studied the sliding and rolling types of friction, proposing experimental validation of their behavior in practical large-scale applications, mainly related to pulleys, capstans, and the launching of ships on slipways (Dowson, 1998).

Figure 3.10 shows the apparatus developed by Coulomb to study the sliding friction between plane surfaces, which is composed of a solid wooden table, serving as the lower surface of the contacting pair, represented by (a), and by a sledge, represented by (b), that can have rails of various widths and materials attached.

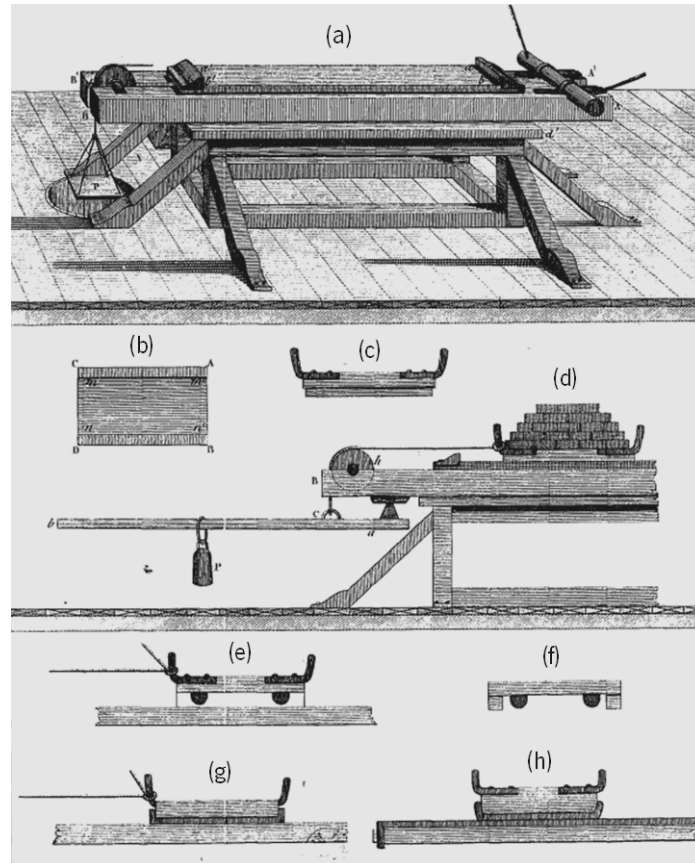


Figure 3.10 – Coulomb's experimental apparatus for the study of sliding friction. Adapted from Coulomb (1821).

The influence of the normal force in the friction force that acts in the reverse direction of the sledge sliding motion was studied by placing different weights on the sledge of Figure 3.10 (d) schematic. The sliding force was exerted by placing a weight along the length of a bar which is connected to a pulley that pushed the sledge over the table through a rope. The data obtained with this experiment was for various combinations of tribological pairs of materials such as oak, green oak, guaiac wood, fir and elm for the woods, and iron and copper for what concerns the metallic materials. Dry and lubricated conditions were analyzed using lubricants such as axle grease, tallow, soot, water and olive oil, both in smooth and rough surfaces, for pressures ranging up to approximately 30.4 MPa and maximum sliding velocities of about 18 km/h. The static state of motion time was also an important variable in Coulomb's theories of friction and his experimental data showed stationary times ranging from 0.5 s up to 4 days (Besson, 2013; Dowson, 1998).

Coulomb investigated the influence of four different types of factors that can have a significant impact on friction: (i) the nature of the materials in contact as well as the effect of dry and lubricated conditions, (ii) the area of contact, (iii) the normal load supported by the surfaces, which Coulomb described as pressure, and (iv) the duration of the contact between surfaces (Besson, 2013; Dowson, 1998). The main contributions to friction knowledge resulting from the experimental findings of Coulomb can then be exposed: (i) the friction exhibits an initial rise but soon reaches a maximum for tribological contacts of wood sliding on wood under dry conditions, being the force of friction essentially proportional to the load; (ii) the friction force for wood sliding on wood is essentially proportional to the load at any speed, but the kinetic friction force, normally associated to the steady state sliding motion, has a lower magnitude compared to the static friction force, especially those seen after long periods of rest; (iii) the friction force for metals sliding over metals without lubricant is proportional to the load and there is no difference between the magnitude of both static and kinetic friction; (iv) for the metal-wood tribological pair under dry conditions, the static friction rises at a small rate, taking some hours or even days to reach its limit, while for metal-metal pairs this limit is reached almost immediately, and for wood-wood pairs this static friction rise takes up a short minute time. It was also verified that for wood-wood and metal-metal pairs, the kinetic friction force magnitude under dry conditions is almost independent of the tangential velocity, while for wood-metal contacts, the friction force magnitude increases with the increase of tangential velocity between contacting surfaces (Dowson, 1998).

The contributions by Coulomb to the development of friction did not only encompass the practical effects of friction, but also delved into the theoretical explanation of friction phenomena. According to Coulomb, friction could have two possible causes, one of them being the interlocking of asperities on surfaces, as represented in Figure 3.11, that can only be separated by bending, breaking or rising above other, or by adhesion/cohesion phenomena verified between the surfaces in contact (Besson, 2013).

Coulomb verified that only the interlocking of surface asperities was responsible for the friction phenomena, while cohesion represents a small influence, since friction is proportional to the normal load and independent of the contact area. The cohesion represents an inevitable increase of friction because there are multiple contact points between surfaces, but the magnitude of the referred contacts, although not being null, can be negligible for a macroscopic analysis (Besson, 2013).

The asperities of contacting surfaces, after the application of a tangential force, deform until a critical point where the opposing asperities of the adjacent surface slip out of the interlocking mesh and sliding motion initiates, as it is represented in the Figure 3.11 (c) scheme. Since the deformation of asperities is assumed to be constant in the sliding phase, the friction force is also constant, fact that led

a third law of friction to be attributed to Coulomb where is stated that “kinetic friction is independent of the sliding velocity” (Dowson, 1998).

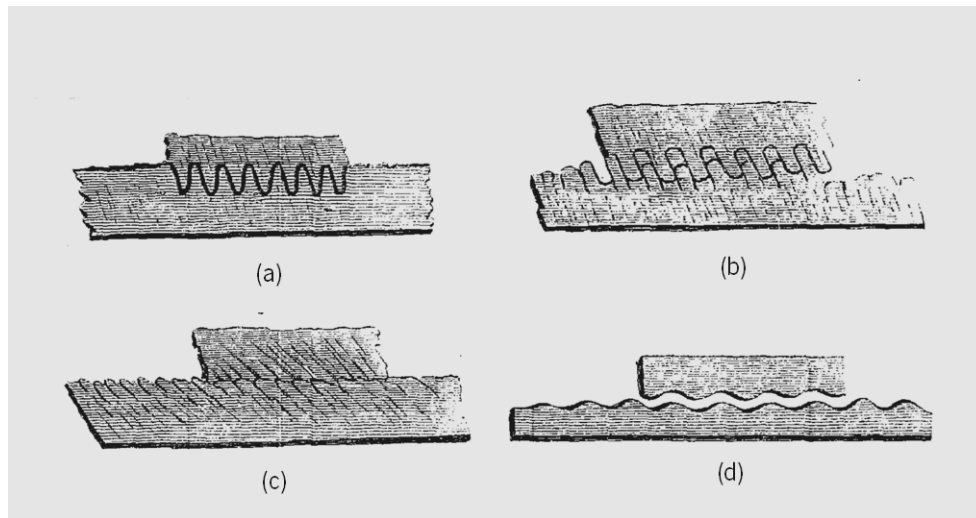


Figure 3.11 – Interlocking phenomena between asperities surfaces. Adapted from (Coulomb, 1821).

Another important aspect of Coulomb's friction discoveries is correlated to experimental apparatuses built to study the governing phenomena of rolling friction. Figure 3.12 shows a workbench projected to study the rolling friction phenomenon, where Coulomb stated that rolling friction was dependent on the normal reaction force, although presenting a magnitude lower than sliding friction and that the resistance to rolling was inversely proportional to the radius of the roller (Dowson, 1998).

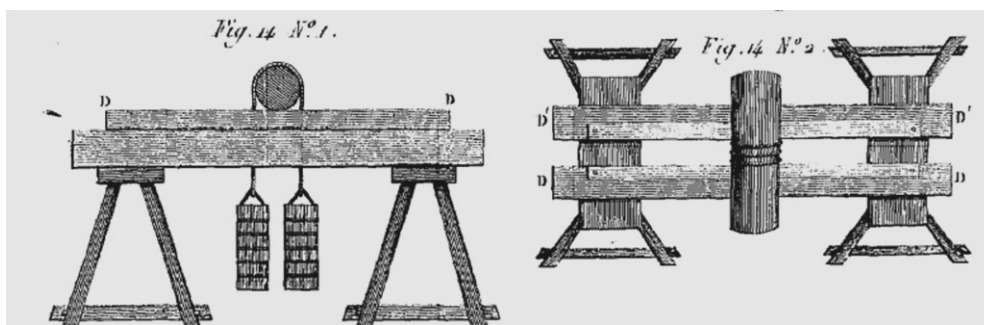


Figure 3.12 – Coulomb's experimental apparatus for the study of rolling friction (Coulomb, 1821).

It is clear that Coulomb was one, if not the most influential researcher in the study of friction, paving the way for the subsequent research findings in the 20th century by introducing and developing the concepts of surface roughness, asperity deformation, adhesion and breaking of asperities, which are arguably the most important concepts in the scientific domain of friction.

3.6 Other Studies on Friction

Following the essential contributions made to the knowledge of friction right after the Industrial Revolution, that remain unchanged or suffered small improvements until present, in this section are briefly presented new interpretations of the friction phenomenon, primarily associated to wear effects.

The initial significant theoretical inquiries into the mechanisms of wear were conducted by Holm, Burwell and Strang, Archard, and Archard and Hirst (Besson, 2013). These researchers presented a coherent theoretical analysis, introducing models to describe and explain wear and providing formulas for calculation of the wear rate as the volume of material removed per unit length of work. More specifically, Holm proposed an adhesive mechanism at the atomic level, suggesting that bodies in contact only made contact at the peaks of the surface irregularities (asperities). This concept, later expanded upon by Bowden and Tabor, involved Holm assuming that with each approach of two atoms between the contacting asperities, existed a constant probability of an atom being detached from the surface of either of the two bodies (Besson, 2013).

The comprehensive synthesis of research on friction, lubrication, and wear can be found in the collective works of Bowden and Tabor, encompassed in their two volumes published in 1950 and 1964 (Bowden & Tabor, 1950, 1964). Despite some knowledge gaps or incomplete aspects, their contributions established the foundation for the modern study of friction and served as a primary reference in the field. Bowden and Tabor provided extensive experimental results, particularly focused on metals and wood, covering fundamental aspects of the subject such as surface structure, topography, sliding and rolling friction, lubrication, and wear (Dowson, 1998).

One of the most significant contributions of Bowden and Tabor was the introduction of the concept of the real area of contact, comprising numerous small regions known as asperities or junctions of contact. These authors demonstrated the strong dependence of friction force between sliding surfaces on the real contact area and developed an interpretation of Amontons' laws based on an adhesion model called the adhesive (plastic) junction model, which remains a fundamental reference in the friction domain. To support their model, Bowden and Tabor delved into the topography of surfaces and asperities, especially in metals (Besson, 2013).

Additionally, Bowden and Tabor explored friction from the perspective of purely elastic sliding processes. In such cases, they applied the theory developed by Hertz in 1881 concerning the deformations of contact between two elastic solids under load. The Hertz contact theory, still considered valid today, forms the basis for calculating asperities' deformations in friction models.

3.7 Summary and Discussion

This chapter presented an overview of the first scientific contributions to the knowledge of friction phenomena. Although there were no scientific methodologies at the time, early civilizations such as the Egyptians recognized the existence of dissipative effects between contacting surfaces when sliding large objects in the ground, leading to the use of ancient lubricants to smooth these processes.

Motivated by improving his designs and comprehend the physical phenomena that governed his designs, Leonardo Da Vinci can be considered the first scientist to propose a research methodology to study friction. The experimental apparatuses of Da Vinci, despite its simplicity, allowed him to being the first to study the influence of apparent contact area upon frictional resistance, creating the embryogenesis of the well-known friction law that specifies that the apparent area of contact does not influence the friction response.

Guillaume Amontons proposed and built more sophisticated experimental workbenches, utilizing mechanisms inspired by industrial tools of that era, such as polishing glass tables. By testing different pairs of contacting materials and the addition of coats of pork fat that acted as lubricants, Amontons was able to formulate two of the most widely recognized laws of friction: (i) the force of friction is directly proportional to the applied load and (ii) the force of friction is independent of the apparent area of contact. Moreover, Amontons was also responsible for associating friction with the deformation of asperities, which constitutes nowadays the fundamental basis of many friction force models.

The theoretical works of Leonhard Euler were mainly associated with the proposal of the interlocking asperity theory to explain the frictional behavior between contacting surfaces. The main contribution of Euler to friction's scientific knowledge is associated with the distinction of static and kinetic friction coefficients, through studies based on inclined planes. The introduction of the Greek letter μ to refer to the friction coefficient is also attributed to this author.

Charles August Coulomb is probably the most widely recognized and associated name with friction, creating many experimental apparatuses based on large scale real-world applications. Coulomb's works relies on the developments by Amontons, although contradicting some theories formulated by the latter.

Although the third fundamental law of friction is attributed to Coulomb, which states that the kinetic friction is independent of the sliding velocity, the first two fundamental laws are associated with Amontons. The fundamental laws of friction should be referred to as Amontons-Coulomb friction laws instead of only attributing credit to the latter, as it is done by many authors. The nature of contacting materials, dry or lubricated conditions, contact area, normal load, and contact duration significantly influence friction were also studied by Coulomb, as well as other key variables include surface roughness,

asperity deformation, adhesion, and the breaking of asperities, forming the fundamental concepts in the scientific study of friction.

To conclude, the studies performed in the past millennium paved the way to a solid scientific research methodology associated with friction, leading to numerous mathematical and phenomenological models that can accurately predict the dynamic behavior of mechanical systems considering the dissipative effects of friction.

4. FRICTION – CONCEPT, DISTINCTION AND INFLUENCING FACTORS

Friction is a phenomenon which is observed in a great variety of situations. This subject is studied in the context of tribology, a word that derives from the Greek root *tribos*, which means “rubbing”, and the suffix *-logy* that represents the “study or science of”, and encompasses the study of lubricants, lubrication, friction, wear and bearings (Dowson, 1998). Since this science covers a wide range of scientific areas, it can be described as an interdisciplinary subject that requires the expertise study of physicists, chemists, and mechanical engineers, as well as materials scientists and metallurgists (Hutchings & Shipway, 2017).

In a more common approach, friction can be universally described as the resistance to relative tangential motion which occurs between two interacting surfaces. There are several laws that attempt to describe the friction phenomenon, being the most relevant ones exposed in Table 4.1.

Table 4.1 – Summary of the most relevant friction laws. Adapted from Mo et al. (2009).

Nature of the friction theory	Friction law
Macroscale	Amonton’s law
	Bowden and Tabor
Single asperity	Non-adhesive – based on Hertz model
	Adhesive
Multi-asperity picture of nanoscale contact	Non-adhesive
	Adhesive

Since the main objective of this work is to make a critical analysis of the friction force models in dynamical systems, the description and analysis will be focused on the macroscale friction concept laws.

The most utilized and accepted macroscale theory to explain friction is related to the existence of asperities representing the topography of a certain surface, known as the roughness of a surface. Bowden and Tabor (1950, 1964) made valuable contributions to tribology science that are still relevant and used today, even if more advanced theories exist nowadays. These authors were responsible for the introduction of the concept of real area of contact, composed of numerous small regions, known as asperities or contact junctions, and demonstrated that the frictional force between sliding surfaces heavily relied on this real contact area. Figure 4.1 exhibits the real area of contact between two surfaces characterized by the black dots.

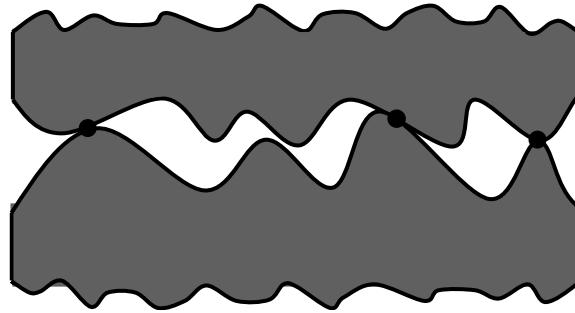


Figure 4.1 - Contact between asperities in the interface of two surfaces. Contact only occurs in the dots, which explains the difference between the real area and the apparent area of contact.

A section view of the graphical representation of the contact between asperities is presented in Figure 4.2 exhibiting the physical difference between the real and apparent area of contact.

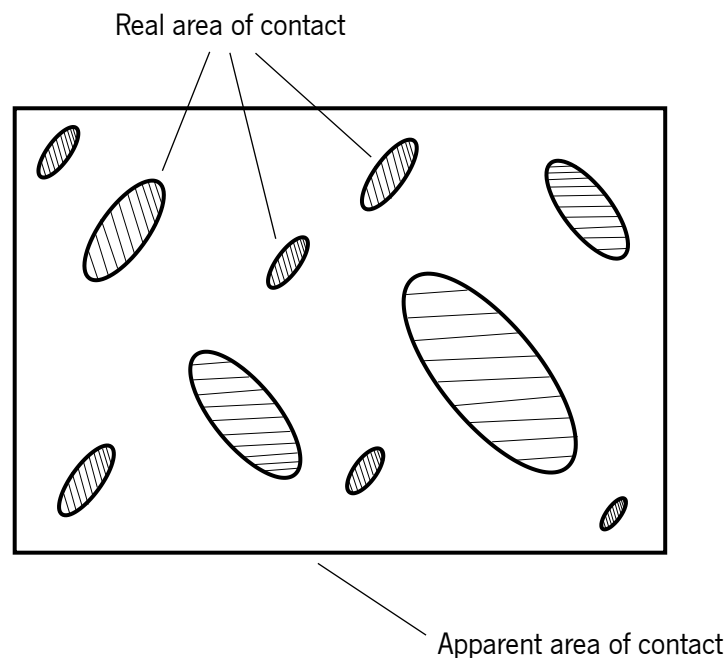


Figure 4.2 – Schematic representation of sectioned asperities, corresponding to the real area of contact, and the apparent area of contact.

It is then intuitive to understand that the real area of contact is a relatively small fraction of the apparent area of contact. Bowden & Tabor (1950) determined for various loads the ratio between the apparent area of contact and the real area of contact in steel-on-steel tribological pairs through electrical conductivity methods, concluding that the real contact of area is 10^{-5} to 10^{-2} times smaller than the apparent area.

These authors developed an interpretation of Amontons' laws based on an adhesion model called the adhesive (or plastic) junction model, which remains a fundamental reference till present days. In this

model, due to surface irregularities, contact occurs primarily between the highest asperities. Consequently, the real contact area is much smaller than the apparent macroscopic area and increases with load, due to the deformation of asperities' tips. The pressure at these small contact areas leads to deformations, high temperatures, and local junctions, possibly resulting in local micro-welding, particularly in metals (Bowden & Tabor, 1950).

The total force required to separate all junctions, that is, the opposite of the friction force, is proportional to the real contact area and is influenced by the deformation – either plastic or elastic – of asperities, as represented in Figure 4.3.

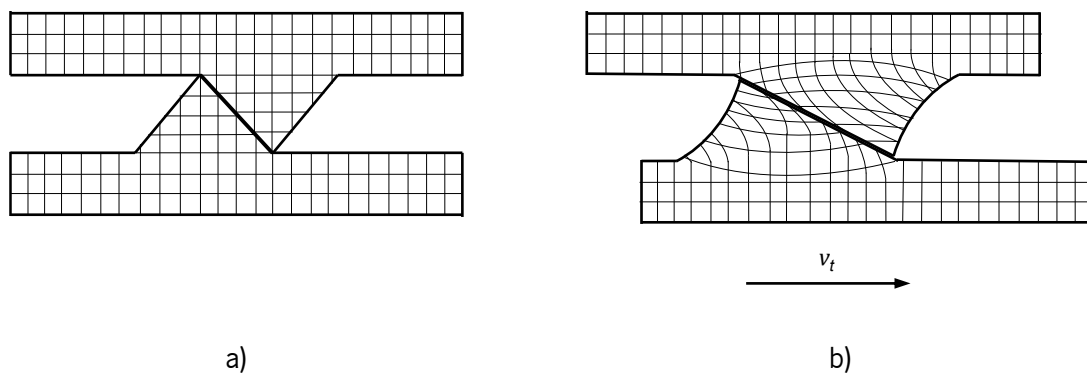


Figure 4.3 – Simple representation of the contact and plastic deformation of asperities (a) initial static position (b) deformation of asperities due to the existence of a tangential velocity, v_t . Adapted from Green (1955).

Dry sliding friction, according to this model, arises from the rupture of numerous interfacial adhesive bonds, which are similar to small spot welds, continually formed as two objects slide against each other. The energy loss in the friction mechanism is attributed to the plastic deformation of the asperities (Bowden & Tabor, 1950). However, the adhesion junctions model alone cannot fully account for dynamic friction, especially during the sliding process. Throughout this motion, some of the rough spots on surfaces will inevitably collide and break off and, in many cases, the deformations of these asperities are not plastic but elastic (Bhushan, 2013). In metals, the plastic adhesive effect of surface roughness typically predominates, resulting in the creation of permanent tracks or scratches due to plastic deformation. On the other hand, for materials such as rubber and polymers, elastic deformations with hysteresis often dominate since the energy absorbed during frictional interaction is dissipated as heat (Bhushan, 2013).

If the interaction between the adhesion and deformation processes during sliding is considered to be negligible, the calculus of the total intrinsic frictional force, f_i , can be determined by combining both phenomena. This total force then comprises the force required to shear adhered junctions, f_a , and

the force needed to provide the energy of deformation, f_d , and can be expressed as described by Bowden & Tabor (1964) as

$$f_i = f_a + f_d \quad (4.1)$$

In what concerns the investigation of friction from a deformation model of surface asperities perspective, for the case of purely elastic contact, the total real area, A , defined by Hertz's theory (1881) is not proportional to the load (W), but to a power equal to $2/3$, as indicated in the following equation

$$A \propto W^{2/3} \quad (4.2)$$

For perfectly plastic behavior of the asperities, the relation between the total real area of contact and the load (W) is as follows

$$A \propto W \quad (4.3)$$

indicating that for this type of material behavior the area of contact is directly proportional to the applied load (Hutchings & Shipway, 2017). These relations between total real area, A , of contact and applied load, W , are simplifications made considering round tip asperities of the same radii which exhibit the same individual area of contact. However, real surfaces exhibit a non-uniform distribution of asperities with varying radii and heights, which need to be determined using statistical distributions. When a load is applied to a real surface, the contact area for each individual asperity increases, and additionally, more asperities come into contact and begin to bear some of the load as shown in Figure 4.4.

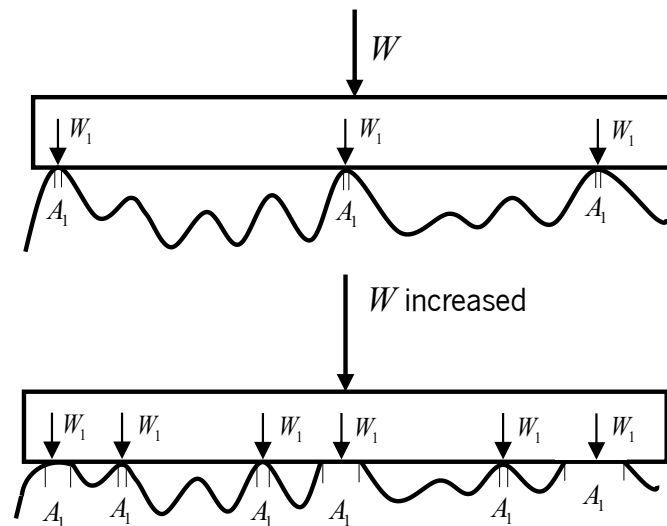


Figure 4.4 – Asperities of a rough surface in contact with a smooth surface to the applied load W . An increase in the value of W leads to the deformation of the contacting asperities and more asperities come into contact with the smooth surface, increasing the total real contact area, A .

Greenwood & Williamson (1966) developed more sophisticated models to describe the contact between rough surfaces, incorporating stochastic distributions of the heights of surface asperities. It was observed that when the height distribution follows an exponential pattern, there exists an exact proportionality between the real contact area and the applied load. Similarly, in the case of a Gaussian height distribution, an approximate relationship of proportionality was found to that of the exponential distribution. This latter finding holds significance, as the authors presented data indicating that the heights of surface asperities in many instances closely approximate a Gaussian distribution.

Although more recent models were developed to describe the behavior of contacting asperities, some of them having the works of Greenwood & Williamson (1966) as a basis of improvement, they will not be explored, since that topic does not fit in the main scope of this work

4.1 Static and Kinetic Friction

4.1.1 Static and Kinetic Coefficient of Friction – a comprehensive practical case

It is intuitive to understand that the force needed to initiate the movement of a certain object is higher than the force required to maintain its motion. However, defining the transition instant between static and motion state is a rather complex problem. The analysis of this situation may start with the concept of inertia described by Isaac Newton's first law of motion which states that a body at rest, remains at rest, and an object in motion remains in motion at a constant velocity, unless acted on by a non-null resultant force. In fact, despite inertia being a property of an object and friction a force that acts on an object, their effects are somehow related since they both offer a resistance to the transition from a static to a kinetic state of motion.

The transition of static to kinetic friction can be analyzed from the perspective of a simple experiment involving the movement of a block sliding along an inclined surface as observed in Figure 4.5.

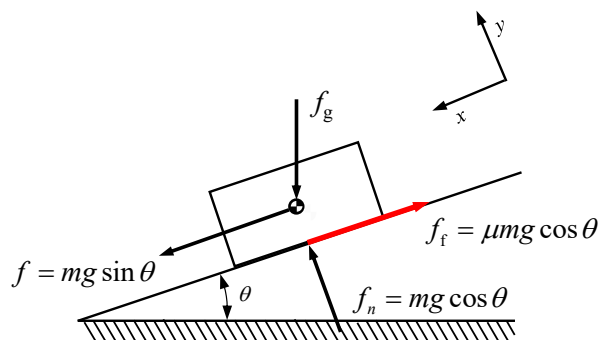


Figure 4.5 – Forces acting on a block placed in an inclined plane.

Considering a situation of static equilibrium, the block is stationary since the resultant of forces is null in both x and y directions. Particularly, in the equilibrium state in x direction, the following equality can be written

$$\sum F_x = 0 \Leftrightarrow mg \sin \theta = \mu mg \cos \theta \quad (4.4)$$

which can be re-written in the form of

$$\mu = \tan \theta \quad (4.5)$$

Although Euler formulated this equation in terms of potential and kinetic energy, so Equation (4.5) has more parameters related to sliding distance and sliding velocity, in this study a more simplified approach will be followed.

In the sequence presented in Figure 4.6, it can be observed that despite f and f_f are in equilibrium until a certain angle θ_3 , these forces suffer a variation of both direction and magnitude. In the last scheme of Figure 4.6, the block starts sliding because the force f acquires a higher magnitude than the friction force f_f , that maintains the block stationary, being safe to affirm that it reaches the slip phase of its motion. In the remaining schemes, the block is stationary, despite the inclination angle of the plane suffering an increase, so the block is said to be in a stick phase.

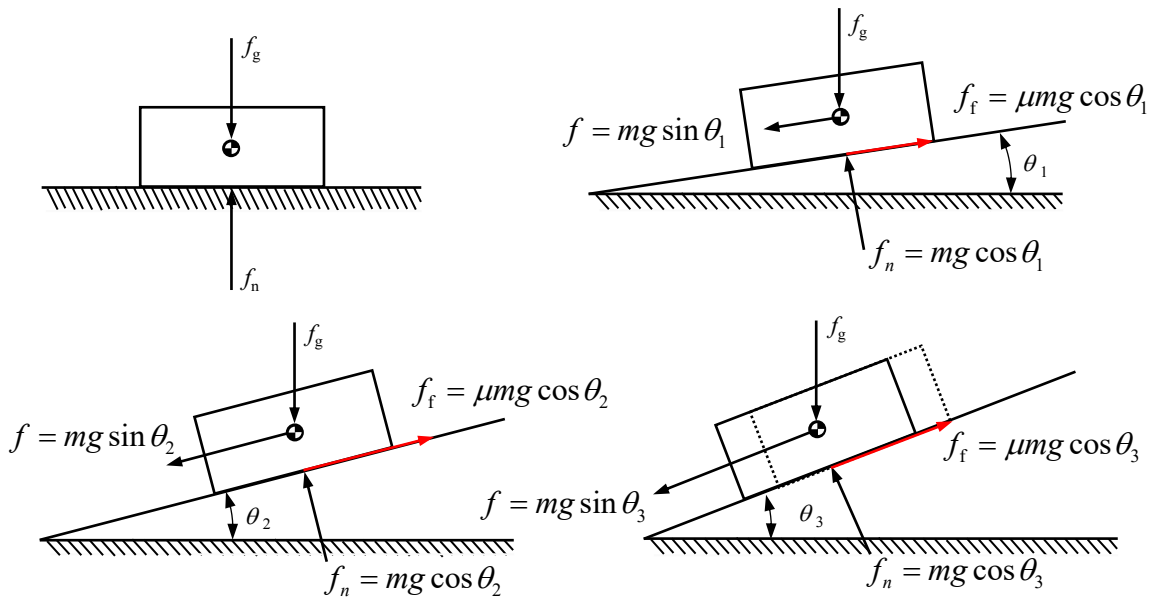


Figure 4.6 – Inclined plane positioned at different angles of inclination. In the bottom right scheme, the block starts sliding along the plane.

Since f depends on the sine trigonometric function of the inclined plane angle θ , $\sin \theta$, it is known that this factor will increase with the increasing value of θ while resulting in a decrease

of the factor $\cos \theta$. Thus, the coefficient of friction, μ , is the factor that compensates this difference between the trigonometric functions of the inclination angle, θ , and increases proportionally to it as demonstrated by Equation (4.5). This explains why the static friction force suffers an increase before reaching the maximum static friction force value, which is materialized by the rising slope of the graph presented in Figure 4.7

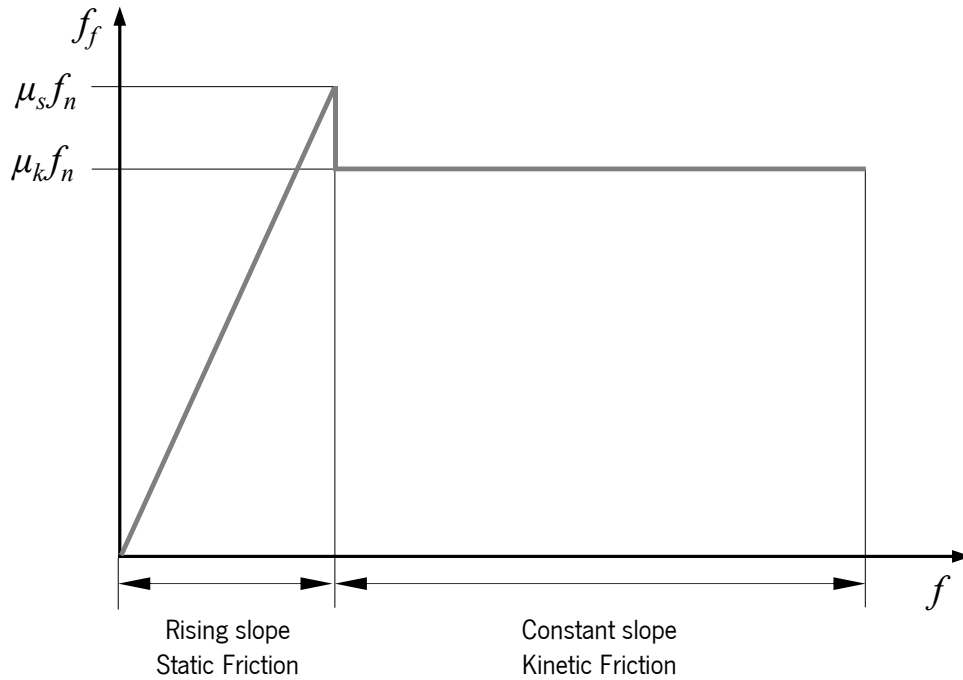


Figure 4.7 – Graph of the variation of friction force between static and kinetic states as a function of the applied external force f .

In the extreme situation, which represents the eminence of motion, the inclined plane reaches an angle θ , that allows the force f to assume a value greater than the friction force f_f , and the block starts sliding along the inclined plane with an increasing velocity. The acceleration of the block can be determined via Newton's second law of motion considering friction using the following formulation

$$a = g(\sin \theta - \mu \cos \theta) \quad (4.6)$$

4.1.2 Coulomb Friction Force Model

The previous analysis of the static and sliding behavior of a block on an inclined plane leads to the representation of the graph in Figure 4.8 of friction force, f_f , as a function of relative tangent velocity, v_t , where the friction force increases until the start of motion, that is, until the instant where the relative velocity becomes non-null.

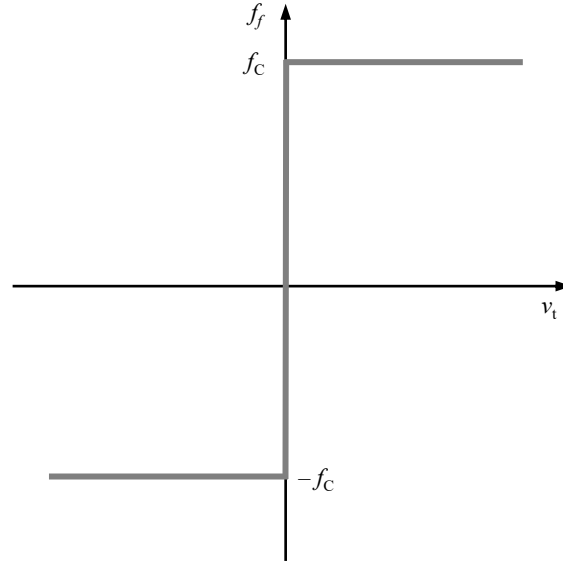


Figure 4.8 – Graphical representation of the Coulomb dry friction force model as a function of tangential velocity.

More specifically, Figure 4.8 describes the dry Coulomb friction model. This model can be considered as the predecessor for all the friction models that exist today and can be described by the following mathematical system of equations

$$f_f = \begin{cases} f_c \operatorname{sgn}(v_t) & \text{if } \|v_t\| \neq 0 \\ \min(\|f_e\|, f_c) \operatorname{sgn}(f_e) & \text{if } \|v_t\| = 0 \end{cases} \quad (4.7)$$

in which

$$f_c = \mu_k \|f_n\| \quad (4.8)$$

and f_n as the normal contact force, μ_k the kinetic coefficient of friction, f_c is the Coulomb friction magnitude, f_e is the resultant of the external forces acting on the block and f_f represents the friction force developed between the contacting surfaces of the block and the inclined plane. Analyzing the first branch of Equation (4.7) it is possible to notice that this model exhibits a dependence on the relative velocity between the sliding surfaces that can be mathematically represented by the signum function

$$\operatorname{sgn}(v_t) = \begin{cases} \frac{v_t}{\|v_t\|} & \text{if } \|v_t\| \neq 0 \\ 0 & \text{if } \|v_t\| = 0 \end{cases} \quad (4.9)$$

with 0 representing a null vector with the same dimensions as v_t . This model is characterized by its simplicity of comprehension and by the fact that it only needs one input coefficient to be fully defined, that is, the kinetic friction coefficient, μ_k . Despite its simplicity, its implementation in computational codes can be quite complex given the uncertainty in specifying a friction force at null velocity, which can

lead to numerical instabilities in dynamic simulations of mechanical systems. This uncertainty in the friction force value in the nearby region of the null value of tangential velocity requires regularization techniques that prevent numerical instabilities from occurring during simulations.

4.1.3 Sliding Friction

The sliding friction is related to the resistance that an object offers when it slips over another. This type of dynamic friction is directly related to the theory that states that the surfaces are composed of asperities with different lengths and heights. The movement is then characterized by the relative motion between the asperities of both surfaces in a relatively large area of contact and the resistance to motion, i.e., the friction force, results from the shearing of junctions formed as a result of adhesion phenomena at the interface (Bowden & Tabor, 1950; Greenwood et al., 1961; Persson, 1998).

In Figure 4.9 is possible to observe the classical example of sliding friction materialized by a sliding block and it is also represented a zoom-in of the shearing deformation of asperities in the interface of both surfaces in contact.

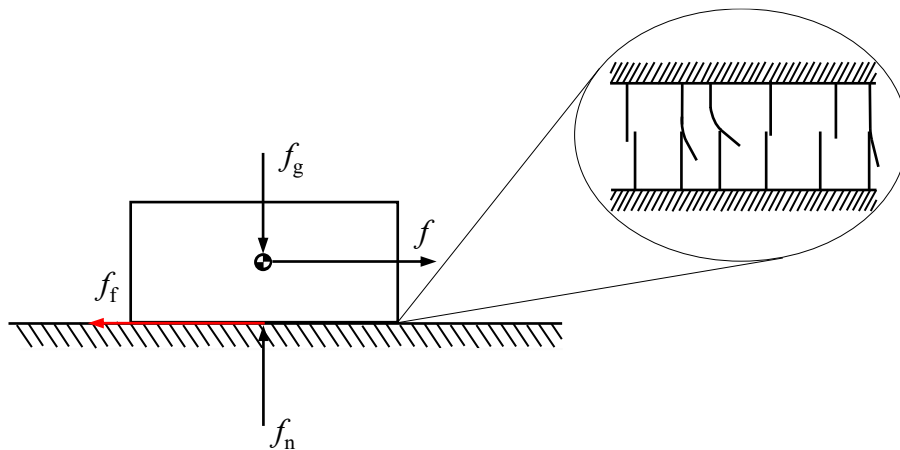


Figure 4.9 – Macroscopic sliding motion of a block and zoom exhibiting the microscopic phenomenon that explains sliding friction as elastic and plastic deformation of asperities.

4.1.4 Rolling Friction

Rolling friction exists when two contacting bodies exhibit rolling motion between them. This type of friction can be described through two different situations that can be verified in Figure 4.10: a) a rolling disc in a stationary state on a plane surface where the contact pressure translates into a normal resultant force, f_n , which is normal to the contacting surface and aligned with the weight acting on the rolling body; b) when the disc starts to roll due to an external force, f , the contact pressure distribution follows the deformation disc-plane, which results in a translation of the contact point by a distance, d , in relation to

the center of gravity of the disc. This displacement of the contact point originates a moment, M_r , named rolling resistance moment, that opposes to the rolling movement of the disc and is directly related to the deformation of the contacting bodies.

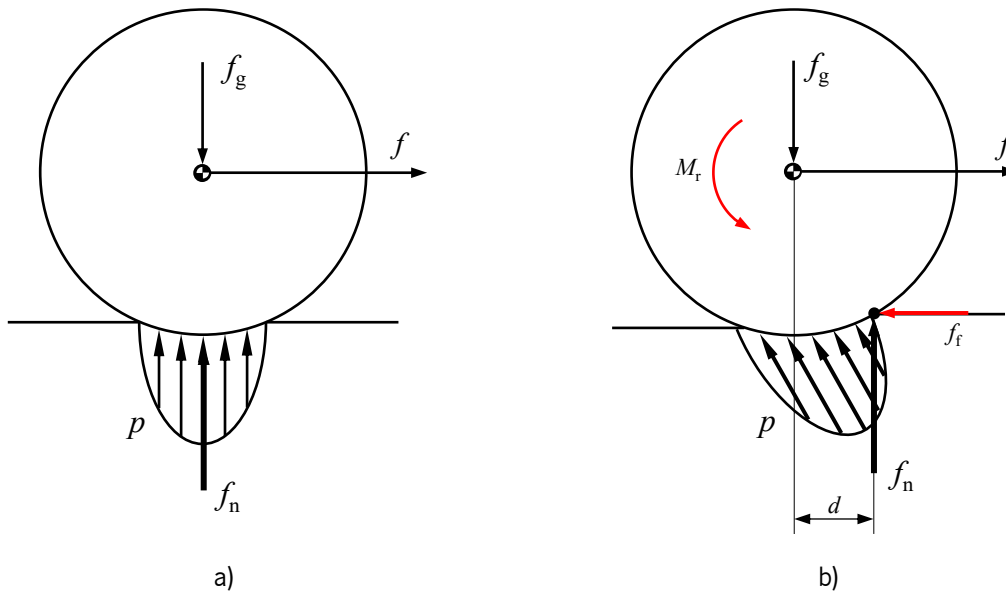


Figure 4.10 – Rolling friction of a disc in a plane a) static situation and pressure distribution b) kinetic situation due to the application of the force f .

Moreover, this surface contact that is generated due to the elastic deformation of the contact surfaces of both bodies can be explained by hysteresis losses. Tabor (1952, 1955) and Attack (1955, 1958) demonstrated that the front portion of the contact of a sphere with the plane translates into elastic work, due to compression stresses, that would be equally compensated by the rear portion of the circle of contact when it recovers elastically and moves the sphere forward, due to tension stresses on the rear of the sphere. That is not verified in reality due to the hysteresis phenomenon, so the force required to move the sphere along the plane is non-null, which then allows to conclude that exists a resistance force acting on the contact surface (Greenwood et al. 1961). According to Dahl (1968), tension and compression properties principally govern the friction process.

The rolling form of motion is more complex than sliding motion and most of the knowledge that exists nowadays is related to Reynolds (1876), Hertz (1886) and Heatcote (1921) (Doménech et al. 1987). One fact that confirms this statement was made by Reynolds where whatever materials that compose the rolling and plane bodies, there will always be slipping motion in the contact surface, although the magnitude of it might be too minor to be measured, so rolling motion then combines rolling and sliding motions which makes it complex to characterize (Reynolds, 1875).

4.2 Factors influencing frictional behavior

The resistance to sliding primarily manifests in the regions near and between solid surfaces. Identifying the specific attributes of contact conditions and materials that contribute significantly to friction force is a significant challenge in developing friction tests and analytical models.

Friction models incorporate various approaches, including geometric considerations such as surface roughness and asperity interlocking, mechanical properties-based variables involving shear properties of solids and substances between surfaces, being the most common situation the existence of lubricants, fluid dynamics principles, electrostatic forces between surface atoms, chemical compatibility, surface temperature, and other variables regarding the dynamics of the system in analysis, such as relative sliding velocity, load or time of contact, for example. Given the diversity of these approaches, the number of potential variables for use in predictive friction models can be substantial. Table 4.2 exhibits a summary of the most common factors influencing frictional behavior accordingly to Blau (2001).

Table 4.2 – Factors influencing frictional behaviour (Adapted from Blau (2001))

Category	Factor
Contact geometry	Conformity of the components: macroscale mating of shapes
	Surface roughness: microscale features – asperity shape, size distribution
Fluid properties and flow	Lubrication regime; boundary, mixed, elastohydrodynamic, hydrodynamic
	Lubricant film thickness, pressure, temperature, shear stress and viscosity
	Oxidation and acidification of lubricants
Relative motion	Unidirectional or reciprocating motion
	Motion variation – accelerations, pauses, stick-slip
Applied forces	Magnitude and variation of the normal force
Third-bodies	Particles entrained in the lubricant and wear particles
	External contaminants
Temperature	Thermal effects on material properties: thermoelastic instabilities, ductility increase
	Thermal effects on lubricant properties
Stiffness and vibrations	Contact compliance
	Damping of frictional or external vibrations

Due to the numerous potential factors influencing friction, it is essential to pinpoint the key variables relevant to each specific case. This allows for the selection of appropriate test methods or simulations tailored to each particular case study (Blau, 2001).

In the context of this work, the effect of each variable presented in Table 4.2 will not be analyzed in detail, since it is not the main focus of the critical and practical analysis of friction force models in dynamical systems conducted in this work.

4.3 Summary and Discussion

In this chapter, a summarized presentation of the main aspects of friction was elaborated, focusing primarily on the concept, distinction and influencing factors that affect the relative motion between contacting bodies.

The distinction between static and kinetic friction coefficients was made considering the practical and intuitive example of the inclined plane, similarly to what has been done in the works of Euler. The conducted analysis allowed to recognize the existence of higher friction forces at static states of motion, compared to the lower and constant value of kinetic friction force associated with sliding motion. Although the exposition of these concepts was made using the inclined plane as a practical example, other not so intuitive examples, such as the pressure cone theory (Flores et al., 2023), could be utilized to explain the differences between static and friction coefficients.

The Coulomb friction force, considered the predecessor of all modern friction force models, was analyzed both in terms of its general graphical representation and mathematical representation. The simplicity of this model is associated with the fact that only needs one input parameter to be fully defined – the kinetic friction coefficient, μ_k . Despite its rudimentary nature, implementing this friction model in computational models is complex due to the non-differentiable behavior near the null value of relative velocity, which can cause numerical instabilities. Regularization techniques, which will be further discussed in the next chapters, are required to ensure stable dynamic simulations.

Both sliding and rolling friction phenomena were also analyzed in a simple and effective manner. In one hand, sliding friction is mainly associated with the theory that surfaces consist of asperities of varying sizes, with movement arising from their relative motion across a large contact area. Friction force originates from the shearing of junctions formed by adhesion at the interface. On the other hand, rolling friction is more complex than sliding friction due to hysteresis, where elastic recovery at the rear region of the contact area does not fully compensate for compression at the front. This results in a non-zero

force associated to the motion of rolling objects. Additionally, rolling motion inherently involves minor slipping at the contact surface, combining rolling and sliding motions, which complicates its characterization.

A summary of actors influencing frictional behavior was also gathered in this chapter, ranging from contact geometry to fluid properties and flow associated with lubricants, relative motion between bodies, applied forces, existence of third-bodies, temperature and stiffness and vibrations, although the effects of each variable are not presented in the context of this work.

5. FRICTION PHENOMENA

Friction is an inherently non-linear phenomenon, shaped by a range of complex sub-phenomena that influence the interactions between contacting surfaces. While many of these sub-phenomena are well understood and supported by mathematical formulations, incorporating them into comprehensive friction force models remains a challenge. This complexity arises from the diversity of materials, lubrication regimes, kinematic variables like relative velocity, and stochastic factors that impact surface interactions. Moreover, characterizing each phenomenon in terms of a numeric value attributed to a model parameter is also a challenging task.

Classical models, such as Coulomb, offer simplified approximations to determine the magnitude of the dissipative effects of friction, but fail to capture critical aspects like static friction exceeding the kinetic friction force levels or the intricate transitions between motion states. Over time, advancements in tribology have uncovered new frictional behaviors, allowing for more detailed and realistic representations of surface contact phenomena. The evolution of computational power has further enabled the accurate simulation of these phenomena, improving predictions of the behavior of dynamic mechanical systems.

This chapter provides a structured exploration of key friction phenomena, progressing in order of increasing complexity. Topics include stiction, breakaway force and dwell-time, viscous effects, the Stribeck effect, stick-slip motion, pre-sliding displacement and micro-slip, frictional lag, and non-reversibility. Together, these phenomena form the foundation for understanding the variables and physical assumptions materialized in the mathematical formulations of friction force models.

5.1 Static friction (stiction), breakaway force and dwell-time

It is intuitive to understand that the force needed to maintain the movement of a certain object is lower than the force required to initiate the motion. With this fact in mind and analyzing the graphical representation of the Coulomb friction force model, it can be contradictory in the first place, since an increasing value of relative tangential velocity between surfaces does not translate into a reduction in friction force, due to the fact that this friction model does not consider any differentiation between the static friction force level and the Coulomb (kinetic) friction force level.

Considering the previously mentioned phenomenon, several studies have been developed to explain it. Morin (1833) introduced the idea of a friction force associated with a static state that is higher than the Coulomb friction and Rabinowicz (1951) was one of the most influential scientists in the study

of the nature of static and kinetic coefficients of friction (Morin, 1833; Rabinowicz, 1951). Through several experiments, the latter author was able to address the transition between sticking and sliding motion and, consequently, between static and kinetic friction as a function of the displacement of a block sliding along an inclined plane.

Due to adhesive bonds that are generated between the asperities of both surfaces in contact and the shear forces developed in the initial phase of motion to overcome the resistance offered by these same asperities, it is possible to explain why the static coefficient of friction, denoted as μ_s , is higher than the kinetic coefficient of friction, μ_k . This means that in the diagram of friction force in function of the tangential relative velocity, there is a static friction force f_s , higher than the Coulomb friction force, f_C , that now represents unequivocally the kinetic friction force according to Figure 5.1.

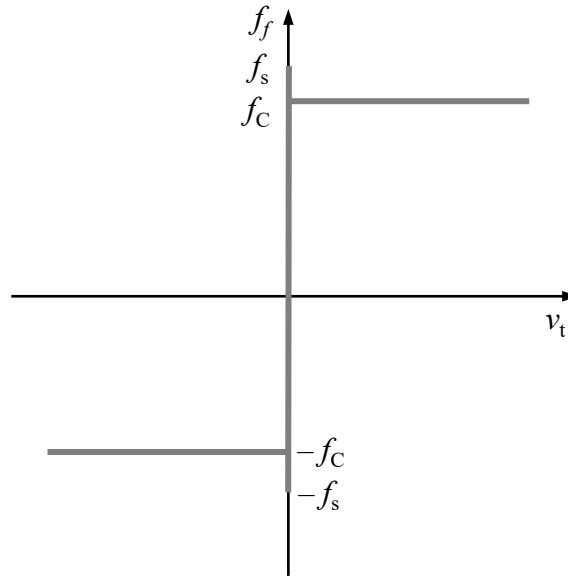


Figure 5.1 – Graphical representation of the Coulomb dry friction model exhibiting the static friction (stiction) phenomena and the Coulomb friction force level for non-null values of tangential relative velocity, v_t .

The introduction of the stiction phenomenon in the Coulomb friction force model leads to the following formulation

$$f_f = \begin{cases} f_C \operatorname{sgn}(v_t) & \text{if } \|v_t\| \neq 0 \\ \min(\|f_e\|, f_s) \operatorname{sgn}(f_e) & \text{if } \|v_t\| = 0 \end{cases} \quad (5.1)$$

where f_s represents the static friction force level (stiction).

Associated with this stiction phenomenon is commonly introduced the breakaway force concept. The breakaway force is associated with the maximum friction force value, thus corresponding to the static

friction force level. Due to the high nonlinearities associated with this phenomenon, the measurement of it is quite challenging and complex, which can explain the inaccuracy sometimes verified in control applications during acceleration and deceleration phases of motion. These inaccuracies arise from the dependency of the static friction on the time of contact, which is referred to as dwell-time dependency. The transition from null relative velocities until reaching the breakaway force level during the stick phase occurs in a finite but continuous time interval, that is often difficult to capture and model. Only highly complex dynamic models such as the one proposed by Gonthier et al. (2004) are capable of modeling and predicting this phenomenon.

5.2 Viscous effect

In the early 19th century, with the increasing interest and development of hydrodynamics theories, the viscous effect was introduced in the Coulomb friction model due to the influence of lubricants in the surface's contact. This behavior is materialized by the fluid lubricant layer that exists between the two rubbing surfaces. In this case, the viscous behavior is considered to be a linear function of the tangential relative velocity as represented in Figure 5.2 (Olsson et al., 1998).

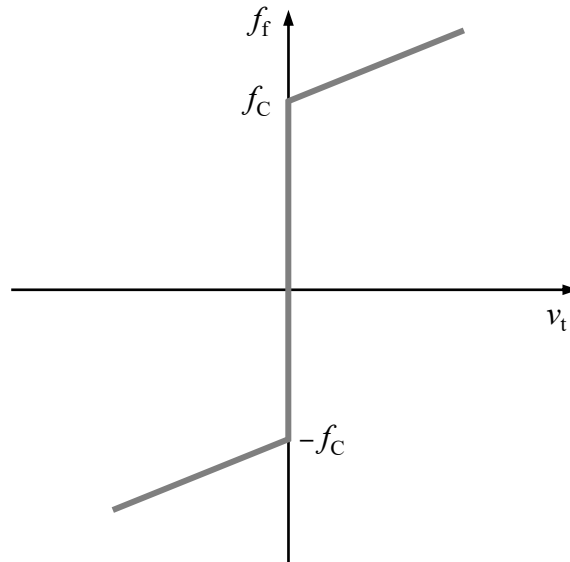


Figure 5.2 – Graphical representation of the Coulomb friction model considering viscous effect.

The linear increasing behavior of the friction force due to viscous effects indicates that the lubrication regime is the full-film lubrication type. The Coulomb friction force model formulation considering the viscous effect component can then be written as

$$f_f = \begin{cases} f_c \operatorname{sgn}(v_t) + f_v v_t & \text{if } \|v_t\| \neq 0 \\ \min(\|f_c\|, f_c) \operatorname{sgn}(f_c) & \text{if } \|v_t\| = 0 \end{cases} \quad (5.2)$$

in which f_v is a viscous friction coefficient related to the viscosity of the fluid.

Although the relation between friction force and velocity is considered to be linear, to have a better fit to experimental data, this effect can be described by a non-linear dependence on velocity using an extra variable, δ_v , which depends on the geometry of the application (Olsson et al., 1998), making the second parcel of the first branch of Equation (5.2) equal to

$$f_v \|v_t\|^{\delta_v} \operatorname{sgn}(v_t) \quad (5.3)$$

5.3 Stribeck effect

The viscous effect mentioned in the previous section described the relationship between friction force and relative tangential velocity in the presence of a lubricated contact between the surfaces as a linear relation, but this phenomenon only captures the behavior of a full-film lubrication situation (Olsson et al., 1998).

To be able to capture multiple stages of the lubrication regime related to different tangential relative velocities, the Stribeck effect was added to the Coulomb dry friction model (Stribeck, 1902). This phenomenon is responsible for ensuring a continuous decrease from static to kinetic friction as can be observed in Figure 5.3.

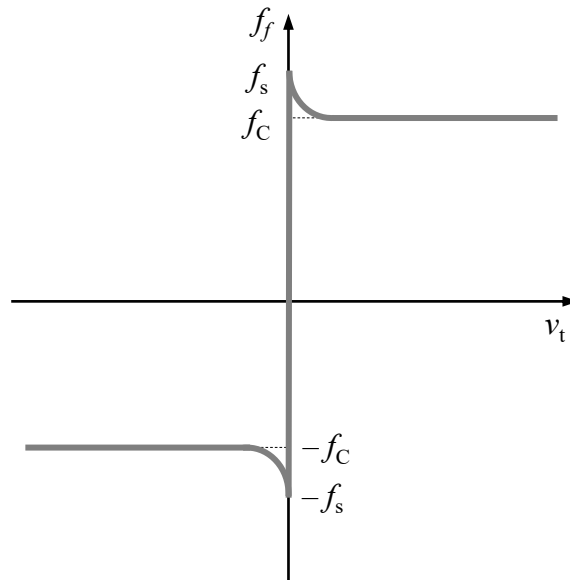


Figure 5.3 – Graphical representation of the Coulomb friction model considering the Stribeck effect.

This continuous drop is characterized by a decreasing friction force with increasing relative tangential velocity, v_t , at low velocity, and is responsible for capturing the transition from a boundary lubrication regime to a mixed-film lubrication, being described numerically by the following formulation

$$f_f = \begin{cases} f(v_t) & \text{if } \|v_t\| \neq 0 \\ \min(\|f_c\|, f_s) \text{sgn}(f_c) & \text{if } \|v_t\| = 0 \end{cases} \quad (5.4)$$

where $f(v_t)$ is an arbitrary velocity dependent friction function. Many authors in literature proposed mathematical expressions to represent the Stribeck effect as a form of a velocity dependent friction function.

Tustin (1947) utilized an exponential expression to establish a relationship between the tangential velocity, v_t , and the Stribeck velocity, v_s

$$f_f = f_c + (f_s - f_c) \left(1 - e^{-\frac{v_t}{v_s}} \right) \quad (5.5)$$

Bo & Pavelescu (1982) also proposed an exponential function to represent the Stribeck effect where the total friction force is determined by the following function

$$f(v_t) = f_c + (f_s - f_c) e^{-\gamma |v_t|^{\delta}} \quad (5.6)$$

where γ is a constant factor defined to control the shape of the Stribeck curve and δ is a factor dependent on the geometry of the contacting surfaces.

Hess and Soom (1990) defined a mathematical expression to numerically describe the Stribeck curve, typically known as Lorentzian model, and that is formulated as

$$f(v_t) = f_c + \frac{f_s - f_c}{1 + \left(\frac{\|v_t\|}{v_s} \right)^2} + f_v v_t \quad (5.7)$$

where $f_v v_t$ is a term used to describe the viscous effect phenomena.

Popp & Stelter (1990) described the decreasing characteristic of the friction force associated with the Stribeck effect using the following formulation

$$f(v_t) = \frac{f_s - f_c}{1 + \lambda |v_t|} + f_s + \alpha v_t^2 \quad (5.8)$$

where λ and α are specific curve fitting parameters, defined as $0.01 \text{ s}^2/\text{m}^2$ and 1.42 s/m , respectively.

Armstrong-Hélouvry (1991) modified the Bo & Pavelescu (1982) exponential function where the Stribeck characteristic velocity is explicitly defined and considered the viscous effect by adding the term $f_v v_t$, yielding the friction force to be calculated as

$$f(v_t) = f_c + (f_s - f_c) e^{-\left(\frac{v_t}{v_s}\right)^\delta} + f_v v_t \quad (5.9)$$

in which the factor dependent on the geometry of the contacting surfaces, δ , is considered to be equal to 2 by the authors. By considering $\delta = 2$, the model represented by Equation (5.9) is a Gaussian model and it is nearly equivalent to the Lorentzian model of Equation (5.7).

It is also very common to associate the Stribeck effect with the viscous effect, which allows the complete transition from a boundary lubrication regime to a full film and also allows the static friction models to simultaneously capture multiple friction phenomena in the same formulation. In Equation (5.7) and (5.9) it is possible to note that the third parcel refers to the viscous effect due to the existence of the viscous friction force, f_v . Figure 5.4 illustrates both viscous and Stribeck effects in the Coulomb friction force model.

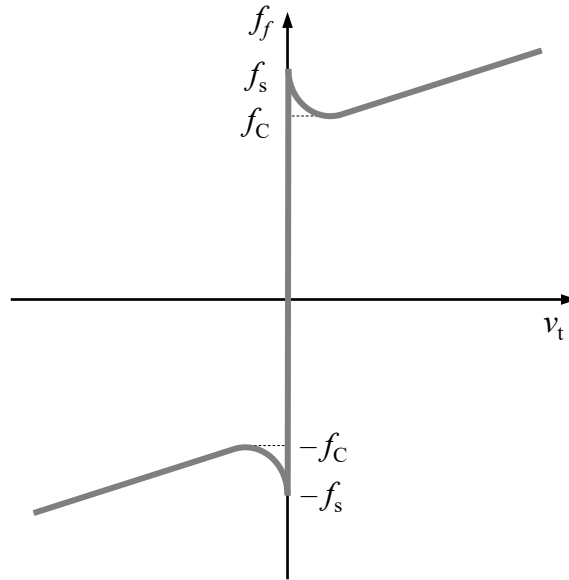


Figure 5.4 – Coulomb friction force model with viscous and Stribeck effects.

5.4 Stick-slip

The stick-slip motion phenomena can be characterized as an oscillatory behavior of sliding or slipping followed by sticking phases between two surfaces during relative motion. This is a highly undesirable phenomenon, since its complex nature affects the precise control and positioning of machine elements and, consequently, industrial machines such as robots and computerized numerical controlled machine tools (CNC) that use them (Armstrong-Hélouvry, 1992; P. E. Dupont & Dunlap, 1993).

Sliding bodies at different velocities and loads, most of the time, do not show a homogeneous sliding movement between them. The occurrence of sticking phases during this relative motion is very well-known but difficult to predict since it manifests in different amplitudes that change with velocity and the force that produces that same relative velocity variation. Figure 5.5 presents the general curve used to describe the stick-slip phenomena at low velocities a) and high velocities b) (Rabinowicz, 1961).

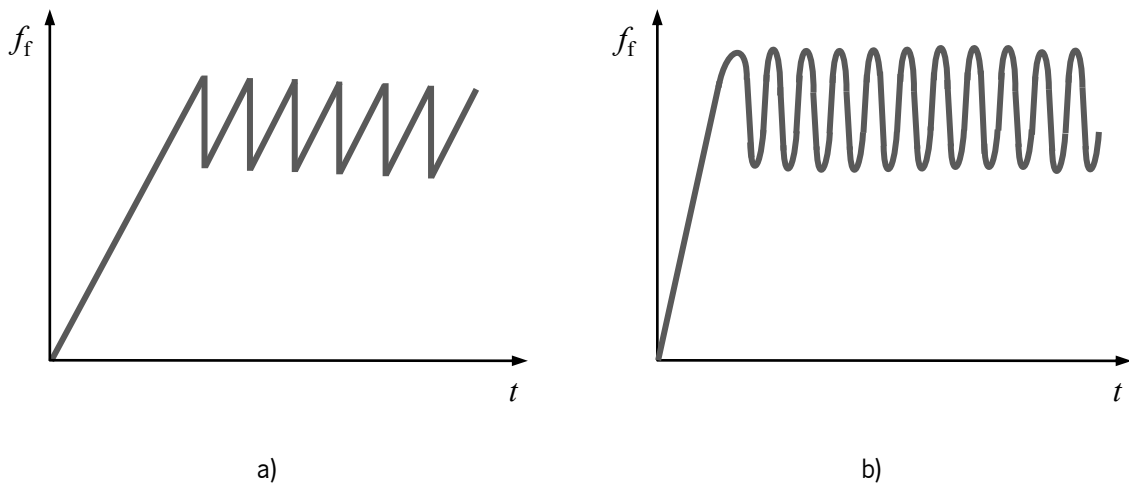


Figure 5.5 – Friction force, f_f , variation as a function of time, t , associated with the stick-slip phenomena for a) low velocities, b) high relative velocities.

The rising slope of the friction-time graphs of Figure 5.5 represents the sticking phases. It is possible to notice that in terms of time, this behavior takes longer to achieve its maximum value of friction force than the slip phase. This can be explained by the friction force dwell-time dependency associated with the stiction phase, since the moving body takes some time, associated with its correspondent displacement and time of contact, to achieve the breakaway force, associated with the stiction friction force level. The kinetic friction force, directly related to the sliding moving phase is, in this case, characterized by the vertical portions of the graph curve in Figure 5.5 a). The sliding phase is much faster and shows an almost instantaneous behavior for low relative velocities (Figure 5.5 (a)). For high relative velocities, the stick-slip phenomenon exhibits a sinusoidal behavior between static and kinetic friction force (Figure 5.5 (b)) (Rabinowicz, 1961).

5.5 Pre-sliding displacement and micro-slip

The studies of Rabinowicz (1951) allowed for a more comprehensive approach concerning the transition between static and kinetic friction as a function of the relative displacement between contacting surfaces.

The sliding motion can be divided into two main regimes: pre-sliding and gross sliding. The gross sliding corresponds to the macroscopic relative motion between two surfaces, commonly known as macro-slip. However, a different type of sliding motion, the pre-sliding, can be verified if the force applied in the sliding body translates into a relative tangential friction force which is below a certain force threshold, i.e., the Coulomb friction force level. Although it is not verified a macroscopic displacement in this case, when the force is decreased to zero, there is a residual displacement that is called pre-sliding, associated with a small motion in the elastic range of the friction force-displacement relationship. This regime can be described by a hysteretic behavior of force and displacement as can be observed in Figure 5.6, since there are points in the interface of the contact area where adhesion bonds have been broken and some points where the adhesion phenomenon is still verified, being this phenomenon known as micro-slip (Chen et al., 2019; Parlitz et al., 2004).

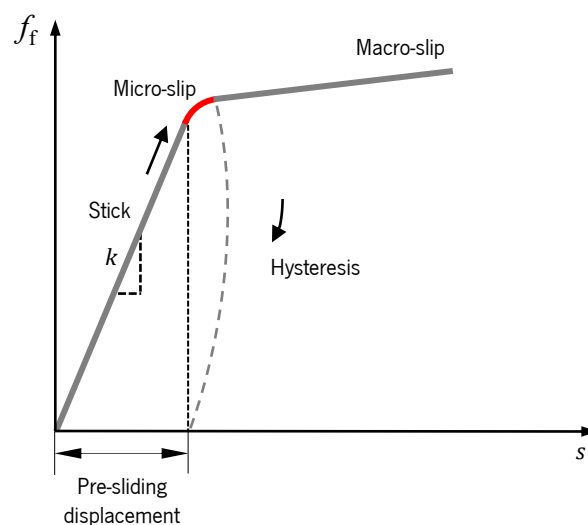


Figure 5.6 – Pre-sliding displacement and association with the micro-slip and macro-slip phenomena.

Figure 5.6 represented above corresponds to the interpretation of Courtney-Pratt and Eisner (1957) for the pre-sliding phenomena considering micro-slip. It is based on the theory of asperity junctions being deformed elastically and then plastically. The plastic deformation corresponding to the breakaway force marks the transition between stiction and sliding friction, since it is considered that all the asperities adhesions have been broken due to shear forces. During the micro-slip phase, when the plastic deformation of asperities begins, if the pulling force ceases, it is verified a small permanent displacement

which is shown by the hysteretic behavior of the dashed line in Figure 5.6. Normally, the pre-sliding behavior, as well as the micro and macro-slip illustrated before, are represented with a hysteresis loop of friction force as a function of relative displacement, but to simplify the presentation of the phenomenon, only the first quadrant of the coordinate plane was used.

The pre-displacement and micro-slip phenomena are very important to consider in the simulation and modelling of control systems since they can negatively impact the control of mechanical and mechatronics systems due to small displacements in certain moving parts translate into significant displacements elsewhere in these mechanisms, thus introducing positioning and precision errors (Rodellar & Griñó, 2005).

5.6 Frictional lag (frictional memory effect)

Similarly to the studies where a displacement delay was found before reaching the maximum static friction force, Rabinowicz (1957) also developed theories based on this same displacement delay to explain the multi-valued friction behavior that affects the stick-slip phenomena. Although this behavior was verified for dry and boundary-lubricated sliding of point contacts, which can rapidly respond to velocity and direction variations, this was not observable to contacts operating mostly under mixed lubrication conditions as was demonstrated by Bell and Burdekin (Hess, 1990).

Hess and Soom (1990) found the need to explain experimentally and mathematically this phenomenon related to the delay in the change of the friction force as a function of a change in the contact interface relative velocity under lubricated conditions. With an experimental apparatus developed by these authors, they have been capable of studying and determining the effects of changes in frequency, applied load and viscosity in the friction force response of their system in function of velocity.

The experimental results show a hysteretic loop behavior of the friction force in relation to velocity, similar to the generic representation presented in Figure 5.7.

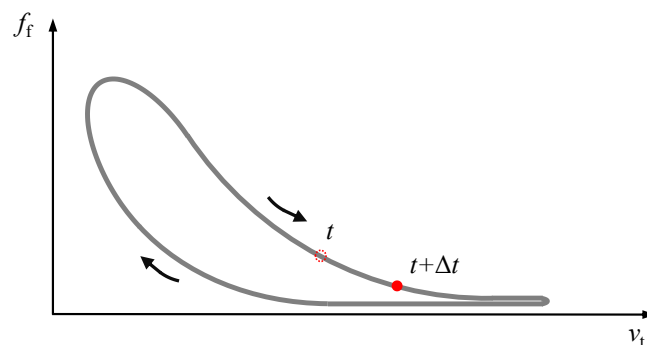


Figure 5.7 – Graphical representation of the frictional lag phenomena.

An analysis of the results obtained by Hess and Soom (1990) allows to conclude that the hysteresis loop associated with the variation of the friction force with the rate of velocity expands when increasing the load applied, increasing the viscosity of the lubricant and/or with an increase of the oscillatory frequency. Besides these conclusions about the multiple variables that affect the size of the hysteresis loop (load, viscosity and frequency), it is possible to notice that frictional lag is a dynamical characteristic resulting in lower friction forces when velocity is decreasing and larger forces for increasing velocities. This fact can easily be verified looking at the red dots placed in the hysteresis loop of Figure 5.7. For the friction force – relative velocity pair of coordinates that originate the filled red dot at the time instant $t + \Delta t$, it is possible to observe that due to the frictional lag phenomena, the friction force value is, in fact, the one correspondent to the dashed dot at the time instant t , which exhibits an higher friction force value, corroborating the theory that for increasing velocities, the friction force assumes larger values and vice versa.

5.7 Non-reversibility

The transition from static to kinetic friction can be described by smooth and gradual transitions explained by viscous and Stribeck effects. When a generalist dynamic system reverses its direction of motion and a transition from kinetic to static friction is verified, another important phenomenon occurs, commonly known as non-reversibility of the friction force-velocity characteristics

Various studies in literature that describe experimental apparatus to measure friction force and identify its characteristics and phenomena present friction force-velocity graphs that exhibit non-reversibility behavior. One example is the work by Bell & Burdekin (1969), in which the stick-slip vibration within a machine tool feed drive was studied and experimental data was presented for various contacts (dry and lubricated), giving clear evidence of the different slopes of friction forces for the acceleration and deceleration phases of motion.

The non-reversibility term was first introduced by Powell & Wiercigroch (1992) after analyzing the dynamics of a base-excited oscillator equipped with an electrorheological damper functioning as a Coulomb damper. The force in the damper transitions from static to dynamic levels during the acceleration phase. Specifically, during this phase, the system moves from one state (point 1) to another (point 2), as depicted in Figure 5.8. During deceleration phases, the relative velocity reverses its direction, reflected by a transition between other states (points 3 to 4 of Figure 5.8), which does not follow the same path exhibited during the acceleration phase.

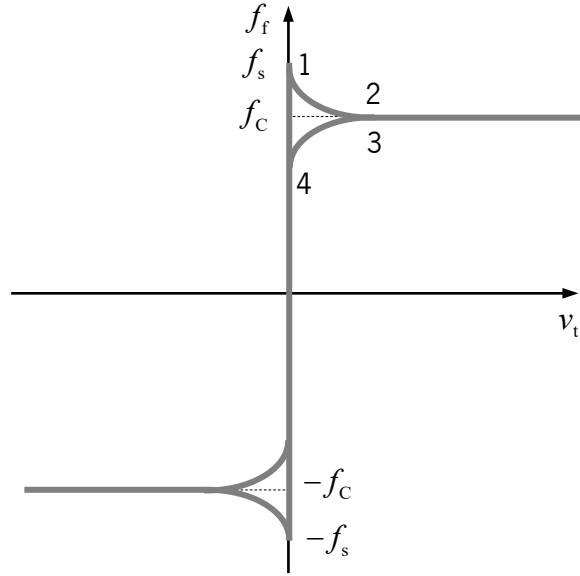


Figure 5.8 – Theoretical schematization of the non-reversibility of friction force. In the direction from 1 to 2, the system is in an acceleration phase, while from 3 to 4 is in a deceleration phase.

These dynamical friction phenomena are effectively modeled using the following exponential function pairs proposed by Wiercigroch et al. (1999)

$$\begin{aligned} f_{fu}(v_t) &= f_c \left[1 + \frac{f_s - f_c}{f_c} \exp(-a|v_t|) \right] \text{sgn}(v_t) \\ f_{fd}(v_t) &= f_c \left[1 - \exp(-b|v_t|) \right] \text{sgn}(v_t) \end{aligned} \quad (5.10)$$

where the first expression denoted as $f_{fu}(v_t)$ represents the friction force behavior for the acceleration phase associated with the upper slope of the curve and $f_{fd}(v_t)$ is associated with the deceleration phase associated with the lower slope of the curve. The constants a and b are defined to adjust the acceleration and deceleration slopes, respectively.

The non-reversibility of friction force is verified because friction behaves asymmetrically during changes in motion. When motion starts or reverses direction, the friction force transitions between static and dynamic states, a process that involves energy dissipation and does not happen instantaneously, as verified earlier in the description of the frictional lag phenomenon. Additionally, the hysteresis loops verified for the friction force-velocity relationships, due to material properties, surface interactions, damping effects and other non-linear and stochastic effects, explains the differences on the friction force behavior during acceleration and deceleration phases of motion.

5.8 Summary and Discussion

In this chapter, a complete and detailed presentation and analysis of the principal phenomena associated with friction that are considered in the mathematical formulations of various friction force models was conducted. Each phenomenon, including stiction, breakaway force, dwell-time, viscous effects, the Stribeck effect, stick-slip motion, pre-sliding displacement and micro-slip, frictional lag, and non-reversibility, was explored to highlight its unique contributions to the overall frictional behavior between contacting surfaces.

Understanding these phenomena is essential for advancing the accuracy of friction force models. Each contributes to the complex dynamics observed in real-world mechanical systems, yet their nonlinear and interdependent nature poses significant challenges for integration into multibody simulations. Accurate modeling requires not only a thorough comprehension of these mechanisms but also robust mathematical formulations that can replicate their effects under varying conditions, such as changes in relative velocity, material properties, and lubrication regimes.

The necessity of incorporating these phenomena into friction models becomes evident when considering the precision required in engineering applications. Failure to adequately represent these behaviors can lead to discrepancies between simulated and observed performance, potentially compromising the reliability of various systems.

With advancements in computational capabilities, it is now possible to incorporate these phenomena into increasingly sophisticated models. Consequently, such enhanced models are indispensable for optimizing system performance, improving durability, and reducing energy losses across a broad range of engineering applications.

In conclusion, the phenomena discussed in this chapter underscore the complexity of friction phenomena and the necessity of accurately predicting their effects on the dynamic behavior of mechanical systems.

6. STATIC FRICTION MODELS

The Coulomb friction model is characterized by a simple graphical shape where the friction force, given by the product between the kinetic friction coefficient, μ_k , and the normal reaction, f_n , assumes a positive value for a certain direction of relative tangential velocity and a symmetric value of friction force when the system in analysis changes its direction. This model can be defined as an empirical model since its graphical representation derived simply from experiments and observations made by Coulomb.

Despite its simplicity at first sight, given by the fact that its implementation only requires one parameter, the kinetic friction coefficient, μ_k , the Coulomb friction model becomes a very complex problem to solve when trying to implement it in computational applications. The observed challenge stems from the discontinuity associated with the frictional force value corresponding to a relative velocity of null magnitude. When the value of relative tangential velocity is null, this friction force model cannot be described by any type of mathematical function since it violates one of the basic principles for a given set of data being described by a function – there is a value of the domain, in this case the null value of tangential velocity, that has multiple and undetermined correspondent range values, in this case friction force values, which means that the model is non-differentiable in the null relative velocity region.

Considering this computational implementation problem, several strategies were developed throughout the years to surpass this numerical instability and “smooth” the transition near the null value of tangential velocity so that friction can be described by a mathematical continuous function.

The models analyzed in this section, which can be fitted in the static category, have a common variable in common – the tangential velocity, v_t . This relative velocity developed in the interface between contacting bodies is very important in the context of this type of models since most of their formulations are made as a function of the aforementioned velocity. The existence of a relatively small number of parameters needed to define these models, limits their capacity of modelling complex phenomena related to physical interaction between the surfaces

The static friction force models studied are divided into two main categories as the Figure 6.1 diagram indicates: static without stiction, where the maximum value of friction force is the kinetic one, also recognized as the Coulomb friction force level, f_C , and the static with stiction models, that as the name suggests, are capable of modelling the maximum value of friction force as the static friction force level, f_s .

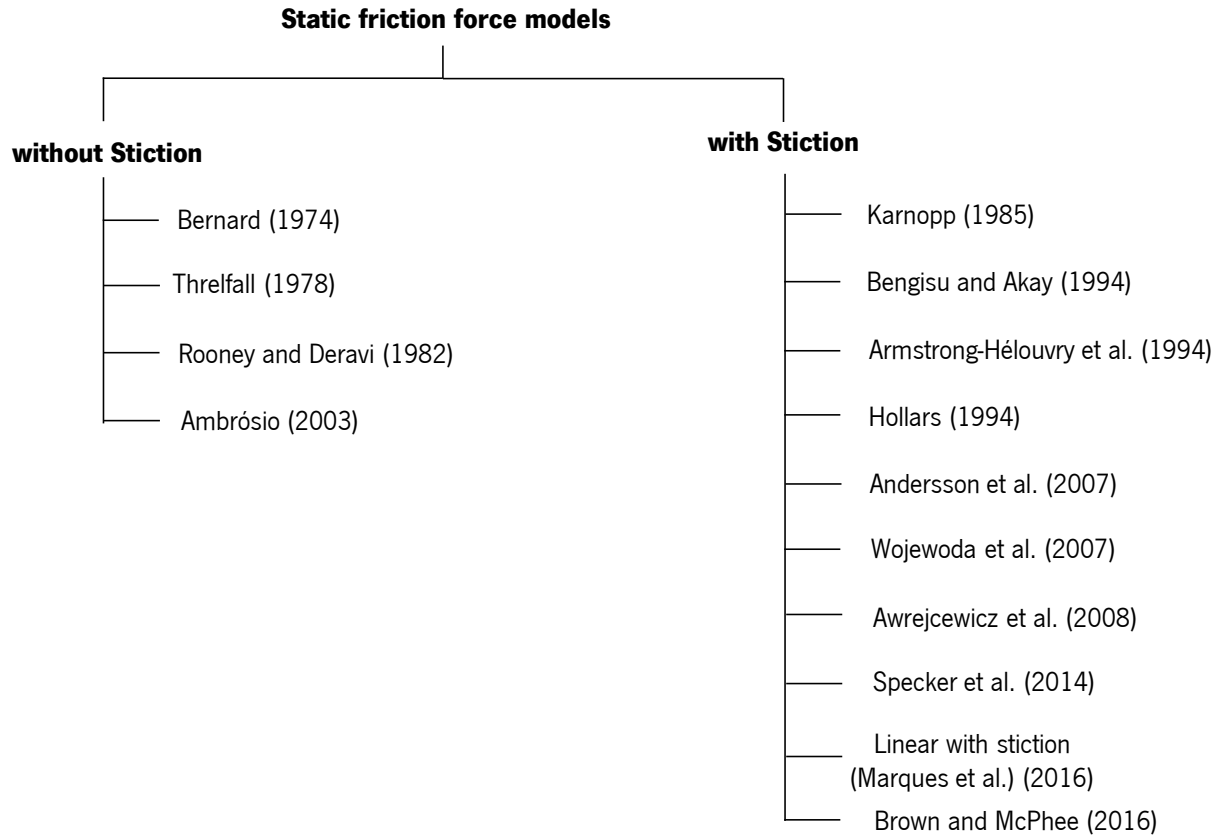


Figure 6.1 – Static friction force models divided by models with stiction and without stiction categories.

It is important to note that the models in each category are presented in an ascending order of appearance, with the oldest models being presented first until reaching the most recent formulations. Furthermore, one of the main objectives of this work is to investigate the computational performance of these friction models, so only regularized approaches to static friction models are studied and simulated through a benchmark problem.

6.1 Static friction force models without stiction

6.1.1 Bernard model (1974)

Bernard (1974) proposed one of the simplest regularization approaches to the Coulomb friction force model by considering a linear transition from a negative value of Coulomb friction force, $-f_c$, to a positive value, $+f_c$. Figure 6.2 shows the smoothened regularization modeled by Bernard.

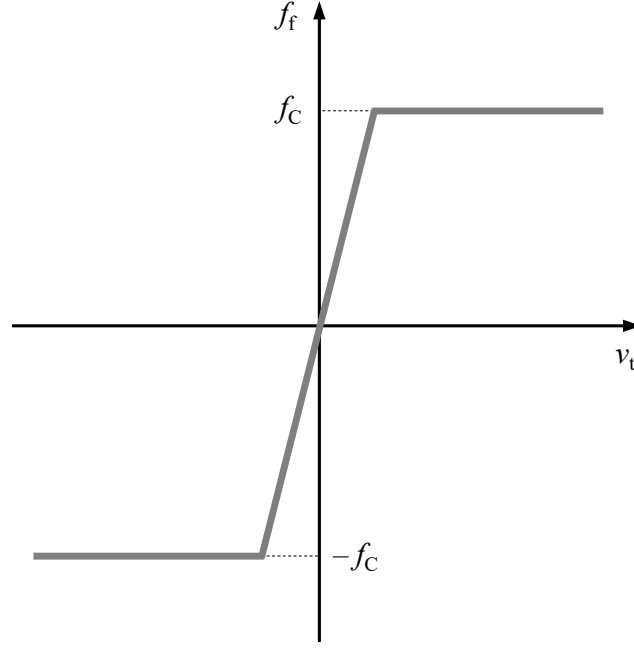


Figure 6.2 – Linear regularization of the Coulomb friction model proposed by Bernard (1974).

In mathematical terms, this model can be described by the following system of equations

$$f_f = \begin{cases} f_C \min(k_1 v_t, 1) & \text{if } v_t \geq 0 \\ f_C \max(k_1 v_t, -1) & \text{if } v_t < 0 \end{cases} \quad (6.1)$$

where f_C is the Coulomb friction force magnitude, k_1 represents a coefficient defined to determine the speed of transition from negative values to positive values of friction force, which in practical terms represents the linear slope of this friction force model and v_t is the relative tangential velocity between the contacting surfaces.

As a clear advantage of this model can be referred the continuous transition between positive and negative values of friction force using a simple (linear) mathematical function that has small computational cost in comparison with other regularization strategies, but the disadvantages related to it are the sharp slope change at the points where the friction force reaches its maximum value, namely in the transition between the linear slope and the Coulomb friction force level, and the fact of not being capable of capture the stiction phenomenon. Despite some of the graphical representations of this model exhibiting a velocity tolerance, which is the value of tangential velocity that marks the transition between sticking and sliding phases, the mathematical formulation does not consider any type of equation to establish this transition, being fully defined by the transition constant slope.

6.1.2 Threlfall model (1978)

As it was previously stated in the Bernard (1974) model section, one of its main disadvantages is the sharp transition between the linear slope and the constant slope associated with the steady-state friction force value, which in this specific case, is represented by the Coulomb friction force level, f_C . To overcome this problem, a curve shape defined by an exponential function was defined by Threlfall (1978), which led the friction force to be mathematically expressed as

$$f_f = f_C \left(1 - e^{-3 \frac{\|v_t\|}{v_0}} \right) \text{sgn}(v_t) \quad (6.2)$$

where the constant 3 is responsible for ensuring that at $v_t = v_0$ the friction force is approximately equal to

$$f_f \approx 0.95 f_C \quad (6.3)$$

being v_0 the velocity at which friction is velocity independent (Threlfall, 1978). When the dynamic system in analysis reaches the v_0 value, the friction force assumes the steady state magnitude, which in the case of this model represents the Coulomb friction force value. Figure 6.3 exhibits the graphical representation of the Threlfall model.

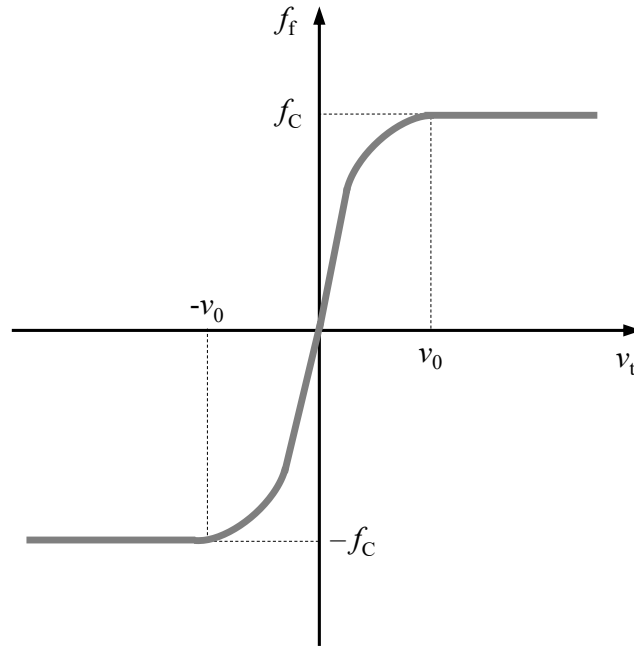


Figure 6.3 – Graphical representation of the Threlfall regularization strategy for the Coulomb friction model.

The value of the parameter v_0 is selected by the model user, however, it should not be very small, i.e., close to zero, because this would result in a behavior similar to the Coulomb friction model, thus introducing a discontinuity near the value of null tangential velocity.

Regarding this model, Marques et al. (2016) proposed an alternative version of it represented formulated as

$$f_f = f_c \left(\frac{1 - e^{-\frac{3\|v_t\|}{v_0}}}{1 - e^{-1}} \right) \text{sgn}(v_t) \quad (6.4)$$

Besides reaching the Coulomb friction force, the modification of the model ensures that the transition between the raising sticking phase and the sliding phase is made in a smoother and continuous manner without yielding numerical instabilities.

6.1.3 Rooney and Deravi model (1982)

Another relatively simple approach, in mathematical terms, of a smoothened Coulomb friction model was proposed by Rooney and Deravi (1982). In this case, a trigonometric function, more specifically a hyperbolical tangent function, $\tanh$, is used, being the friction force defined as

$$f_f = \begin{cases} f_c \tanh\left(\frac{v_t}{v_0}\right) & \text{if } v_t < v_0 \\ f_c & \text{if } v_t \geq v_0 \end{cases} \quad (6.5)$$

where v_0 represents the velocity tolerance. This parameter defines by itself the width of the region in which the friction force is dependent on the tangential velocity. Figure 6.4 exhibits the graphical representation of the Rooney and Deravi regularization approach to the Coulomb friction force model.

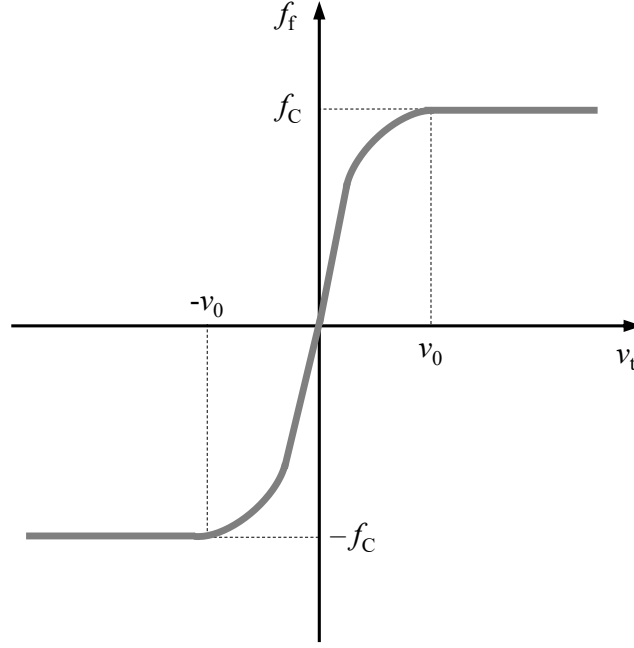


Figure 6.4 – Rooney and Deravi's static friction model graphical representation.

It is possible to notice that the Rooney and Deravi's model considers that for tangential velocities superior to the velocity tolerance ($v_t \geq v_0$), the friction force is equal to the Coulomb friction force, as it is demonstrated by the second branch of Equation (6.5), not having a static threshold friction force to represent the stiction phenomenon. The positive aspects of considering this friction force model are related to the simple computational implementation, the need of only defining two parameters, the kinetic friction coefficient, μ_k , and velocity tolerance, v_0 , and its very suitable for high-speed cyclic motions which makes it very applicable to automotive (engine, transmission and drivetrain's moving parts) and industry applications (Rooney and Deravi, 1982). Downsides to the use of this friction model can be related to the inexperience or lack of sensitivity of the user in defining an adequate value of the velocity tolerance, v_0 , that provides both precise results and minimizes the computational time.

6.1.4 Ambrósio model (2003)

The friction force model proposed by Ambrósio (2003) for application in multibody contact problems characterizes itself as a form of regularization of the Coulomb friction model based in a linear relation between the symmetric maximum values of friction force, $-f_C$ and $+f_C$, respectively. The formulation of this model is made using the following piecewise function

$$f_f = \begin{cases} 0 & \text{if } \|v_t\| \leq v_0 \\ \frac{\|v_t\| - v_0}{v_1 - v_0} & \text{if } v_0 < \|v_t\| < v_1 \\ f_c \operatorname{sgn}(v_t) & \text{if } \|v_t\| \geq v_1 \end{cases} \quad (6.6)$$

where v_0 and v_1 are velocity tolerances that establish the beginning of the stiction phase where the friction force assumes a non-null value and the transition from the sticking phase to the sliding state, respectively. This model prevents the friction force from changing direction near the null value of relative tangential velocity, but as a disadvantage, it also considers a null value of friction force for small sliding velocities. Also, it is not capable of modelling the stiction phenomenon, reason why is inserted in the without stiction static friction force models category. Figure 6.5 represents the graphical aspect of this friction model.

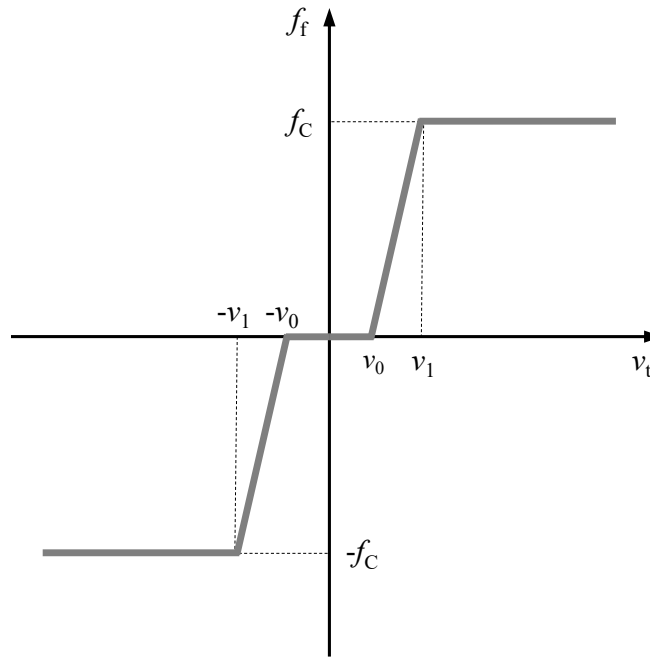


Figure 6.5 – Graphical representation of Ambrósio's regularization approach to Coulomb friction model.

Another disadvantage that arises from the use of this model is related to the sharp transitions verified between the thresholds v_0 and v_1 .

6.2 Static friction force models with stiction

6.2.1 Karnopp model (1985)

In order to reduce the probability of existence of numerical instabilities near the value of null tangential relative velocity, Karnopp (1985) proposed a model where the friction force, f_f , is a function of the tangential velocity, v_t , for what is considered the steady state of the dynamical system in analysis. In the nearby region of null tangential velocity, a small velocity range is defined by the conjunction of conditions $-v_0 \leq v_t \leq v_0$, where v_0 represents a velocity threshold.

Since this model is a regularization of the Coulomb friction model, then it can be expressed as

$$f_f = \begin{cases} f(v_t) & \text{if } \|v_t\| \geq v_0 \\ \min(\|f_c\|, f_s) \operatorname{sgn}(f_c) & \text{if } \|v_t\| < v_0 \end{cases} \quad (6.7)$$

where $f(v_t)$ is a function of tangential velocity, that in practical terms, represents the value of Coulomb friction force, $f(v_t) = f_c$, for values of tangential velocity outside the range defined by the velocity threshold v_0 . Figure 6.6 presents a graphical representation of this friction model.

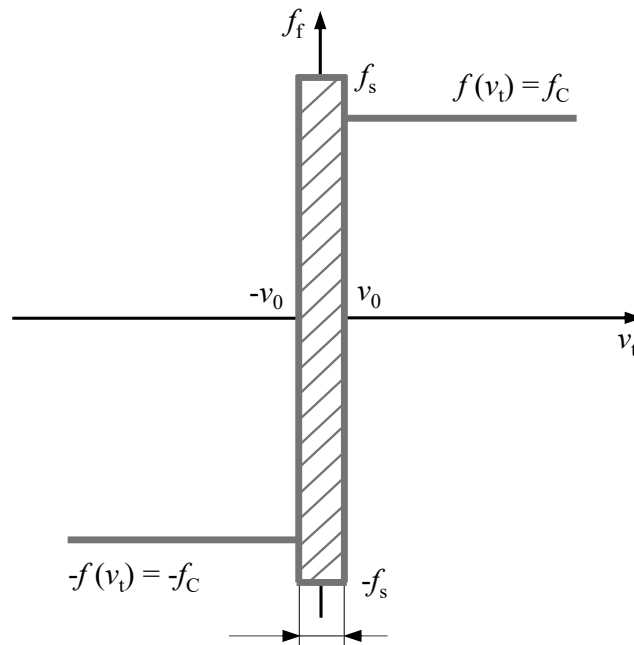


Figure 6.6 – Graphical representation of the Karnopp static friction force model.

Inside the hatched region of the Karnopp graphical representation, which is delimited by the velocity threshold value and its symmetrical value, v_0 and $-v_0$, respectively, the tangential velocity is

considered to be null, letting the system change its state and its corresponding response from the kinetic friction force value to a static (stiction) friction force value, f_s . Thus, the friction force is determined as (i) the value needed to keep the system at zero velocity, that is, the static friction force level, f_s , or (ii) the breakaway force threshold that marks the transition to the kinetic friction force value, f_C , which is associated with the sliding phase.

6.2.2 Bengisu and Akay (1994)

As the Karnopp model allowed the modelling of the stiction phenomenon, a similar strategy was adopted by Bengisu and Akay (1994) in order to model the Stribeck effect, with the purpose of being able to correctly and more realistically represent the friction phenomena in practical applications. Mathematically, the proposed model can be described by the following function

$$f_f = f_C \operatorname{sgn}(v_t) (1 - e^{-\beta \|v_t\|}) [1 + (f_r - 1) e^{-\alpha \|v_t\|}] \quad (6.8)$$

where α and β are parameters that can be either related to the surface roughness or the bonding ability of the surfaces and f_r represents the ratio of the asymptotic value of the curve to its maximum, that is, the ratio between the static and kinetic friction force values

$$f_r = \frac{\mu_s f_n}{\mu_k f_n} \quad (6.9)$$

Figure 6.7 exhibits the graphical representation of this dry friction model.

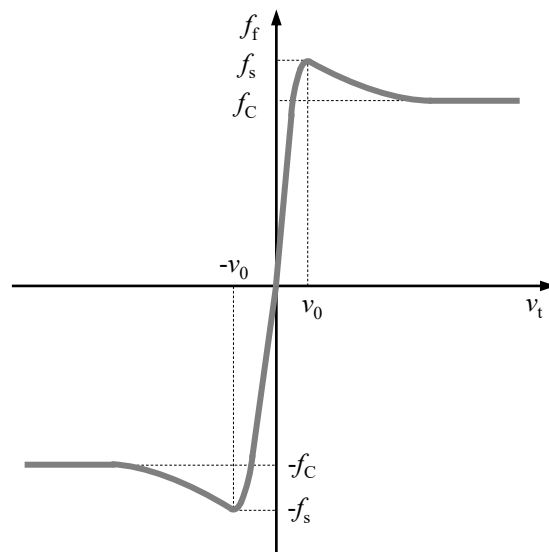


Figure 6.7 – Graphical representation of the Bengisu and Akay static friction force model.

Due to the difficulties associated with determining adequate values for the α and β parameters, Marques et al. (2016) proposed a variant of this model that uses two equations

$$f_t = \begin{cases} \left(-\frac{f_s}{v_0^2} (\|v_t\| - v_0)^2 + f_s \right) \text{sgn}(v_t) & \text{if } \|v_t\| < v_0 \\ \left(f_c + (f_s - f_c) e^{-\xi(\|v_t\| - v_0)} \right) \text{sgn}(v_t) & \text{if } \|v_t\| \geq v_0 \end{cases} \quad (6.10)$$

being the first related to the slope used to represent the stiction phase and the second is used to represent the Stribeck effect which utilizes an exponential function, being this equation very similar to the one found on Bo and Pavelescu (1982) static friction force model, where ξ is a parameter that must be positive, and which represents the negative slope of the sliding state associated with the Stribeck effect.

This model allows an approximate modelling of the transition from stiction to slip motion conditions when a suitable value of slope is chosen, because large values of slope at zero velocity result in difficulties to capture the friction force value at low velocities, requiring small time steps increments, and, consequently, more computational time is consumed. In addition, these models can still be considered as static models since the elastic deformation of the contact surfaces in the tangential direction is not physically represented.

6.2.3 Armstrong-Hélouvry et al. model – the seven parameter model (1994)

The Armstrong-Hélouvry et al. (1994) model, also known as the seven parameter model, is characterized by its complexity due to the fact that considers a wide range of friction phenomena in its formulation, such as pre-sliding displacement, Stribeck effect, viscous effect, stiction, Coulomb friction and frictional lag. In what concerns its mathematical implementation, the model uses an expression to represent the pre-sliding displacement

$$f_t(x) = -k_t x \quad (6.11)$$

where k_t represents the tangential stiffness of the static contact and x is the corresponding displacement.

The sliding phase, which considers the Stribeck and viscous effects, as well as the Coulomb friction force and frictional lag is represented by the following expression

$$f_f = - \left(f_c + f_v \|v_t\| + f_s(\gamma, t_2) \frac{1}{1 + \left(\frac{v_t(t - \tau_L)}{v_s} \right)^2} \right) \text{sgn}(v_t) \quad (6.12)$$

where f_v corresponds to the viscous friction force, v_t is the tangential velocity of sliding, f_c is the magnitude of the Coulomb friction force, γ is a temporal parameter related to the rising slope of the static friction, t_2 is the dwell time which in reality corresponds to the time at zero velocity, τ_L is the time constant of frictional lag and v_s is the characteristic velocity of Stribeck friction.

The magnitude of the rising static friction at the breakaway force, denoted by f_s , can be written as

$$f_s(\gamma, t_2) = f_{s,a} + (f_{s,\infty} - f_{s,a}) \frac{t_2}{t_2 + \gamma} \quad (6.13)$$

where $f_{s,a}$ is the magnitude of the Stribeck friction at the end of the previous sliding period and $f_{s,\infty}$ is the magnitude of the Stribeck friction if the force was applied at an infinitely small rate, leading the system to a long time in a static rest state.

A first look at this model might induce its hypothetical user to think that this is the most complete and suitable static friction model in existence nowadays, since it captures the effect of multiple friction phenomena, but the need to define the value of seven parameters, which is a considerable large number in comparison with the models previously analyzed, leads to big difficulties in calibrating the value of each of them and, consequently, the calibration of the whole model. This fact makes the model very dependable on experimental data from various sources, which makes it unpractical for a wide range of engineering applications. Another disadvantage is related to the transition from the rising static friction to the sliding phase which is not unequivocally defined by any parameter or equation, which can lead to numerical instabilities and a poor computational performance of the model.

6.2.4 Hollars model (1994)

Another form of regularization which can be fitted into the static friction model's category is the Hollars et al. (1994) model. This friction force model was formulated to be implemented in the Symbolic Dynamics software SD/FAST, which is nowadays part of PTC, the owner company of the well-known CAD

software PTC Creo. SD/FAST is a software that allows the simulation of articulated systems of rigid bodies and the correspondent deriving of the nonlinear equations of motion (PTC® SD/FAST).

The mathematical formulation of this model is expressed as

$$f_f = \left[\min\left(\frac{v_t}{v_0}, 1\right) \left(f_c + \frac{2(f_s - f_c)}{1 + \left(\frac{v_t}{v_0}\right)^2} \right) + f_n \mu_{Hv} v_t \right] \quad (6.14)$$

where v_t is the tangential velocity, v_0 is a tolerance/transition velocity that allows the definition of a transition between sticking and sliding states, f_c is the Coulomb friction force level, f_s is the static friction (stiction) force level, f_n is the normal contact force and μ_{Hv} is a specific viscous coefficient of friction defined by Hollars that must be defined as a function of the normal force f_n , considering for that the average value of it, $f_{n,ave}$, being the coefficient expressed as

$$\mu_{Hv} = \frac{\mu_v}{f_{n,ave}} \quad (6.15)$$

in which μ_v is the viscous coefficient of friction.

This model is also implemented to describe the tangential friction component of the Hunt and Crossley contact force model in Simbody™, which is an open-source high-performance C++ library for the sophisticated treatment of multibody systems mainly related to biomedical simulation (Sherman and Eastman).

6.2.5 Andersson model (2007)

Since there are a wide variety of sliding contacts that are lubricated, it is of great importance to introduce the lubricant interface behaviour in the models formulated to describe tangential interaction between contacting bodies.

Andersson et al. (2007) proposed a friction force model with the aim of considering the Coulomb friction and the viscous and Stribeck effects. The original Stribeck force model also presents numerical indeterminacies when changing the tangential sliding direction, that is, in the close regions of the null value of tangential velocity. To regularize the friction force function, a hyperbolic tangent function was

chosen to model the transition between the sliding phases for positive and negative values of tangential velocity. The friction force can then be expressed as

$$f_f = \left(f_C + (f_s - f_C) e^{-\left(\frac{\|v_t\|}{v_s}\right)^p} \right) \tanh(k_{\tanh} v_t) + \mu_v v_t \quad (6.16)$$

where f_C is the Coulomb friction force level, f_s is the static friction (stiction) force level, v_t is the tangential velocity, v_s is a sliding velocity coefficient, p is a parameter that affects the decaying shape of the transition from static to kinetic friction, $k_{\tanh}$ is a coefficient that determines the rate of increase from no friction at zero velocity to static friction, and μ_v is the viscous friction coefficient.

Figure 6.8 shows the generic graphical representation of Andersson' friction model.

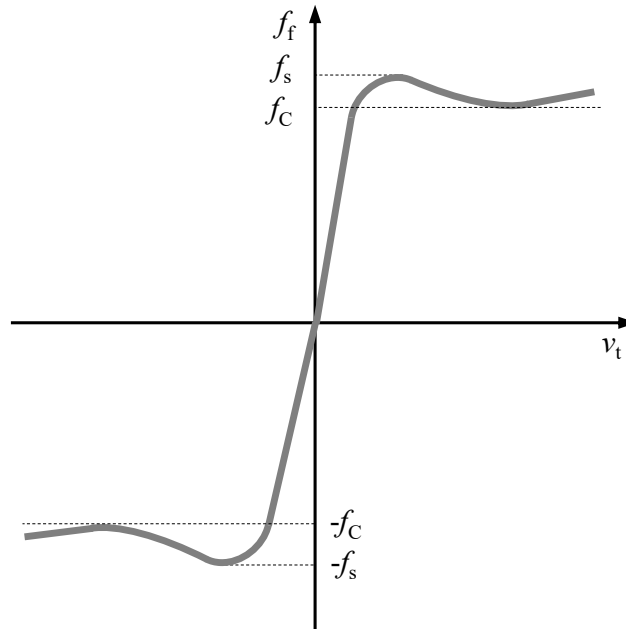


Figure 6.8 – Andersson et al. static friction force model graphical representation.

The main advantages of this model are the regularized approach to the friction force problem and the capability of capturing the Stribeck effect. The main disadvantages are mainly related to the non-intuitiveness in determining the value of v_s , p and $k_{\tanh}$, particularly because the last two parameters do not have any correlation with any physical phenomena.

6.2.6 Wojewoda et al. model (2007)

Another friction force model studied in the context of the static friction model's category is the one formulated by Wojewoda et al. (2008). This model was formulated with the intention of providing an easy to implement approach in terms of numerical calculations and the capacity of modelling the friction phenomena with high precision (Wojewoda et al., 2008).

Although the goal was to create a simple model, by introducing phenomena like the hysteresis associated with the pre-sliding and breakaway force, frictional lag and stochastic effects that can be attributed to the differences in roughness between the sliding elements, the model can be considered quite complex.

This model has the capability of describing both the static friction related to the sticking phase, using for that one equation, and the kinetic friction related to the sliding state of motion, which uses two different equations, one for the acceleration phase and another for the deceleration phase, as it is expressed by the following expressions

$$f_f(v_t, \dot{v}_t) = \begin{cases} f_s \operatorname{sgn}(v_t) & \text{if } f_t < f_{d+} \wedge \operatorname{sgn}(v_t \cdot \dot{v}_t) > 0 \\ f_{d+} \operatorname{sgn}(v_t) & \text{if } f_t > f_{d+} \wedge \operatorname{sgn}(v_t \cdot \dot{v}_t) > 0 \\ f_{d-} \operatorname{sgn}(v_t) & \text{if } \operatorname{sgn}(v_t \cdot \dot{v}_t) < 0 \end{cases} \quad (6.17)$$

where

$$f_s = \frac{1}{2} k_s \frac{v_t^2}{\|\dot{v}_t\|} - f_0 \quad (6.18)$$

and

$$\begin{cases} f_{d+} = f_c \left[1 + \frac{f_s(v_t) - f_c}{f_c} (g(v_t, \dot{v}_t) + f_R(x_t, v_t)) \right] \\ f_{d-} = f_c \left[1 - \frac{f_s - f_c}{f_c} (g(v_t, \dot{v}_t) + f_R(x_t, v_t)) \right] \end{cases} \quad (6.19)$$

In Equation (6.18) is described the determination of the static friction force, f_s , which is a function of the contact stiffness, k_s , the tangential velocity, v_t , the tangential acceleration, $\dot{v}_t$, and of a friction force, f_0 , corresponding to the initial value of the sticking force. In what concerns the meaning of the variables of Equation (6.19), f_{d+} denotes the kinetic friction force related to the acceleration phase,

f_d is the value of the kinetic friction force related to the deceleration phase, f_c is the Coulomb friction force and $f_s(v_t)$ is the breakaway force, described by the following equation

$$f_s(v_t) = f_s + \Delta f_s \frac{1}{1 + \left| \frac{v_t}{v_A} \right|} \quad (6.20)$$

in which Δf_s determines the range of the breakaway force variation and v_A is an adjusting parameter representing an average value of the maximum Stribeck velocity, normally considered equal to $v_A \approx 0.5v_s^{\max}$.

Continuing the analysis of the variables presented in Equation (6.19), $f_R(x_t, v_t)$ is a function representing the stochastic component of the friction force, which is dependent of the tangential displacement and velocity, and $g(v_t, \dot{v}_t)$ is a nonlinear function of the tangential velocity, v_t , and tangential acceleration, $\dot{v}_t$, which can be described by the following equation

$$g(v_t, \dot{v}_t) = \frac{1}{1 + \left(\frac{v_t - \tau \dot{v}_t}{v_A} \right)} \quad (6.21)$$

in which τ is a time delay parameter.

The analysis of this model allows to conclude that although it tries to model a wide range of complex friction phenomena, the large number of parameters and their associated unknown values, which must be obtained and calibrated via experimental data, makes it rather complicated to implement in the analysis of multibody systems.

6.2.7 Awrejcewicz et al. model (2008)

Motivated by the work of Leine et al. (1998), Awrejcewicz et al. (2008) proposed a friction model with the goal of surpassing the problem related to the fact that at a null value of tangential velocity, most of literature models consider the tangential friction force also null, which does not correspond to the reality of the friction phenomena. In this model, the friction force, which is dependent of both relative velocity, v_t , and the resultant external force exerted on the system, f_e , is defined by four different equations, being one of them used to describe the sticking phase, two equations are used to describe the transition between stick and slip states and one for the sliding phase, as it is exposed

$$f_f(v_t, f_e) = \begin{cases} A(-v_t + f_s \operatorname{sgn}(v_t)) + f_e & \text{if } \|v_t\| \leq v_0 \wedge \|f_e\| \leq f_s \\ (2A-1)f_s \operatorname{sgn}(v_t) & \text{if } \|v_t\| \leq v_0 \wedge \|f_e\| \leq f_s \wedge f_e v_t < 0 \\ f_s \operatorname{sgn}(f_e) & \text{if } \|v_t\| \leq v_0 \wedge \|f_e\| \leq f_s \wedge f_e v_t \geq 0 \\ f(v_t) \operatorname{sgn}(v_t) & \text{if } \|v_t\| > v_0 \end{cases} \quad (6.22)$$

where

$$A = \frac{\|v_t\|^2}{v_0^2} \left(3 - 2 \frac{\|v_t\|}{v_0} \right) \quad (6.23)$$

and f_s represents the maximum value of static friction force and v_0 being a velocity tolerance similarly to the ones defined in previously analyzed models to mark the transition between sticking and sliding phases.

It is verifiable that for values of tangential velocity below the threshold defined by the velocity tolerance, v_0 , the friction force is not only dependent of the tangential velocity, but also dependent of the external resultant force, f_e , as it was stated in the beginning of this section. Figure 6.9 shows the graphical representation of this model in which the black lines represent the possible behaviors of this friction model for the minimum and maximum values of external resultant force.

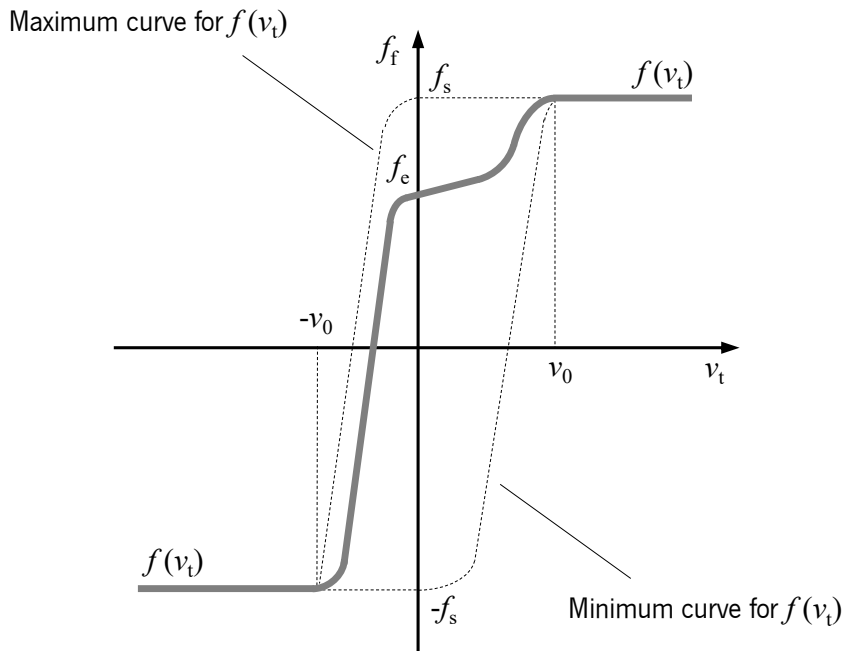


Figure 6.9 – Graphical representation of Awrejcewicz et al. static friction force model.

6.2.8 Specker et al. model (2014)

The static friction model proposed by Specker et al. (2014), alongside the Coulomb friction, is capable of modeling the viscous and Stribeck effects through continuous functions, which makes it ideal for computational applications due to its capabilities of avoiding numerical instabilities.

The Coulomb friction in the analyzed model is described by the following formulation

$$f_c = \mu_k f_n \tanh\left(\frac{v_t}{v_0}\right) \quad (6.24)$$

where μ_k is the kinetic friction coefficient, f_n is the normal force, v_t is the tangential velocity and v_0 is a transition velocity that marks the shift from sticking to sliding phase. This velocity is one of the arguments of the hyperbolic tangent function used to regularize the Coulomb friction force model, thus being responsible for the shape of this function.

In what concerns the Stribeck effect, this model considers that it can be modeled as

$$f_\Delta = \left[f_s - f_c \tanh\left(\frac{v_{sp}}{v_0}\right) - c_v v_{sp} \right] \frac{v_t}{v_{sp}} e^{\frac{1}{2} \left[1 - \left(\frac{v_t}{v_{sp}} \right)^2 \right]} \quad (6.25)$$

where f_s is the static friction force level, f_c is the Coulomb friction force level, v_{sp} is the Stribeck peak velocity that defines the friction force decay due to the Stribeck effect, and c_v is a damping friction coefficient.

The final term of this friction force model is related to the viscous effect and is expressed as

$$f_v = c_v v_t \quad (6.26)$$

where the meaning of each variable has already been described before.

The complete formulation of this friction force model can be obtained by summing the Coulomb, Stribeck and viscous effect terms as follows

$$f_f = \underbrace{\mu_k f_n \tanh\left(\frac{v_t}{v_0}\right)}_{\text{Coulomb}} + \underbrace{\left[f_s - f_c \tanh\left(\frac{v_{sp}}{v_0}\right) - c_v v_{sp} \right] \frac{v_t}{v_{sp}} e^{\frac{1}{2} \left[1 - \left(\frac{v_t}{v_{sp}} \right)^2 \right]}}_{\text{Stribeck effect}} + \underbrace{c_v v_t}_{\text{Viscous effect}} \quad (6.27)$$

Like other friction models, the disadvantage of the Specker et al. model resides in the definition of the parameter values, since it can only be made in a precise way via experimental setups. Nonetheless, the transition velocity and the Stribeck peak velocity can be approximated as

$$v_s \approx 4v_0 \approx 2v_{sp} \quad (6.28)$$

being v_s the Stribeck velocity.

An interesting fact of this model is that the authors performed an adaptation of it to a dynamic friction model by introducing a linear parameter-varying (LPV) system in a form of a first-order lowpass filter, which allowed to capture hysteresis and memory effects that are complex phenomena normally only modelled by dynamic friction force models (Specker et al. 2014).

6.2.9 Linear static with stiction model (2016)

Another regularization approach to the Coulomb friction force model discontinuity at null relative tangential velocity is the linear friction model with stiction (Marques et al., 2016). In practical terms, this model considers a peak of friction force corresponding to the value of static friction which occurs at the velocity threshold, v_0 , which marks the transition where the friction force is independent of the tangential velocity, v_t .

The transition from the static friction force level, f_s , to kinetic friction force, f_c , verifiable in the Coulomb friction force with stiction, is also characterized by another numerical discontinuity, since this transition occurs for the null value of tangential velocity. To overcome this problem, the strategy followed was to consider a linear function between v_0 and a second velocity threshold named v_1 , that allows a continuous transition from static to kinetic friction force level.

The model can be mathematically expressed by the following system of equations (Marques et al., 2016)

$$f_f = \begin{cases} \left(\frac{\|v_t\|}{v_0} f_s \right) \text{sgn}(v_t) & \text{if } \|v_t\| \leq v_0 \\ \left[f_s - \frac{\|v_t\| - v_0}{v_1 - v_0} (f_s - f_c) \right] \text{sgn}(v_t) & \text{if } v_0 < \|v_t\| < v_1 \\ f_c \text{sgn}(v_t) & \text{if } \|v_t\| \geq v_1 \end{cases} \quad (6.29)$$

Figure 6.10 shows the graphical representation of the linear friction model with stiction.

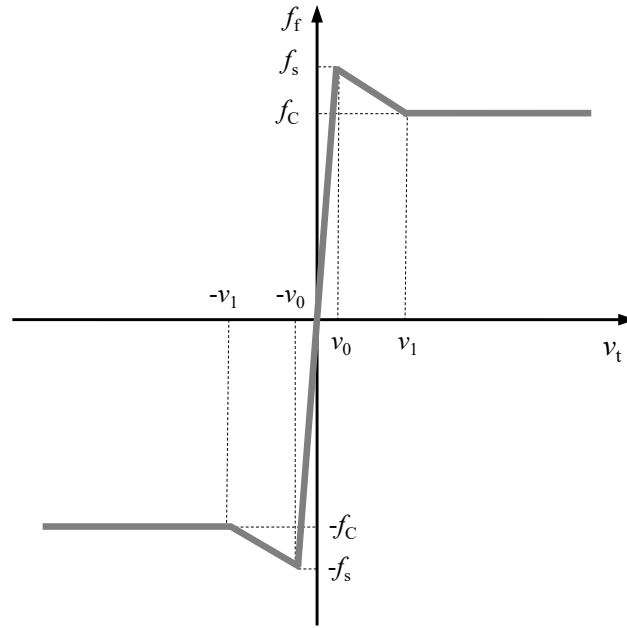


Figure 6.10 – Graphical representation of the linear static friction force model with stiction.

An overview of the presented model allows to conclude that its mathematical formulation is very simple to implement in a computational code and the type of equations used (linear) lead to very low computational costs. The disadvantages shown by this model are the lack of capability of modelling more complex phenomena such as Stribeck and viscous effects, and the sharp transitions between the different branches of the model's mathematical formulation, that could lead to numerical instabilities.

6.2.10 Brown and McPhee model (2016)

Brown and McPhee (2016) proposed a velocity-based friction force model capable of capturing static, dynamic and viscous behaviors. These three different behaviors are represented by three non-coupled terms which can be expressed as

$$f_f = \underbrace{f_c \tanh\left(4 \frac{v_t}{v_0}\right)}_{\text{Coulomb steady state dynamic friction}} + \underbrace{(f_s - f_c) \frac{\frac{v_t}{v_0}}{\left(\frac{1}{4} \left(\frac{v_t}{v_0}\right)^2 + \frac{3}{4}\right)^2}}_{\text{Static friction (stiction)}} + \underbrace{\mu_v v_t \tanh\left(4 \frac{f_n}{f_{nt}}\right)}_{\text{Viscous effect}} \quad (6.30)$$

where the first term represents the steady state dynamic friction, which corresponds to the Coulomb friction force level, the second term the static friction (stiction) and the third one the viscous friction. This non-coupled approach is very useful and simple to implement when certain friction phenomena, for

example the viscous effect, is not a variable of interest for the analysis of a certain system, which is done by removing the corresponding term from Equation (6.30).

In what regards the meaning of the variables, f_c represents the Coulomb friction force level, v_t is the tangential velocity, v_0 is a threshold or tolerance velocity that marks the transition from stick to slip phase, which occurs for the maximum value of static friction force, f_s , μ_v is the viscous coefficient of friction, f_n is the normal contact force and f_{nt} is a transition normal force which is used to detect when the viscous friction effects should be accounted. When $f_n > f_{nt}$, the viscous friction term has a significant impact on the friction force value. Without considering viscous effects, the third term of Equation (6.30) can be neglected and the friction force can be simply calculated as

$$f_f = f_c \tanh\left(4 \frac{v_t}{v_0}\right) + (f_s - f_c) \frac{\frac{v_t}{v_0}}{\left(\frac{1}{4} \left(\frac{v_t}{v_0}\right)^2 + \frac{3}{4}\right)^2} \quad (6.31)$$

This friction force model reveals a simple approach to regularize the Coulomb friction model, through the use of a hyperbolic tangent function and is able to capture stiction and viscous effect in an intuitive to implement model.

6.3 Benchmark problem – static friction force models

To evaluate the behavior and computational efficiency of the static friction force models presented and analyzed in the previous sections, it was necessary to select a dynamic problem with significant relevance and well documented in literature, that is, a benchmark problem. This simple system is widely used in literature to validate friction force models, and it was first proposed by Rabinowicz (1956).

Figure 6.11 exhibits the one degree of freedom system composed by a block with mass m which is connected to a fixed reference by a spring with stiffness k and is placed on a conveyor belt that moves with a linear speed v_b .

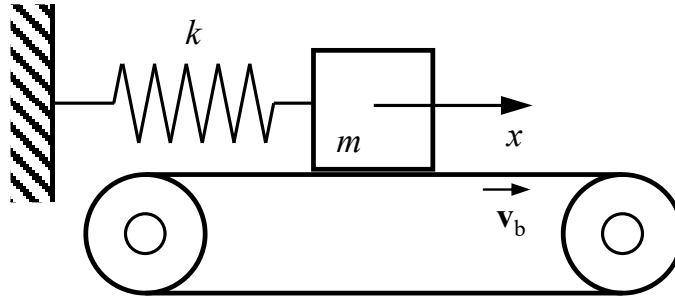


Figure 6.11 – Schematic representation of the one degree of freedom benchmark problem.

Using the equations of dynamics it is possible to describe the mass movement of this mechanical system through the following differential equation

$$m \frac{d^2 x}{dt^2} = -kx - f_f \quad (6.32)$$

where f_f represents the tangential friction force.

The simulation parameters chosen to study the performance, behavior and efficiency of the static friction force models analyzed in this section are presented in Table 6.1, being particularly based on the ones used by Gonthier et al. (2004) and Marques et al. (2016).

Table 6.1 – Benchmark problem simulation parameters.

Parameter	Symbol	Value	Parameter	Symbol	Value
Mass of the block	m	1 kg	Initial velocity	v_0	0.1 m/s
Gravity acceleration	g	9.81 m/s ²	Time-step	-	0.0001 s
Conveyor belt speed	v_b	0.1 m/s	Simulation time	-	40 s
Spring stiffness constant	k	2 N/m	ODE Solver	-	ode45, ode15s, ode23s, ode78
Kinetic friction coefficient	μ_k	0.1	Solver Relative Tolerance, RelTol	-	1e-8
Static friction coefficient	μ_s	0.15	Solver Absolute Tolerance, AbsTol	-	1e-8
Initial position	x_0	0 m			

These parameters were then introduced in the interface shown in Figure 6.12 which is an integrant part of a computational code dedicatedly programmed in MATLAB to study the behavior of a variety of friction force models in an intuitive way.



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Coded by: Francisco Novais V1.0.2024

Rabinowicz 1 degree of freedom benchmark problem

Initial conditions

Initial position, x_0 [m]

Initial velocity, v_0 [m/s]

Simulation time [s]

Time step [s]

General Parameters

Block mass, m [kg]

Gravity acceleration, g [m/s²]

Spring constant, k [N/m]

Kinetic friction coefficient, μ_k [-]

Static friction coefficient, μ_s [-]

Conveyor belt speed, v_b [m/s]

Solver

☐ ode45

☐ ode15s

☐ ode23s

☐ ode78

Solver options

Relative tolerance, RelTol

Absolute tolerance, AbsTol

NOTE TO USERS on Solver options:

In what concerns the relative tolerance, RelTol and the absolute tolerance, AbsTol, MATLAB considers the default values of 1e-3 and 1e-6, respectively. To prevent

Next

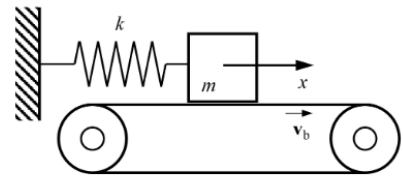


Figure 6.12 – Main window of the computational code created in MATLAB to study the benchmark problem.

After introducing all the parameters in the main window and hitting the *Next* button, the user is taken to another window, presented in Figure 6.13, where it is possible to choose the models that want to be studied.



Friction force models selection



Choose the friction force models to be analysed

Static friction force models without stiction

☐ Coulomb

☐ Rooney and Deravi

☐ Bernard

☐ Ambrosio

☐ Threlfall

☐ Threlfall modified (Marques et al.)

Static friction force models with stiction

☐ Karnopp

☐ Specker

☐ Bengisu and Akay

☐ Linear with stiction (Marques et al.)

☐ Andersson

☐ Hollars

☐ Awrejcewicz

☐ Brown and McPhee

Dynamic friction force models with stiction

☐ Dahl

☐ Elasto-Plastic

☐ Reset Integrator

☐ Gonthier

☐ LuGre

☐ Liang Bristle

Define parameters
for static friction force
models without stiction

Define parameters
for static friction force
models with stiction

Define parameters
for dynamic friction
force models

Figure 6.13 – Friction force models selection window of the MATLAB code created.

In the next sections a more detailed study of the static friction force models with and without stiction will be made using the programmed MATLAB code.

6.3.1 Performance analysis of static friction force models without stiction

After selecting all the static friction force models without stiction inserted in the MATLAB code, it is necessary to define the values of certain specific parameters of each model. Table 6.2 displays the various parameters considered for this category of models.

Table 6.2 – Static friction force models without stiction specific parameters.

Parameter	Symbol	Value
Transition coefficient (Bernard)	k_1	2000
Velocity threshold (Threlfall, Rooney and Deravi, Threlfall modified)	v_0	0.001 m/s
Velocity threshold (Ambrósio)	v_0	0.0001 m/s
Velocity threshold (Ambrósio)	v_1	0.001 m/s

The previous parameters were then defined in the static friction force without stiction window in each respective box as visible in Figure 6.14.

Static friction force models without stiction

For the previously selected static friction force models without stiction, fill the following fields with the correspondent numerical value of each specific parameter

Bernard

Transition coefficient, k_1

Threlfall original

Velocity threshold, v_0 [m/s]

Rooney and Deravi

Velocity threshold, v_0 [m/s]

Ambrósio

Velocity threshold, v_0 [m/s]

Velocity threshold, v_1 [m/s]

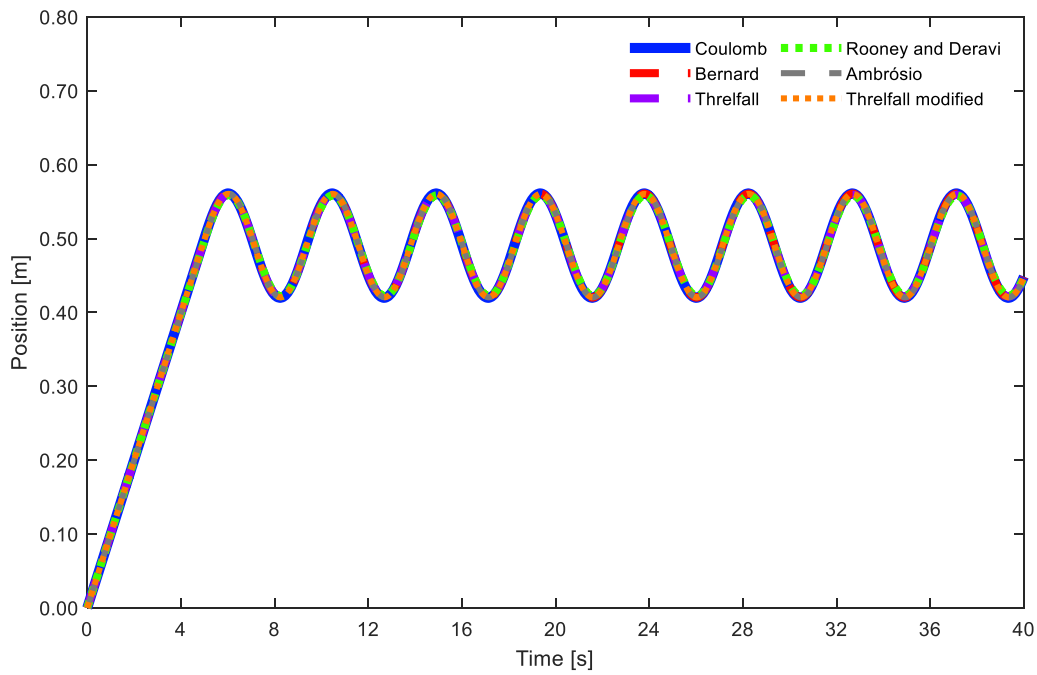
Threlfall modified

Velocity threshold, v_0 [m/s]

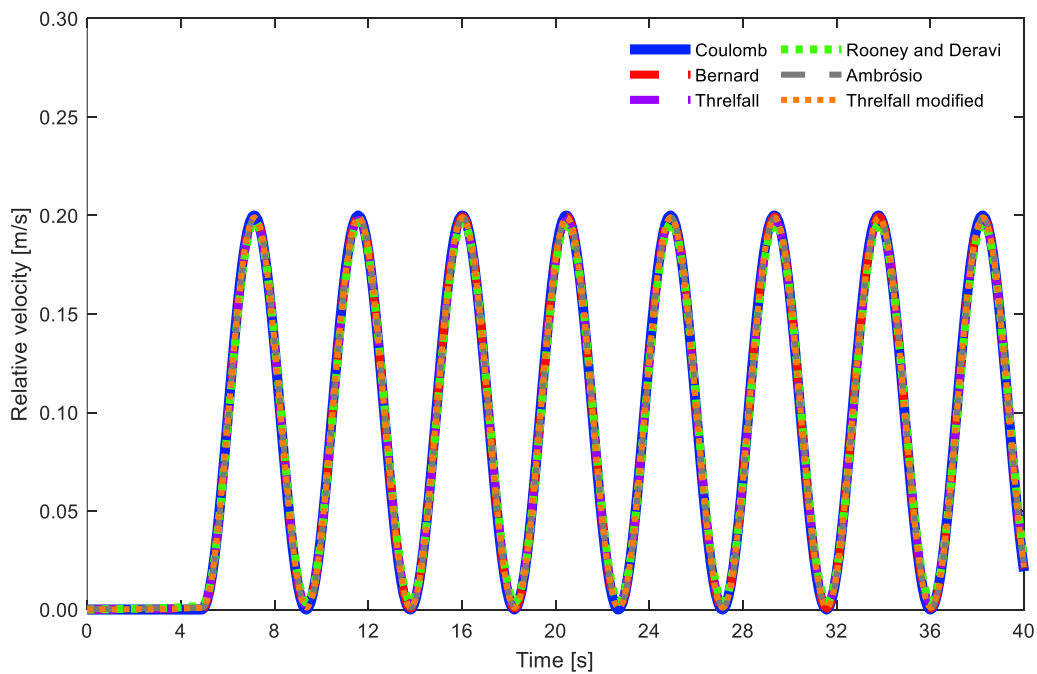
Solve

Figure 6.14 – Static friction force models without stiction window from the MATLAB computational code to define the specific parameters of each model.

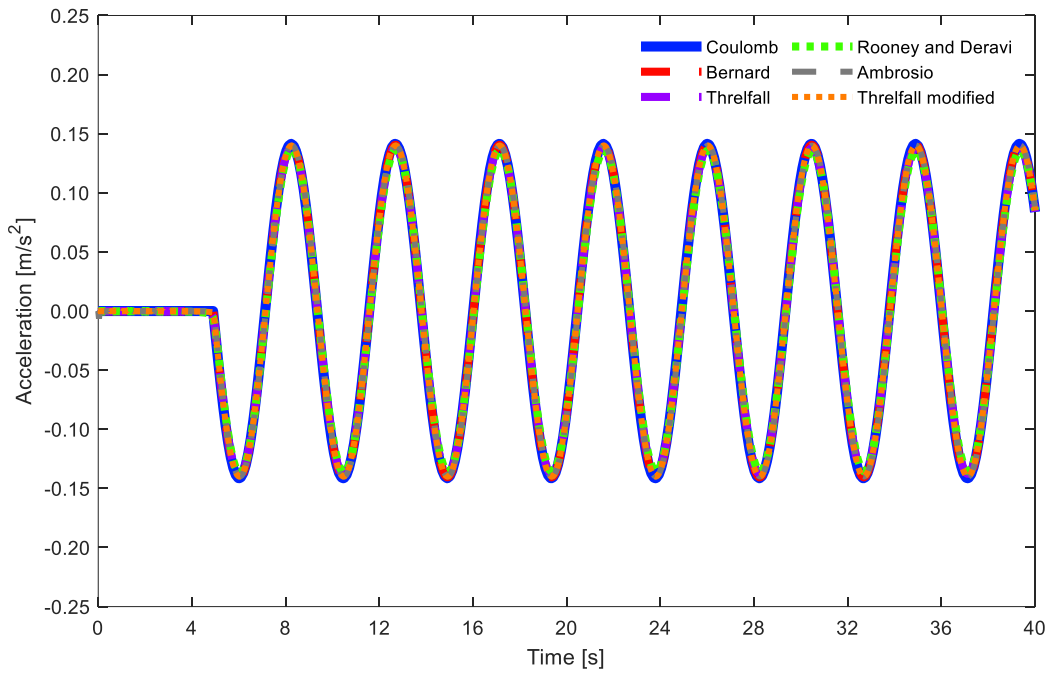
After clicking the *Solve* button, it is possible to obtain the graphs that quantify the behavior of the system in analysis in terms of position, relative velocity, acceleration, and friction as functions of the simulation time and which are represented in Figure 6.15 from a) to d).



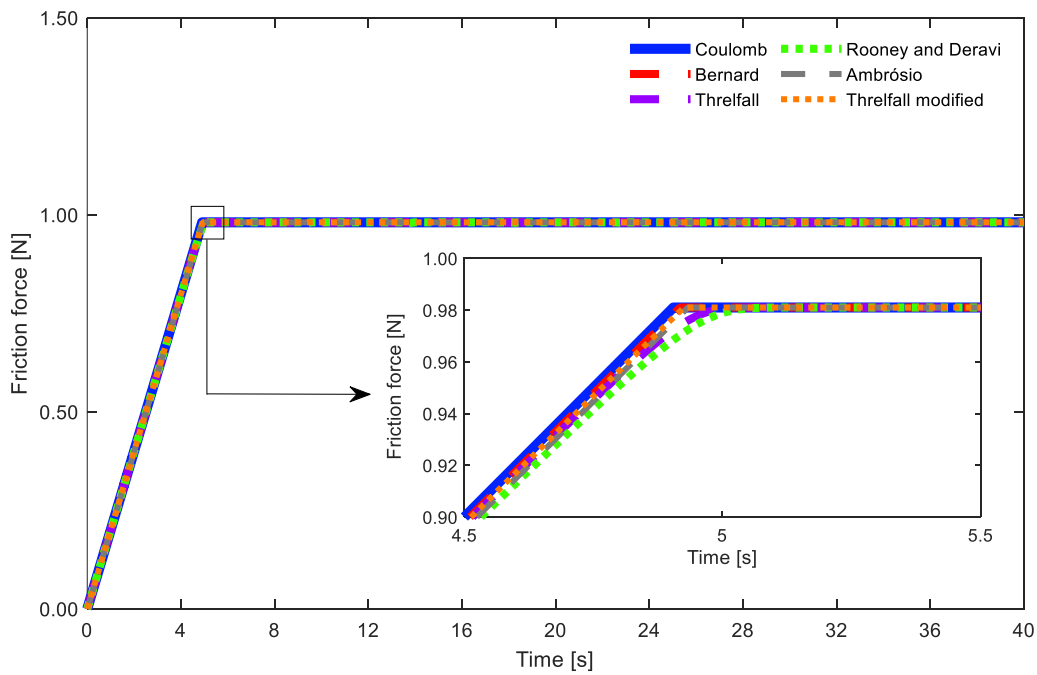
a)



b)



c)



d)

Figure 6.15 – Graphical results of the benchmark problem for the static friction force models without stiction a) Position [m] vs Time [s] graph b) Relative velocity [m/s] vs Time [s] graph c) Acceleration [m/s^2] vs Time [s] graph d) Friction force [N] vs Time [s] graph using the ode45 solver.

A general overview of all the graphs allows to conclude that all the static models without stiction have remarkably similar behaviors. Figure 6.15 a) exhibits an initial linear slope that corresponds to the

sticking phase of the block's motion in which it moves jointly with the conveyor belt, since the spring's force is not capable of making the block slide in the opposite direction. At approximately $t \approx 6$ s, the block starts sliding in the opposite direction of the belt velocity and from this instant until the end of the simulation ($t = 40$ s) the block exhibits an oscillatory stick-slip motion. This oscillatory motion is also confirmed by analyzing the relative velocity and acceleration graphs represented in Figure 6.15 b) and c)

In what concerns the friction force vs time graph, represented in Figure 6.15 d), it is possible to confirm that the studied static models without stiction are not capable of modelling the stiction phenomena, since that for the parameters selected for the simulation, the maximum stiction force should be equal to $f_s = 1.47$ N, but all the models are only capable of reaching a maximum value of 0.98 N, which corresponds to the Coulomb friction force level. Analyzing the zoomed section of the previously mentioned graph, it is possible to observe that all the models reach the Coulomb friction force level at very close time instants, these differences being related to the values chosen for each specific parameter of each model.

To evaluate the computational efficiency of each of the static friction force models studied, a group of simulations was realized using different MATLAB ODE solvers. The selected solvers were the ode45, which is the most commonly used ODE solver of MATLAB, the ode15s for being a recommended solver for stiff problems that should be tried when ode45 fails to converge, the ode23s due to its capability of solving stiff problems more efficiently than ode15s and using worse tolerances and the ode78 for being a solver with high accuracy (MathWorks, 2023).

The simulations were performed on a laptop equipped with an Intel(R) Core (TM) i7-9750H 2.60 GHz CPU and 16 GB of RAM, being the results presented in Table 6.3. The average simulation time was obtained by running three simulations for each model and determining the mean value. It is possible to observe that the most efficient model in terms of simulation time was the Threlfall modified with an average time of 1.2801 s using the ode23s solver and the slower computational time was also verified for the Threlfall modified model with a mean time of 2.3350 s but using the ode15s solver. Although not represented in Table 6.3, the average value of the simulation times of all the solvers yields the Bernard friction model as the most efficient model in terms of computational elapsed time, being the Coulomb friction force model the least efficient. The results also show that the ode23s is not a very reliable solver since that for the Coulomb friction force model did not even converge to a solution.

Table 6.3 – Solver and average simulation time for the static friction force models without stiction.

Model	Solver	Average Simulation Time [s]	Model	Solver	Average Simulation Time [s]
Coulomb	ode45	1.5734	Rooney and Deravi	ode45	1.4223
	ode15s	2.1877		ode15s	2.3275
	ode23s	DNC*		ode23s	1.2810
	ode78	2.2297		ode78	2.0536
Bernard	ode45	1.4075	Ambrósio	ode45	1.4375
	ode15s	2.2250		ode15s	2.3073
	ode23s	1.3076		ode23s	1.3605
	ode78	2.0337		ode78	2.1710
Threlfall	ode45	1.3994	Threlfall modified (Marques et al.)	ode45	1.4218
	ode15s	2.3081		ode15s	2.3350
	ode23s	1.2860		ode23s	1.2801
	ode78	2.2161		ode78	2.1284

*DNC – Did Not Converge

6.3.2 Performance analysis of static friction force models with stiction

In this section the results from the static friction force models with stiction will be analyzed. The reason that justifies the distinction between static models without stiction and with stiction is related to capability of the latter ones of modeling the static friction level.

Using the interface of Figure 6.13 presented in the previous section, it is possible to choose all the static with stiction friction force models implemented in the MATLAB code. After checking all the boxes associated with this category of models, it is necessary to define the specific parameters of each friction force model, being the values used in the analysis presented in this work defined in Table 6.4.

Table 6.4 - Static friction force models with stiction specific parameters.

Parameter	Symbol	Value	Parameter	Symbol	Value
Velocity threshold (common to all models)	v_0	0.001 m/s	Decaying shape exponent (Andersson)	p	7
Velocity threshold (Linear with stiction)	v_1	0.01 m/s	Hyperbolic tangent factor (Andersson)	$k_{\tanh}$	10000
Slope of the sliding state (Bengisu and Akay)	ξ	100	Transition normal force (Brown and McPhee)	f_{nt}	0 N
Viscous friction coefficient (Andersson, Brown and McPhee)	μ_v	0.1 N·s/m	Stribeck peak velocity (Specker)	v_{Sp}	0.0025 m/s
Viscous friction coefficient (Hollars)	μ_{Hv}	0	Viscous damping coefficient (Specker)	c_v	0.1 N·s/m

Similarly to what has been done for the static friction models without stiction, the dedicated window shown in Figure 6.16 was programmed in the MATLAB programmed code for the analysis of this benchmark problem to introduce the previously presented specific parameters in a fast and intuitive way.

Static friction force models with stiction

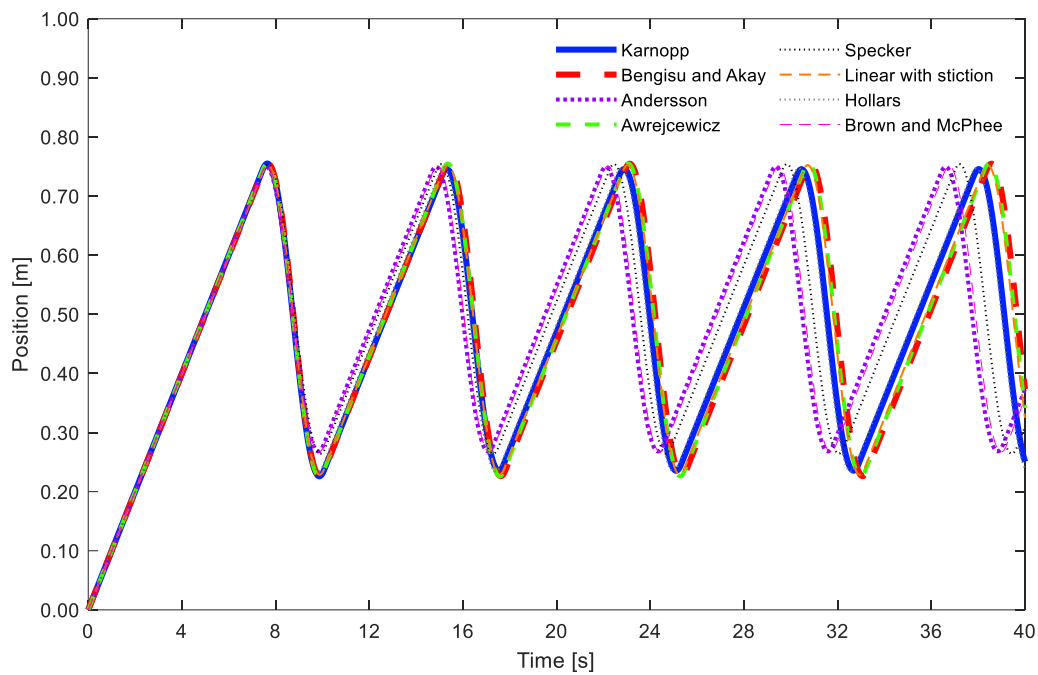
For the previously selected static friction force models with stiction, fill the following fields with the correspondent numerical value of each specific parameter

Karnopp Velocity threshold, v_0 [m/s] 0	Andersson Velocity threshold, v_0 [m/s] 0
Linear friction with stiction Velocity threshold, v_0 [m/s] 0	Decaying shape exponent, p [-] 0
Velocity threshold, v_1 [m/s] 0	Hyperbolic tangent factor, $k_{\tanh}$ [-] 0
Bengisu and Akay Velocity threshold, v_0 [m/s] 0	Viscous friction coefficient, μ_v [N · s/m] 0
Slope of the sliding state, ξ [-] 0	Brown and McPhee Velocity threshold, v_0 [m/s] 0
Hollars Velocity threshold, v_0 [m/s] 0	Viscous friction coefficient, μ_v [N · s/m] 0
Viscous friction coefficient, μ_{Hv} [N · s/m] 0	Transition normal force, F_{nt} [N] 0
Awrejcewicz Velocity threshold, v_0 [m/s] 0	Specker Velocity threshold, v_0 [m/s] 0
	Stribeck peak velocity, v_{Sp} [m/s] 0
	Viscous damping coefficient, c_v [N · s/m] 0
	Solve

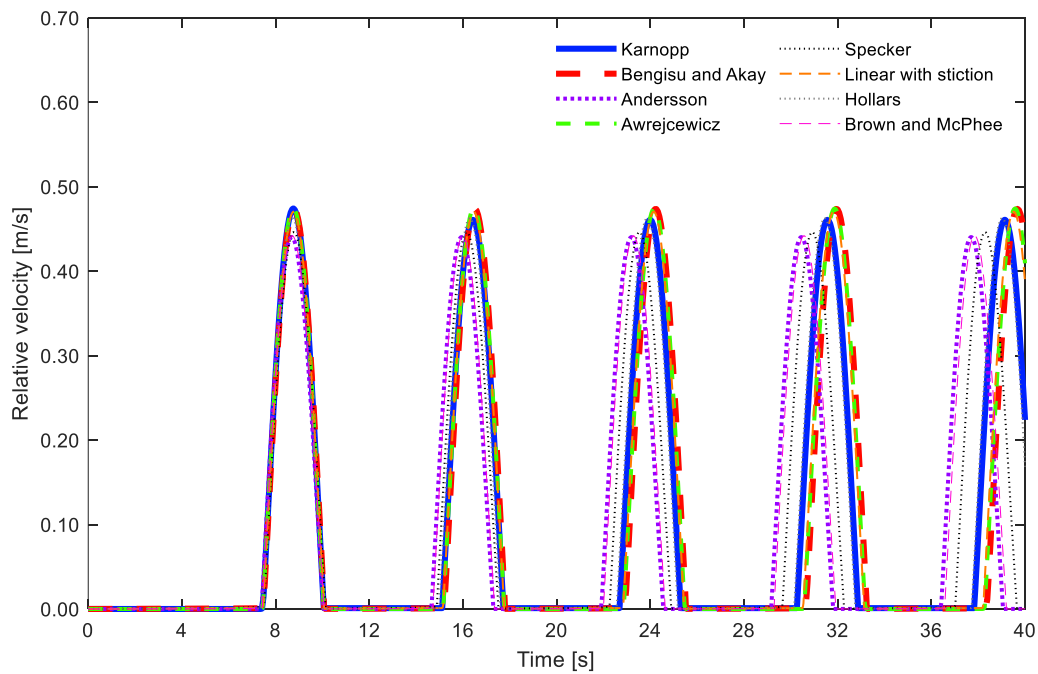
Figure 6.16 – Static friction force models with stiction window from the MATLAB computational code to define the specific parameters of each model.

To obtain the results of the simulation of the benchmark problem considering the static friction force models with stiction, the *Solve* button was pressed, which allowed the plot of the different graphs

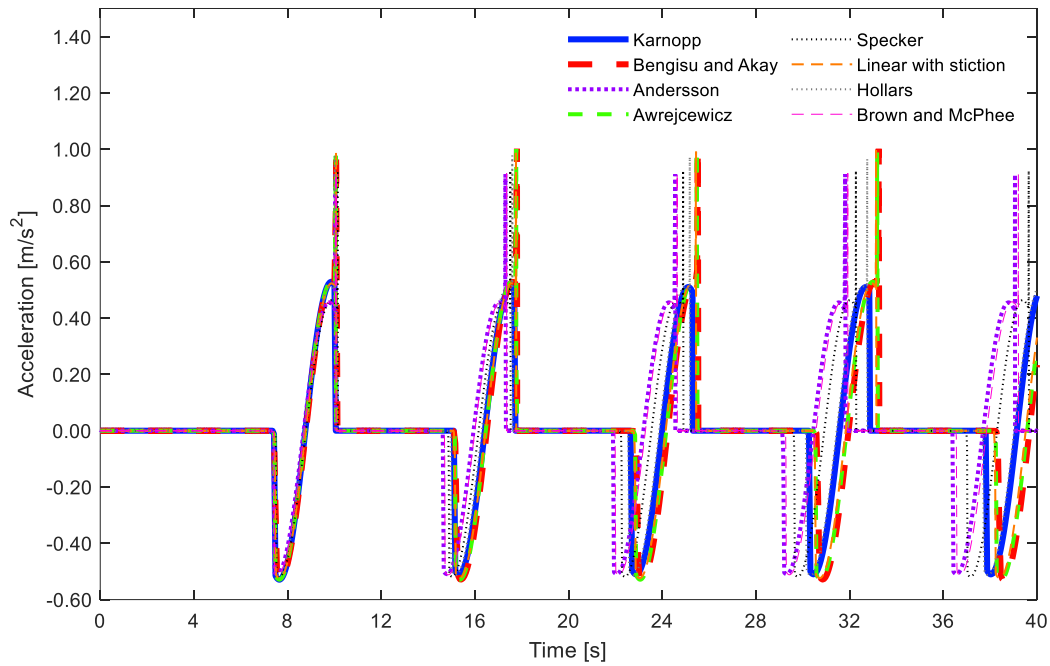
that quantify the behavior of the system in analysis in terms of position, relative velocity, acceleration and friction as functions of the simulation time and which are represented in Figure 6.17 from a) to d).



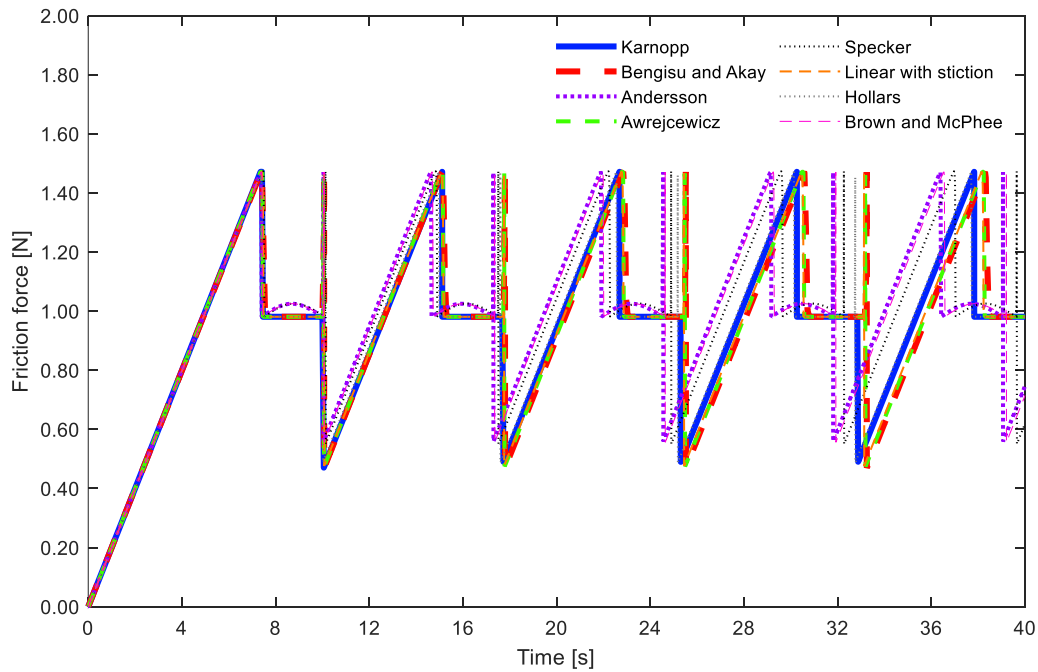
a)



b)



c)



d)

Figure 6.17 – Graphical results of the benchmark problem for the static friction force models with stiction a) Position [m] vs Time [s] graph b) Relative velocity [m/s] vs Time [s] graph c) Acceleration [m/s^2] vs Time [s] graph d) Friction force [N] vs Time [s] graph using the ode45 solver.

Opposed to what has been stated for the static without stiction friction force models, the static models with stiction results show some differences in the behavior of the different models analyzed. In

general, all the friction models of this category are capable of modelling the stiction phenomena, fact that is corroborated by the position vs time graph results of Figure 6.17 a) which show that the maximum position reached by all models is approximately equal to 0.77 m instead of the 0.57 m exhibited by the static friction models without stiction and the maximum friction force value reached is equal to 1.47 N, which corresponds to the static friction force level, according to the graph of Figure 6.17 d). Since the maximum friction force level is higher than the previously analyzed models (static without stiction), the initial slope of the position vs time graph has a lower value, and it is capable of reaching a longer distance due to the prolonged sticking phase. The sliding phase of these models shows a different behavior from the static without stiction models, since it is more abrupt and faster than the sticking phase, while on the previously analyzed models the sticking and sliding phases showed an oscillatory behavior between the sticking and sliding phases, almost resembling a sinusoidal function.

Another interesting fact worth of notice is the particular behavior of Andersson et al. (2007), Brown & McPhee (2016) and Specker et al. (2014) models. After the first peak of the maximum position reached in the graph of Figure 6.17 a) it is possible to observe that the sliding phase is shorter than the other static with stiction friction force models. This phenomenon is verified due to the capabilities of these models to model viscous and Stribeck effects (for the Specker et al. (2014) case), which results in a damping effect of the block oscillations, preventing it from sliding longer distances and it explains the small lag verified between the behavior of these models in terms of position, relative velocity, acceleration and friction force values in comparison to the other models, as well as the smaller peak values of relative velocity and acceleration exhibited in Figure 6.17 b) and c), respectively.

Despite the Hollars et al., 1994) model being capable, in theory, of modelling viscous effects, the implementation shown in this work did not consider the use of the viscous friction coefficient for this model, since it yielded non-feasible results that do not correspond to the physical behavior of the system.

The static friction force models with stiction were also the object of study in terms of computational efficiency using the same solvers and the same equipment mentioned before in the previous section, the results being presented in Table 6.5.

Analyzing the results from the simulations of the dynamic system using the different static with stiction friction force models it is possible to state that the most efficient model in terms of simulation time was the Linear with stiction (Marques et al.) with a mean time of 1.1405 s using the ode23s solver and the slower computational time was verified for the Andersson et al. model with a mean time of 4.2416 s using the ode78 solver.

Although not represented in Table 6.5, the average value of the simulation times of all the solvers yields the Awrejcewicz friction model as the most efficient model in terms of computational elapsed time, while the Andersson et al. model is the least efficient. These results show once again that the ode23s is not a very reliable solver, since that for the Karnopp friction force model, the problem did not converge to a solution.

Table 6.5 – Solver and average simulation time for the static friction force models with stiction.

Model	Solver	Average Simulation Time [s]	Model	Solver	Average Simulation Time [s]
Karnopp	ode45	1.4239	Specker	ode45	1.5924
	ode15s	1.9092		ode15s	2.2854
	ode23s	DNC*		ode23s	1.4331
	ode78	2.2380		ode78	2.4739
Bengisu and Akay	ode45	1.7400	Linear with stiction	ode45	1.7217
	ode15s	2.4848		ode15s	2.3644
	ode23s	1.3448		ode23s	1.1405
	ode78	2.6984		ode78	2.6100
Andersson	ode45	2.6665	Hollars	ode45	1.6364
	ode15s	2.3516		ode15s	2.2919
	ode23s	1.2756		ode23s	1.2626
	ode78	4.2416		ode78	2.5369
Awrejcewicz	ode45	1.4188	Brown and McPhee	ode45	1.8278
	ode15s	2.2201		ode15s	2.5609
	ode23s	1.1813		ode23s	1.4607
	ode78	2.1865		ode78	2.8658

*DNC – Did Not Converge

6.4 Summary and Discussion

This chapter has presented a comprehensive analysis of 14 static friction force models, comprising 4 static models without stiction and 10 static models with stiction. Each model represents a regularized alternative to the Coulomb friction model, that is, they do not exhibit a numerical discontinuity near the value of null tangential velocity. The analyses focused primarily on the computational performance, physical realism, and ability of these models to incorporate phenomena such as viscous effects and the Stribeck curve.

The static friction models without stiction demonstrated remarkable computational efficiency, with behaviors that are largely consistent across all models. However, their inability to simulate stiction phenomena limits their applicability in scenarios where a realistic representation of static friction is required. These models reach only the kinetic friction force level during simulations, as evident from their friction force vs. time responses. While the absence of stiction simplifies their implementation and improves computational performance, it also constrains their use in simulations demanding higher fidelity.

In contrast, the static friction models with stiction offer enhanced capabilities to represent friction phenomena more realistically. These models successfully simulate the stiction phase, achieving the expected maximum static friction force level in line with theoretical expectations. The distinction between sticking and sliding phases was evident, with the sliding phase exhibiting faster transitions compared to the oscillatory stick-slip behavior observed in static models without stiction. This improved realism, however, comes at the cost of increased computational complexity, which is still relatively low.

The computational efficiency of the models was evaluated using a benchmark dynamic problem based on the 1 DoF problem proposed by Rabinowicz, for which an intuitive graphical interface was created in MATLAB. For static models without stiction, the Threlfall modified model emerged as the most computationally efficient, achieving an average simulation time of 1.2801 s using the ode23s solver. However, the Coulomb friction model, while simple in concept, exhibited the least computational efficiency and failed to converge under certain conditions, underscoring its limitations in numerical applications due to instabilities in the non-differentiable region of null relative velocity.

Among static models with stiction, the Linear with stiction model demonstrated the shortest average simulation time of 1.1405 s with the ode23s solver. Conversely, the Andersson et al. model was the least computationally efficient, with an average time of 4.2416 s using the ode78 solver. The results also indicated that the reliability of solvers like ode23s can vary significantly, as evidenced by non-convergence in some scenarios.

The performance discrepancies among solvers, as observed in this study, further underscore the importance of selecting appropriate numerical solvers for friction simulations. The results suggest that while ode23s often yields faster computations, its reliability can be problematic, necessitating careful consideration of solver choice based on the specific friction model and dynamical system in analysis.

The findings of this chapter reinforce the importance of tailoring static friction force models to the specific needs of a given application. Static friction models without stiction are well-suited for scenarios requiring high computational efficiency and moderate fidelity, while static models with stiction provide a better representation of real-world friction phenomena, by being able of representing the static friction force phenomenon, having similar computational demands.

7. DYNAMIC FRICTION MODELS

Static friction models are not capable of capturing certain complex phenomena, such as hysteresis or frictional lag. With this problem in mind, a more complex and precise category of friction models is introduced – the dynamic friction force models.

Dynamic friction models not only depend on the tangential relative velocity between contacting surfaces but also incorporate a state variable, often represented by bristle deflection, as most dynamic models are formulated at the asperity level. A state variable can be defined as a mathematical variable that captures the condition or configuration of a certain dynamic system in a certain instant of time. This variable is then explicitly updated using an update expression at the beginning or end of each time step of the numerical simulation.

The use of a state variable enables the mathematical formulation of dynamic friction models to be described by continuous functions that not only correctly describe the friction phenomenon at the asperity level, but also eliminate the numerical instabilities normally verified in the proximity region of the null value of tangential velocity.

Despite its enhanced precision in predicting the friction force acting on the contact interface between bodies that exhibit relative motion, their increased computational cost and complexity make some of these models hardly usable for practical engineering tasks. Another major problematic associated with these asperity level models is the difficulty to choose parameter values to model the physical behavior of surface bristles, which often requires calibration procedures utilizing experimental data.

In this section, similarly to what has been done for the static models, the dynamic friction force models analyzed appear in chronological order of appearance, being the studied ones presented in the schematic of Figure 7.1.

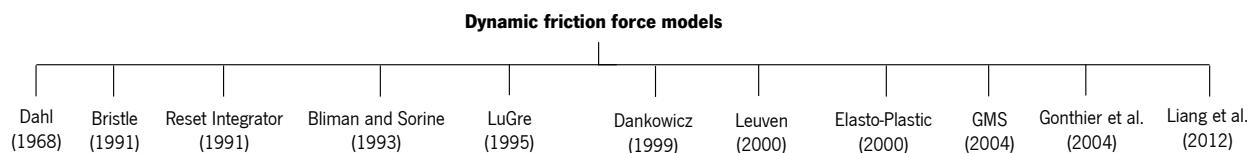


Figure 7.1 – Analyzed dynamic friction force models.

7.1 Dynamic friction force models analyzed – formulations

7.1.1 Dahl model (1968)

A new mathematical model of solid friction was formulated and presented by Dahl (1968) with the main objective of describing friction for both sliding and rolling friction that can be used in simulations of dynamic systems involving mechanical elements that are subjected to friction.

The Coulomb friction concept was introduced based on experimental data and it is widely accepted as a physical macro-phenomenon. In this model, a new theory emerged by establishing a comparison between frictional behavior and the quasi-static properties of materials. When an external force is applied to a material, due to the inertial properties of it, a resistance to the deformation is offered by the material. In frictional terms, this resistance to deformation can be thought of as the friction force that develops at the interface between the bodies and prevents the tangential relative motion between them.

With that being said, stiction, the maximum value of static friction, can be directly correlated with the ultimate tensile stress, σ_{UTS} , of a stress-strain curve, since from this point onwards, the applied force will lead to a continuum plastic deformation of the bristles between both contacting surfaces until Coulomb friction force is reached. This kinetic friction force level can be directly correlated with the rupture point of a material in a stress-strain curve where the contacting surfaces will exhibit relative tangential motion, that is, will start to slide (Dahl 1968).

It was found that this analogy to deformation of materials can be distinct when brittle and ductile materials are in contact due to differences between the stress-strain curves of these types of materials. Thus, for materials that have a brittle behavior, the stiction (static friction) and Coulomb friction are indistinguishable, as it can be observed in

Figure 7.2 a) and for ductile materials, the ultimate stress point will correspond to the maximum static friction value (stiction) and Coulomb friction force will be correlated with rupture point as it is represented in

Figure 7.2 b) and as it was stated in the previous paragraph.

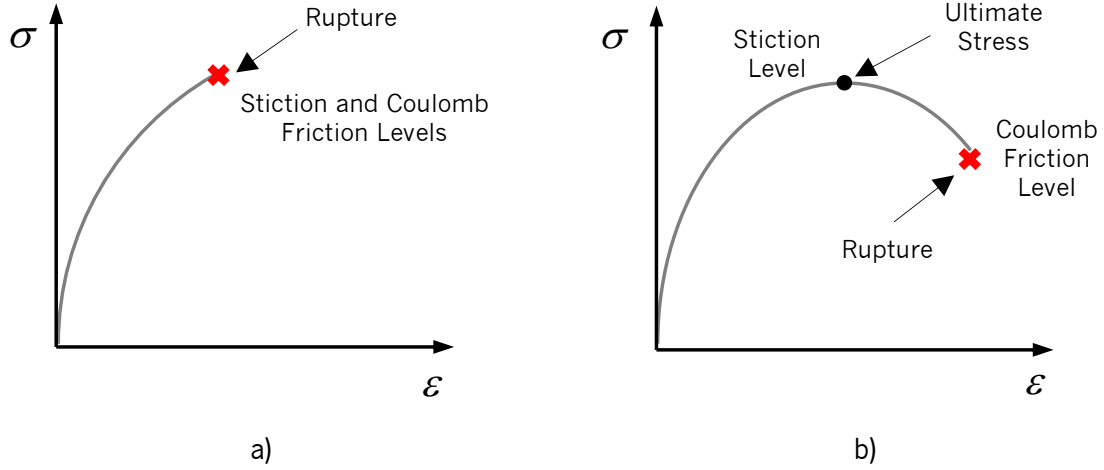


Figure 7.2 – Analogy of bristle elastic and plastic deformation with material's behavior a) brittle material b) ductile material. Adapted from Dahl (1968).

In what concerns the mathematical formulation of this model, Dahl (1968) described this deformation and hysteretic behavior of materials as dependent of both the relative displacement and velocity of the contacting surfaces, being the stress strain curve defined by a differential equation as

$$\frac{df_f}{dx} = \sigma_0 \left| 1 - \frac{f_f}{f_c} \right|^\alpha \operatorname{sgn} \left(1 - \frac{f_f}{f_c} \operatorname{sgn}(v_t) \right) \quad (7.1)$$

where f_f corresponds to the friction force, x is the displacement, σ_0 denotes the stiffness of the bristles and α is a parameter that defines the shape of the material curve, being dependent of the material and usually varying between 0 and 1.

It is possible to notice that Equation (7.1) is a function of displacement, but according to Dahl (1968) it can be differentiated in order to time, leading to

$$\frac{df_f}{dt} = \frac{df_f}{dx} \frac{dx}{dt} = \frac{df_f}{dx} v_t = \sigma_0 \left| 1 - \frac{f_f}{f_c} \right|^\alpha \operatorname{sgn} \left(1 - \frac{f_f}{f_c} \operatorname{sgn}(v_t) \right) v_t \quad (7.2)$$

In the case of considering the parameter α being equal to 1, which is the most common case, then Equation (7.2) can be written as

$$\frac{df_f}{dt} = \sigma_0 \left(1 - \frac{f_f}{f_c} \operatorname{sgn}(v_t) \right) v_t \quad (7.3)$$

The Dahl model is considered to be a dynamic friction model, so it must have a state variable which describes the dynamic behavior of the system in analysis. As this formulation considers the

existence of bristles, then the state variable is the bristle deflection, represented by the parameter z , and assuming that these entities exhibit a linear spring behavior that can be described as

$$f_f = \sigma_0 z \quad (7.4)$$

then the rate of bristle deflection is defined as

$$\frac{dz}{dt} = \left(1 - \frac{\sigma_0 z}{f_c} \text{sgn}(v_t) \right) v_t \quad (7.5)$$

which when the system reaches the steady state yields

$$f_f = f_c \text{sgn}(v_t) \quad (7.6)$$

being Equation (7.6) simply the Coulomb friction force model.

The analyzed approach allows to model both pre-sliding displacement as well as Coulomb friction, through the introduction of a state variable related to the bristle's deflection, but the main drawbacks of this model are its incapability of modelling the Stribeck effect and to capture stiction, which led to the formulation of new friction models based on the Dahl model, such as the Bliman and Sorine (1993, 1991) and the LuGre model (Canudas De Wit et al., 1995).

7.1.2 Bristle model (1991)

The continuous search for a friction force model able to realistically model the physical phenomena and simultaneously exhibit good computational performance was the main goal of Haessig and Friedland (1991). These authors stated that friction force should be defined as a function of displacement rather than tangential velocity, so the model proposed must be able to describe the sticking phase of friction as a function of displacement and during the slipping phase the friction force could be described as a function of tangential velocity.

The friction between surfaces is then given by the interaction between a N number of bristles in which each bristle contributes with an infinitesimal fraction of the total friction force. With that being said, the total friction force can be expressed as

$$f_f = \sum_{i=1}^N \sigma_i (x_i - b_i) \quad (7.7)$$

where σ_i is the stiffness of each bristle, b_i is the reference position related to the initial bond of the bristles and x_i is the relative position of the bristles.

Figure 7.3 illustrates the physical interaction between two bristles and the corresponding deflection of the lower one.

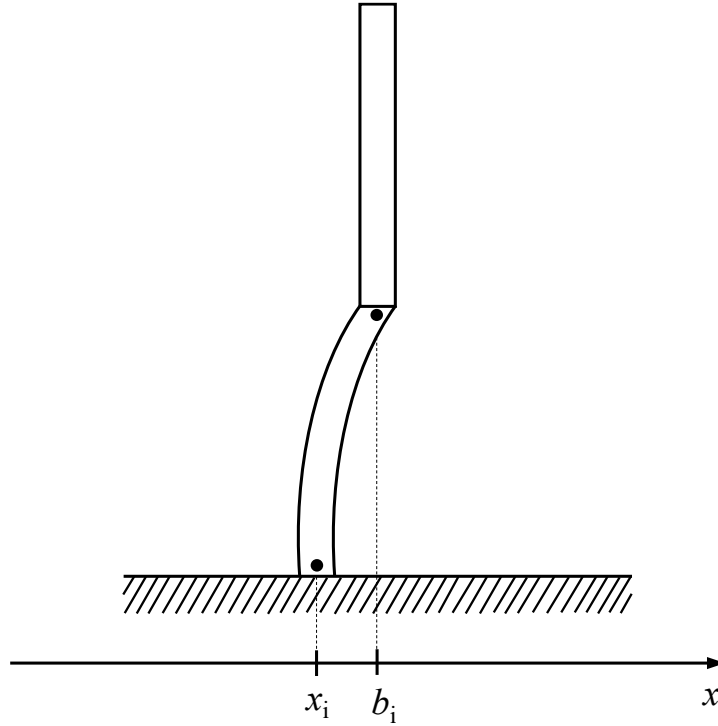


Figure 7.3 – Schematic representation of bristle deflection and location of the bond between asperities.

The difference between the last two variables indicated before (b_i and x_i) represents the relative displacement that should be used as a state variable to describe the sticking phase of the relative motion between surfaces. When the strain at a certain bristle exceeds its limit, $\Delta = (x_i - b_i) > \delta_i$, the rupture of that bond occurs, and a new bond is formed in a new location, which is calculated from the previous one.

The bond between a rigid bristle and an elastic bristle is destroyed when a critical deflection point is reached and then a new bond is randomly created between the rigid bristle and another different elastic bristle. The new bristle location can be determined as

$$b_{i'} = b_i + \text{uniform}(\Delta) \text{sgn}(\delta_i) \quad (7.8)$$

where b_i is the reference bond location and Δ the new bristle range.

Although the bristle model was designed for accuracy, this comes at a cost, that in this case is the computational efficiency. Despite considering a finite relatively small number of bristles, that should be less than 50 as stated by the authors (Haessig & Friedland, 1991), because of the spacing between bristles and the discontinuities related to the snapping of bristle bonds, this model becomes

computationally inefficient, requiring very small integration steps to provide precise results. This is the main reason why this model is not commonly used.

7.1.3 Reset Integrator model (1991)

The same authors of the bristle model exposed in the previous section proposed a new approach to reduce the computational time requirements, while still being capable of accurately representing and modelling the stick-slip friction phenomena. Haessig and Friedland (1991) formulated a model that does not consider the use of multiple bristles to accurately represent the bonding effect of stick-slip phenomena in the sticking phase, but instead considers this stick phase of the friction force as a function of position for a single bristle.

The input of the integrator used in this model is the strain rate of the bristle deformation, that can assume the following values

$$\frac{dz}{dt} = \begin{cases} 0 & \text{if } z \cdot v_t > 0 \wedge \|z\| \geq z_0 \\ v_t & \text{if } z \cdot v_t \leq 0 \wedge \|z\| < z_0 \end{cases} \quad (7.9)$$

where v_t is the tangential relative velocity, z is the bristle deflection and z_0 the pre-determined maximum value of the bristle deflection.

To determine the correspondent friction force of this model it is necessary to use two different equations: one to represent the sticking phase until reaching the breakaway force and another to describe the system's behavior during the sliding phase as it is expressed below

$$f_f = \begin{cases} \sigma_0(v_t)(1+a)z + \sigma_1 \frac{dz}{dt} & \text{if } \|v_t\| < z_0 \\ \sigma_0(v_t)z \operatorname{sgn}(z) & \text{if } \|v_t\| \geq z_0 \end{cases} \quad (7.10)$$

where σ_0 is the bristle stiffness, a is a coefficient related to the stiction phenomena, normally representing the ratio between the stiction load, f_s , and the sliding steady-state load, f_C , and σ_1 is a damping coefficient used to eliminate unwanted oscillations in the sticking phase and can be defined as

$$\sigma_1 = 0.707\sqrt{\sigma_0 m} \quad (7.11)$$

where m represents the mass of the sliding body.

In conclusion, this model requires significantly less computational time than the previously proposed model (Bristle model, 1991), since it exhibits a smooth and continuous behavior, so the small time steps associated with breaking the bristles is not a mandatory requirement.

7.1.4 Bliman and Sorine model (1993)

The friction models developed by Bliman and Sorine (1993; 1991) are dynamical models based on the works of Rabinowicz (1951; 1957; 1961). The models presented in this section fulfil both dependence on the sign function of the tangential velocity, $\text{sgn}(v_t)$, and the space variable s , which results in the integral form

$$s = \int_0^t \|v_t(\tau)\| d\tau \quad (7.12)$$

to replace the time variable, t , with the space variable, s , through a variable transformation process.

Considering that x_s is the absolute relative displacement of the contacting bodies, then the model can be expressed as a linear system of equations

$$\begin{cases} \frac{dx_s}{ds} = Ax_s + Bv_s \\ f_f = Cx_s + Dv_s \end{cases} \quad (7.13)$$

This model depends on the sign function of the tangential velocity, $\text{sgn}(v_t)$, which in this case is mathematically represented by the variable v_s , being $v_s = \text{sgn}(v_t)$

The first model analyzed is a first-order model, characterized by its reduced complexity, which uses the following variables

$$A = -\frac{1}{\varepsilon_f} \quad B = f_1 \varepsilon_f \quad C = 1 \quad D = 0$$

which results in the model form

$$\frac{df_f}{dt} = \frac{df_f}{ds} \frac{ds}{dt} = \|v_t\| \frac{df_f}{ds} = \frac{f_1}{\varepsilon_f} \left(v_t - \|v_t\| \frac{f_f}{f_1} \right) \quad (7.14)$$

This formulation is remarkably similar to the Dahl model, where

$$f_c = f_1 \quad \sigma = \frac{f_1}{\varepsilon_f} \quad \alpha = 1$$

making this model incapable of modeling the Stribeck effect and it is not also able to exhibit stiction. With this limitation in mind, a second-order model was proposed considering A, B and C as

$$A = \begin{pmatrix} -\frac{1}{\eta\varepsilon_f} & 0 \\ 0 & -\frac{1}{\varepsilon_f} \end{pmatrix} \quad B = \begin{pmatrix} \frac{f_1}{\eta\varepsilon_f} \\ -\frac{f_2}{\varepsilon_f} \end{pmatrix} \quad C = (1 \quad 1)$$

The second-order model can be viewed as a parallel connection of a fast and a slow model and by subtracting the slow model force from the fast model, which exhibits a higher steady-state friction, it is possible to model a stiction peak. The presentation of this model is merely informative, which means it will not be studied in practical applications that are explored in this work, since it is not usual to consider this dynamic friction force model in multibody applications.

7.1.5 LuGre model (1995)

Despite the well understood theoretical aspects and its rather simple application, the main drawbacks of the Dahl model are the incapability of reaching the stiction level and to capture the Stribeck effect. To adapt this effect into the Dahl model, Canudas De Wit et al. (1995) proposed a dynamic friction model based on the behavior of bristles that exist in every material surface.

In theory, considering asperities as microscopic entities that exhibit an elasto-plastic behavior, if the tangential acting force is sufficiently large, some of the bristles will deflect until they start slipping as it is represented in Figure 7.4.

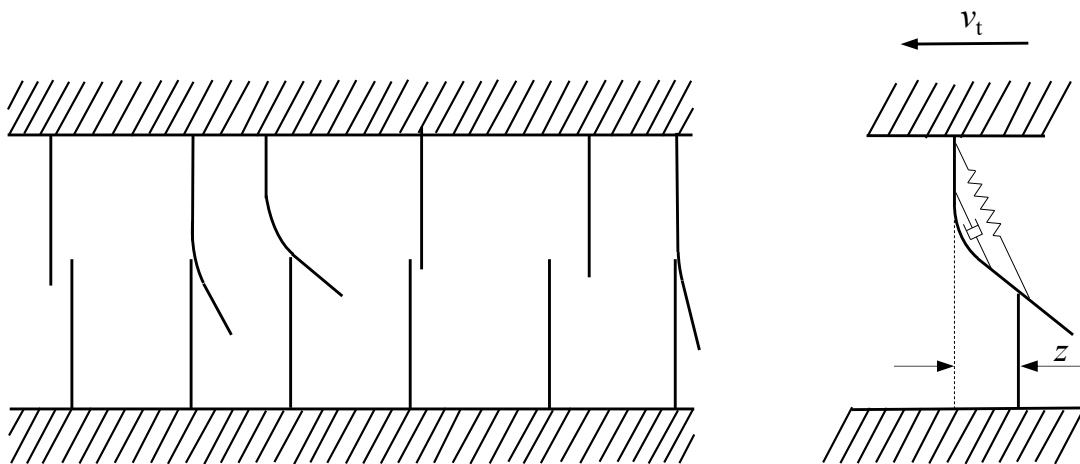


Figure 7.4 – Physical interpretation of bristle interaction between surfaces and bristle deflection, z , which is influenced by the bristle stiffness and damping, represented by the spring and the damper elements, respectively.

Since this model considers the average deflection of the bristles, then this phenomenon can be modeled as

$$\frac{dz}{dt} = v_t - \frac{\|v_t\|}{g(v_t)} z \quad (7.15)$$

where $\frac{dz}{dt}$ is the bristle deflection time rate, z represents the average deflection of the bristles, v_t is the tangential velocity and $g(v_t)$ is a positive function that depends on factors like the material properties, lubrication and temperature and can be expressed as

$$g(v_t) = \frac{f_c + (f_s - f_c) e^{-\left(\frac{v_t}{v_s}\right)^2}}{\sigma_0} \quad (7.16)$$

being f_c the Coulomb friction force, f_s the maximum static friction force (stiction), v_s the Stribeck velocity, and σ_0 is the stiffness of the bristles. Given the variables used in this equation and the presence of the exponential function, it is intuitive to understand that this term represents the Stribeck effect, since higher values of tangential velocity, v_t , lead to a decaying behaviour of the friction curve. It is also worth mentioning that in steady-state conditions (constant tangential velocity), the average deflection of the bristles can be described as

$$z_{ss} = \frac{v_t}{\|v_t\|} g(v_t) = g(v_t) \text{sgn}(v_t) \quad (7.17)$$

The complete formulation of this dynamic friction force model based on the bending behavior of the bristles is expressed as

$$f_f = \sigma_0 z + \sigma_1 \frac{dz}{dt} + \sigma_2 v_t \quad (7.18)$$

where σ_1 is a bristle damping friction coefficient and the third parcel, $\sigma_2 v_t$, is a linear viscous friction function, where σ_2 is a viscous friction coefficient. In steady state, replacing Equation (7.17) in Equation (7.18) gives

$$f_{fss} = g(v_t) \text{sgn}(v_t) \sigma_0 + \sigma_2 v_t = \text{sgn}(v_t) \left[f_c + (f_s - f_c) e^{-\left(\frac{v_t}{v_s}\right)^2} \right] + \sigma_2 v_t \quad (7.19)$$

A detailed analysis of the equations that mathematically describe this model allows to conclude that it is capable of capturing and modelling the friction phenomena related to pre-sliding displacement and stiction, frictional lag, breakaway force and stick-slip.

7.1.6 Dankowicz model (1999)

Following the works of Dahl (1968, 1976) and Canudas De Wit et al. (1995), Dankowicz (1997,1999) further investigated the idea of internal state variables, proposing a model which establishes that the local deformations at the contact interface, as well as changes in the surface separation, lead to dynamic variations in the frictional force.

The local deformations and surface separation are based on the asperities model in order to represent the static and quasi-static forces. Considering the asperity compression representation presented in Figure 7.5 a), it is possible to define d as the separation between the reference planes of the interacting surfaces and h as the relative height to a reference plane, being $h > d$.

In what concerns the asperity shearing deformation to account for the quasi-static behavior, represented in Figure 7.5 b), it is introduced a state variable of equivalent shearing deformation, z , that describes the collective deformation. This theory considers the asperities as linear springs, thus the tangential load applied to an individual asperity being represented as

$$f_s(\delta) = \sigma_0 \delta \quad (7.20)$$

where δ is the asperity's deformation and σ_0 its stiffness.

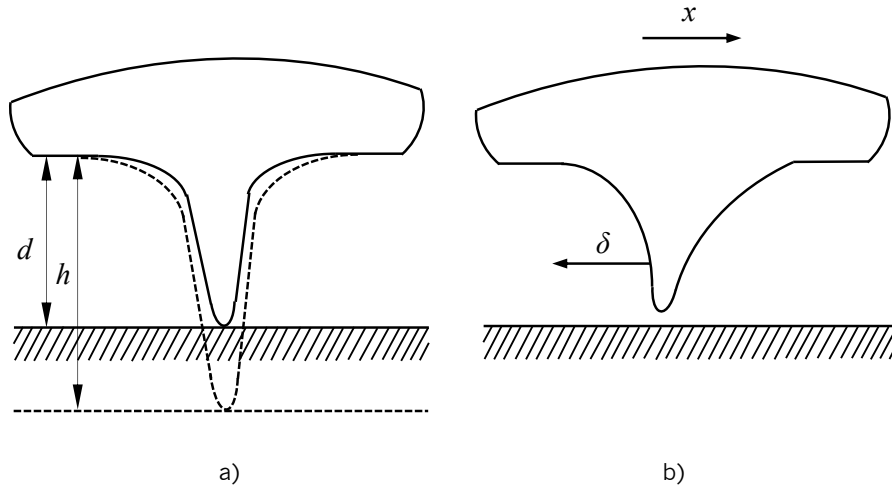


Figure 7.5 – a) Schematic representation of an asperity with d being the distance between reference planes of contacting surfaces and h the relative height to a different reference plane b) Asperity's deformation, δ , due to tangential relative motion.

All these variables and phenomena yield in the equation that describes the collective rate of equivalent deformation as

$$\frac{dz}{dt} = v_t \left(1 - \text{sgn}(v_t) \frac{z}{\delta} \right) \quad (7.21)$$

where v_t is the macroscopic tangential velocity.

Considering the dynamic interactions related to the impact of asperities, which can be studied in more detail in Dankowicz (1999) article, the total tangential component of the surface interactions, that corresponds to the friction force can be formulated as

$$f_f = -f_n \left[\alpha v_t^2 \operatorname{sgn}(v_t) \left(1 - \frac{\Delta d}{d^\infty} \right) + \mu_s \frac{z}{\bar{\delta}} \right] e^{-\frac{\Delta d}{\sigma}} \quad (7.22)$$

where f_f denotes the friction force, f_n represents the normal load, α is a specific parameter which the value must be determined via experimental data, d^∞ is the separation between the reference planes of the interacting surfaces when this distance tends to infinite, μ_s is the static coefficient of friction, σ is the stressed induced in the asperities due to the application of the tangential load, and $\bar{\delta}$ is the average deformation, which can be simply expressed as

$$\bar{\delta} = \mu_s \sigma \quad (7.23)$$

The model described is only presented in an informative way, since it will not be studied in any practical applications analyzed in this work, the reason being the lack of complete comprehensive information in literature about its parameters and respective values.

7.1.7 Leuven model (2000)

The Dahl and LuGre models, despite their capabilities of modelling friction with sufficiently precision compared experimental data, do not attempt to include hysteresis with nonlocal memory in the presliding phase of motion. The Leuven model (Swevers et al., 2000) presented in this subsection is a dynamic friction model which allows the accurate modelling of sliding and presliding regimes while being able to capture, simultaneously, breakaway force variations, Stribeck effect, frictional lag and stick-slip behaviour. The main goal of this model is to present a better alternative friction force model that is able to correctly predict friction values at velocity reversals.

In this model, the friction force equation can be expressed as

$$f_f = F_h(z) + \sigma_1 \frac{dz}{dt} + \sigma_2 v_t \quad (7.24)$$

where σ_1 is a damping coefficient, σ_2 is a viscous damping coefficient, v_t is the tangential velocity and $F_h(z)$ is a nonlinear hysteretic friction force function with nonlocal memory, responsible for hysteretic

behavior. This function is referred as nonlocal memory since the transition curves are only determined by the maximum previous value of $F_h(z)$, thus the intermediate variation of the bristle deflection, z , verified between limit values is not considered. This function can be expressed as

$$F_h(z) = F_b + F_d(z) \quad (7.25)$$

where F_b is the hysteretic behavior of the force in the beginning of the transition curve and $F_d(z)$ is a point-symmetrical strictly crescent function similar to a piecewise linear function which represents the transition curve active at a certain time instant.

The nonlinear state equation regarding the velocity of bristle deflection in this model is expressed as

$$\frac{dz}{dt} = v_t \left(1 - \operatorname{sgn} \left(\frac{F_d(z)}{S(v_t) - F_b} \right) \cdot \left| \frac{F_d(z)}{S(v_t) - F_b} \right|^n \right) \quad (7.26)$$

where $S(v_t)$ is a function that models the constant velocity behavior in sliding, being expressed as

$$S(v_t) = f_c + (f_s - f_c) e^{-\left(\frac{v_t}{v_s}\right)^\psi} \quad (7.27)$$

with v_s being the Stribeck velocity and ψ an arbitrary exponent. It is possible to notice that this last equation resembles the $g(v_t)$ function of LuGre friction force model (Canudas De Wit et al. 1995).

The same authors proposed later (Lampaert et al. 2002) an adapted nonlinear state equation of bristle deflection formulated as

$$\frac{dz}{dt} = v_t \left(1 - \operatorname{sgn} \left(\frac{F_d(z)}{S(v_t)} \right) \cdot \left| \frac{F_d(z)}{S(v_t)} \right|^n \right) \quad (7.28)$$

where the F_b term has been removed to eliminate some numerical instabilities that occurred with the application of this model when transitioning from sticking to sliding phases.

7.1.8 Elasto-Plastic model (2000)

The Elasto-Plastic model, based on the elasto-plastic material models proposed by Prandtl to represent the behavior of solids under stress, was proposed by Dupont et al. (2000;2002). The primary objective of this approach was to model pre-sliding displacement and stiction.

The analyzed model in this section divides the displacement, x , of a sliding body into two different components: an elastic displacement portion, z , and a plastic displacement, represented by the implicit variable w , that correlates as

$$x = z + w \quad (7.29)$$

With the equation given above it is possible to describe both sticking and sliding phases as

$$\begin{cases} \frac{dw}{dt} = 0 & \text{sticking - elastic displacement} \\ v_t = \frac{dz}{dt} & \text{sliding - plastic displacement} \end{cases} \quad (7.30)$$

which means that the elastic displacement, z , manifests during the sticking phase and the plastic displacement, w , during the sliding phase, as it is possible to visualize in Figure 7.6.

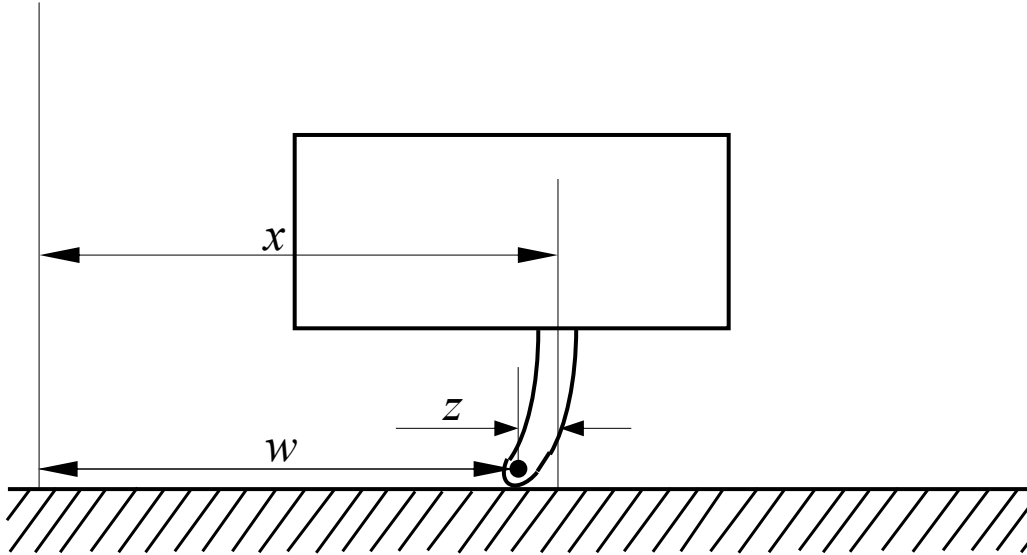


Figure 7.6 – Physical representation of the elasto-plastic dynamic friction model where the displacement x of the sliding body is divided into an elastic displacement portion z and a plastic displacement w .

The mathematical formulation of the friction force in this model is very similar to the LuGre model

$$f_f = \sigma_0 z + \sigma_1 \frac{dz}{dt} + \sigma_2 v_t \quad (7.31)$$

where σ_0 is the stiffness of the bristle, σ_1 is a damping coefficient and σ_2 a viscous coefficient. In this case, the deflection rate of the bristle is equal to

$$\frac{dz}{dt} = v_t \left(1 - \alpha(z, v_t) \frac{\sigma_0}{F_{fss}(v_t)} \text{sgn}(v_t) z \right)^i \quad (7.32)$$

in which $F_{f_{ss}}(v_t)$ is the friction force value in steady state, i is an integer coefficient and $\alpha(z, v_t)$ is a piecewise continuous function used to capture stiction, since it allows the elastic displacement to be developed until reaching the breakaway force. This function depends on the regime (elastic or plastic) of the displacement as it is represented

$$\alpha(z, v_t) = \begin{cases} 0 & \text{if } \text{sgn}(v_t) = \text{sgn}(z) \wedge \|z\| \leq z_{ba} \\ 0 < \alpha < 1 & \text{if } \text{sgn}(v_t) = \text{sgn}(z) \wedge z_{ba} < \|z\| < z_{ss}(v_t) \\ 1 & \text{if } \text{sgn}(v_t) = \text{sgn}(z) \wedge \|z\| \geq z_{ss}(v_t) \\ 0 & \text{if } \text{sgn}(v_t) \neq \text{sgn}(z) \end{cases} \quad (7.33)$$

where z_{ba} is the breakaway bristle deflection and $z_{ss}(v_t)$ is the steady-state bristle deflection, which

corresponds to the maximum value of deflection, $z_{ss}(v_t) = z_{max}$. The relation $\frac{z_{ba}}{z_{ss}(v_t)} \approx 0.7$ can be used

to determine the value of the breakaway deflection, z_{ba} .

In the specific case of small vibrations, the $\alpha(z, v_t)$ function for the $z_{ba} < \|z\| < z_{ss}(v_t)$ interval can be equal to

$$\alpha(z, v_t) = \frac{1}{2} \sin \left(\pi \frac{z - \left(\frac{z_{ss}(v_t) + z_{ba}}{2} \right)}{z_{ss}(v_t) - z_{ba}} \right) + \frac{1}{2} \quad (7.34)$$

The use of the sine function is explained by the fact of being very useful to study the transition behavior from static to kinetic friction force levels in friction force models. Despite its accuracy, this model still presents some problems related to the determination of the mathematical description of $\alpha(z, v_t)$ function, since its shape cannot be intuitively chosen due to the inexistence of a physical relation with the friction phenomena.

7.1.9 Generalized Maxwell Slip model (2003)

The Generalized Maxwell Slip Model (GMS), proposed by Lampaert et al. (2003), is a friction force model based in physical properties of interacting surfaces which simulates the contact physics at the asperity levels. This model can predict pre-sliding regime, Stribeck effect, frictional lag, breakaway force and stiction behaviors.

There are two main categories of phenomena introduced in this model: (i) friction mechanics and (ii) the asperity contact scenario. In what concerns the friction mechanics, it is possible to evaluate and analyze phenomenological mechanisms such as normal creep of contacting asperities, adhesion raising and hysteresis losses due to asperity deformation. Regarding asperity contact, this category can be divided into three different phases: (i) the asperity of one contact surface is free, thus not contacting or suffering any kind of bending solicitation from the other surface; (ii) the asperity of one surface is in contact with the opposed surface which leads to a deformation that can be expressed as a function of the relative displacement between the interacting surfaces; (iii) the surfaces are no longer in contact and all the deformation energy is dissipated.

This analysis leads to the representation of friction as the parallel interaction of N elementary single state friction models as it is represented in Figure 7.7.

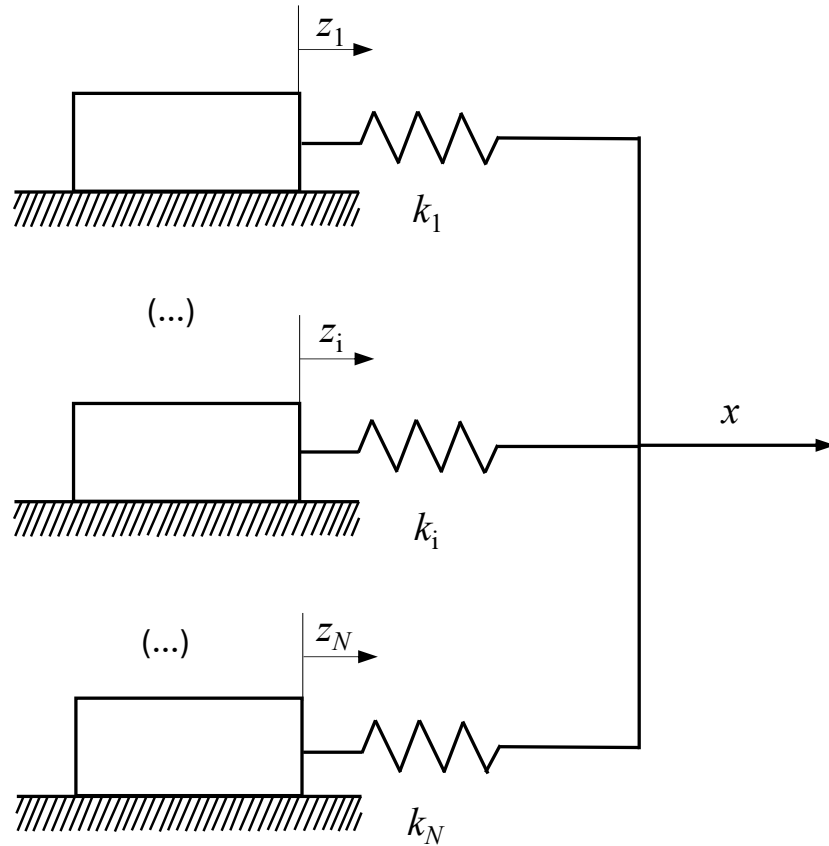


Figure 7.7 – Physical interpretation of the Generalized Maxwell Slip friction model as the parallel interaction of N elementary single state friction models.

Each one of the elements of this model is described by the same dynamical model, which is materialized by the bending behavior of bristles as springs, where the displacement and velocity applied to each one of the elementary models is the same. The sticking and sliding phases of each one of the elementary friction models is evaluated to describe the general response of the system in analysis.

The GMS model is based on three friction properties: (i) the Stribeck curve for constant velocities, (ii) the hysteresis function with non-local memory in the pre-sliding regime, and (iii) the frictional memory in the sliding regime (Lampaert et al. (2003)).

Al-Bender et al. (2005) stated that if an individual friction element i is sticking, then the bristle deflection rate can be expressed as

$$\frac{dz_i}{dt} = v_t \quad (7.35)$$

where v_t is the tangential velocity. This regime maintains until the deflection of the bristle equals the value of the Stribeck velocity in sliding, $z_i = s_i(v_t)$.

For the slipping phase of an individual friction element i , the bristle deflection rate can be described as

$$\frac{dz_i}{dt} = \text{sgn}(v_t) C_i \left(1 - \frac{z_i}{s_i(v_t)} \right) \quad (7.36)$$

where C_i is an attraction parameter which represents a gain, similar to the proportional term of a PID controller, that determines how fast z_i converges to s_i . Equation (7.36) is valid until the relative velocity of the elementary model reaches a null value.

Since the sticking and slipping equations of the state variable z are written for each individual friction element i , then to determine the total friction force acting on a system is necessary to sum all the outputs of the N elementary friction models as

$$f_f = \sum_{i=1}^N \left(k_i z_i(t) + \sigma_i \frac{dz_i}{dt} \right) + f(v_t) \quad (7.37)$$

where k_i is the stiffness of each elementary model spring, σ_i is a damping coefficient and $f(v_t)$ is a viscous function which is defined to model the viscous component. This function is referred to as velocity strengthening, since it is proportional to the tangential velocity.

It is possible to notice that each element is characterized by four different parameters: a stiffness k_i , a damping coefficient σ_i , an attraction parameter C_i and a Stribeck velocity function s_i . The definition of s_i introduces one more unknown parameter α_i , which is the reciprocal of the Stribeck velocity for element i . To solve this problem without introducing additional unknown parameters nor

sacrificing the essence of the problem, a v_i variable must be defined which acts as a scale parameter for all the elemental friction models existing on the system.

Although being very precise and faithful to experimental data, this dynamic friction force model has the disadvantage of requiring a lot of computational resources, needing around 1000 equivalent asperities to be modeled to obtain smooth responses. Thus, its main applications can be associated to the validation of simpler friction force models which are more suitable for practical control and multibody applications.

7.1.10 Gonthier et al. model (2004)

The Gonthier model, proposed by Gonthier et al. (2004), is a regularized three-dimensional contact force model with asymmetric damping and dwell-time dependent friction. Since the focus of this work is to describe the friction force models in a comprehensive way, only the tangential component of this model will be studied in detail.

In this model, the friction force is also defined as a function of bristle deflection, similarly to the approach proposed in the LuGre model by Canudas De Wit et al. (1995), being expressed as

$$f_{br} = \sigma_0 z + \sigma_1 \frac{dz}{dt} \quad (7.38)$$

where σ_0 is the stiffness of the bristles, z the bristle deflection, σ_1 a damping coefficient and $\frac{dz}{dt}$ the rate of bristle deflection.

The novelty introduced with this model is directly related to the bristle state of deflection. This deflection was then divided into two components: a static rate of deflection defined for the sticking phase $\frac{dz_{st}}{dt}$, and a dynamic rate of deflection defined for the sliding phase $\frac{dz_{sl}}{dt}$. The general formulation of the deflection rate is presented below

$$\frac{dz}{dt} = s \frac{dz_{st}}{dt} + (1 - s) \frac{dz_{sl}}{dt} \quad (7.39)$$

where s is a sticking state function defined as

$$s = e^{-\left(\frac{v_t}{v_s}\right)^2} \quad (7.40)$$

in which v_t represents the tangential relative velocity and v_s the Stribeck velocity.

It is then intuitive to understand that when v_t tends to zero, the deflection rate becomes equal to the static deflection rate, $\frac{dz}{dt} = \frac{dz_{st}}{dt}$, and when s tends to zero, the deflection rate is practically equal to the dynamic deflection rate, $\frac{dz}{dt} \approx \frac{dz_{sl}}{dt}$. Moreover, in the sticking regime, the bristle deflection will correspond to the tangential velocity between the interacting surfaces, $\frac{dz_{st}}{dt} = v_t$, and in the sliding regime the bristle deflection will be defined in terms of the Coulomb friction force f_C , since it represents the steady state of the system in analysis.

The Coulomb friction force opposes the external force applied to a system and can be defined as

$$f_C = f_C \text{dir}_0(v_t, v_0) \quad (7.41)$$

where f_C represents the Coulomb friction force level and $\text{dir}_0(v_t, v_0)$ is a piecewise function of both tangential velocity v_t and a small velocity tolerance v_0 which returns a unit vector along the v_t direction expressed as

$$\text{dir}_0(v_t, v_0) = \begin{cases} \frac{v_t}{\|v_t\|} & \text{if } \|v_t\| \geq v_0 \\ \frac{v_t}{v_0} \left(\frac{3}{2} \frac{\|v_t\|}{v_0} - \frac{1}{2} \left(\frac{\|v_t\|}{v_0} \right)^3 \right) & \text{if } \|v_t\| < v_0 \end{cases} \quad (7.42)$$

The velocity tolerance, v_0 , is used to reduce the discontinuities in velocity direction and should be at least one tenth of v_s , being commonly considered to be equal to one hundredth of v_s .

Combining Equation (7.39) with the assumption that $\frac{dz}{dt} \approx \frac{dz_{sl}}{dt}$ results in a dynamic bristle deflection equal to

$$\frac{dz_{sl}}{dt} = \frac{1}{\sigma_1} f_C - \frac{\sigma_0}{\sigma_1} z \quad (7.43)$$

which leads to a general bristle deflection equation formulated as

$$\frac{dz}{dt} = e^{-\left(\frac{v_t}{v_s}\right)^2} v_t + \left(1 - e^{-\left(\frac{v_t}{v_s}\right)^2}\right) \left(\frac{1}{\sigma_1} f_C - \frac{\sigma_0}{\sigma_1} z \right) \quad (7.44)$$

To correctly model the transition existent in the stick-slip motion due to the frictional lag associated with the dwell-time, it is necessary to include a new state variable s_{dw} . This variable is a function defined by two time constants that allow the correct modelling of dwell-time effect on the maximum static friction force and can be expressed as

$$\dot{s}_{dw} = \begin{cases} \frac{1}{\tau_{dw}}(s - s_{dw}) & \text{if } s - s_{dw} \geq 0 \\ \frac{1}{\tau_{br}}(s - s_{dw}) & \text{if } s - s_{dw} < 0 \end{cases} \quad (7.45)$$

where τ_{dw} is the dwell-time dynamics time constant and τ_{br} is the bristle dynamics time constant which can be defined as $\tau_{br} = \frac{\sigma_1}{\sigma_0}$.

The maximum static friction (stiction) force can be defined as

$$f_{\max} = f_C + (f_s - f_C)s_{dw} \quad (7.46)$$

Considering that the friction force applied by the bristle f_{br} should be equal to the Coulomb friction force f_C when the contacting bodies are sliding, $f_{br} = f_C$, then the friction force can be modelled as

$$f_f = \begin{cases} f_{br} + \sigma_2 v_t & \text{if } \|f_{br}\| \leq f_{\max} \\ \frac{f_{br}}{\|f_{br}\|} f_{\max} + \sigma_2 v_t & \text{if } \|f_{br}\| > f_{\max} \end{cases} \quad (7.47)$$

where σ_2 is a viscous damping coefficient.

This friction model is then characterized by eight (or seven, if it is considered that the velocity tolerance, v_0 , is defined as a function of the Stribeck velocity, v_s) different parameters: μ_k , μ_s , σ_0 , σ_1 , σ_2 , v_0 , v_s and τ_{dw} .

7.1.11 Liang et al. (2012)

The Liang et al. (2012) bristle model is an extension of the bristle model proposed by Haessig and Friedland (1991) to three dimensional multibody systems that can be used to describe the tangential contact between general rigid or near-rigid body contact dynamics applications.

This model consists in treating the contact between interacting surfaces as multiple individual contacts, materialized by bristles. In Figure 7.8 is represented the contact between two different bodies, Body 1 and Body 2 in two different time instants, t and $t + \Delta t$.

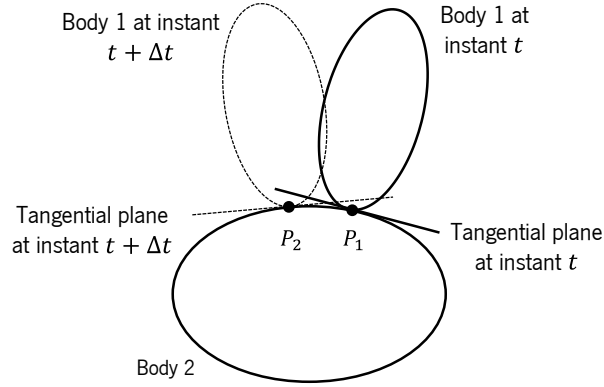


Figure 7.8 – Contact between two different bodies (Body 1 and Body 2) in two different instants, t and $t + \Delta t$.

For a time instant, t , the contact point between both bodies is the point P_1 . If the system is in a sticking phase, then for the instant $t + \Delta t$ the two bodies remain in the same relative position, so the point of contact between bodies 1 and 2 is still P_1 . On the other hand, if one body moves relatively to the other, then the contact point will change from P_1 to P_2 . Then, the motion of these bristles can be described as a spring like behavior that exhibits stretch and rotation around the common tangential plane, being the friction force computed as the average displacement of these elements, as depicted in Figure 7.9 a) and b).

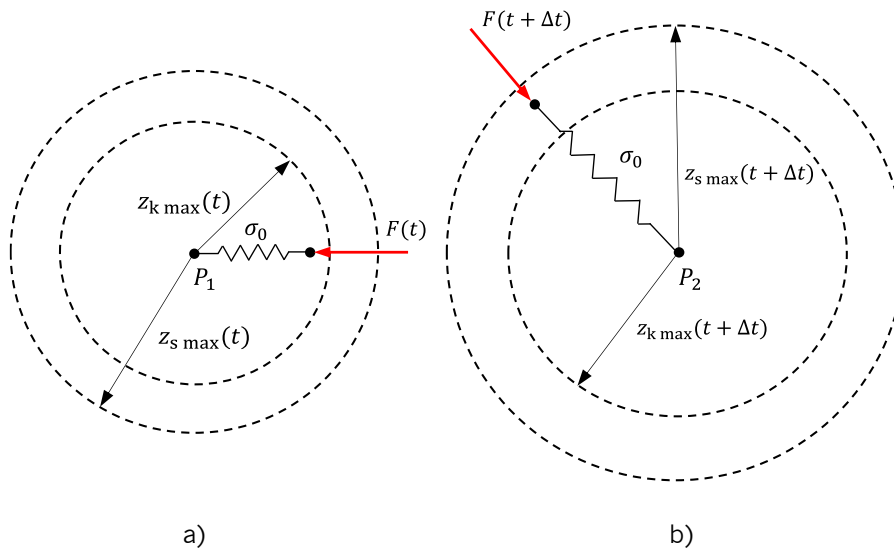


Figure 7.9 – a) Representation of maximum static ($z_{s \max}$), and kinetic ($z_{k \max}$) deflections for a given acting force F in a certain time instant t and b) for another time instant given by $t + \Delta t$, where the bristle deflection shows an expansion and rotation around the common tangential plane.

The main advantages of this model are that it is a standalone model, so it can be easily implemented and integrated into an existing general dynamics solver for multibody dynamical systems, and it is able to describe both static and dynamic friction in the same model and simulate usual frictional phenomena (Liang et al. 2012).

In what concerns each contact point, the three-dimensional friction force can be described by Hooke's law in the form of

$$f_f = -\sigma_0 z \quad (7.48)$$

where σ_0 is the bristle stiffness and z is a three-dimensional vector that represents the average bristle deflection or displacement, which can be expressed as

$$z = \begin{cases} z(t_0) + \int_{t_0}^t v_t(t) dt & \text{if } \|z\| < z_{\max}(t) \\ z_{\max}(t) \frac{v_t(t)}{\|v_t(t)\|} & \text{if } \|z\| \geq z_{\max}(t) \end{cases} \quad (7.49)$$

where t_0 is the initial time instant corresponding with the beginning of contact, t is the current time of contact, $v_t(t)$ represents the three-dimensional vector of tangential velocity and $\|v_t(t)\|$ is simply the magnitude of tangential velocity.

The maximum bristle deflection in a given time instant, $z_{\max}(t)$, can be defined as

$$z_{\max}(t) = \begin{cases} z_{k\max}(t) = \frac{f_c(t)}{\sigma_0} & \text{if } \|v_t\| > v_0 \\ z_{s\max}(t) = \frac{f_s(t)}{\sigma_0} & \text{if } \|v_t\| \leq v_0 \end{cases} \quad (7.50)$$

in which f_c is the Coulomb friction force acting in each instant, v_0 is a velocity threshold that marks the transition between sticking and sliding phases and f_s is the static friction force acting in each instant. It is also important to notice that this definition of the maximum bristle deflection makes a distinction between values for the maximum static deflection $z_{s\max}(t)$, and maximum kinetic deflection $z_{k\max}(t)$, as it is visible in Figure 7.9.

The model formulation and the corresponding illustration presented in Figure 7.9 a) and b) show that this model approximates the bristle behavior to the one exhibited by a linear spring. Thus, this can lead to numerical instabilities in the transition between stick and slip state of motion due to excessive

stiffness and/or oscillatory behavior of the system. To prevent the noise generated by high frequency friction force variation, that can lead to numerical discontinuities, it is introduced a bristle damping coefficient σ_1 which leads the friction force mathematical expression to be equal to

$$f_f = -\sigma_0 z - \sigma_1 \frac{dz}{dt} \quad (7.51)$$

This friction force model reduces to the Coulomb friction force model when the friction is in sliding regime and while the system in analysis is in the sticking phase, the friction force magnitude is below the one corresponding to the stiction threshold. To fully define this friction force model it is necessary to specify the values of five parameters: the kinetic coefficient of friction μ_k , the static coefficient of friction μ_s , the velocity threshold v_0 , the bristle stiffness σ_0 , which should be defined, by rule of thumb, with a value of one order of magnitude (10^1) higher than the largest dominant stiffness of the system in analysis, and the bristle damping coefficient σ_1 .

7.2 Benchmark problem – dynamic friction models

After presenting the most commonly utilized dynamic friction force models in literature, it becomes imperative to assess the computational behavior and efficiency of this category of models. The MATLAB computational program briefly described in the previous chapter about static friction force models, was also adapted to study the one degree of freedom dynamic system of the sliding block using dynamic friction force models.

The selection of this type of model implies the definition of the specific parameters of each model. Table 7.1 displays the various parameters considered for the dynamic category of friction force models.

Table 7.1 – Dynamic friction force models specific parameters.

Parameter	Symbol	Value	Parameter	Symbol	Value
Initial bristle deflection (common to all models)	z_0	0 m	Stiction- steady state ratio (Reset Integrator)	a	0.52
Stiffness coefficient (common to all models)	σ_0	10^5 N/m	Integer power (Elasto-Plastic)	i	1
Damping coefficient (LuGre, Reset Integrator, Elasto- Plastic, Gonthier, Liang)	σ_1	$\sqrt{10^5}$ N·s/m	Initial dwell-time (Gonthier)	s_{dw}	0
Viscous coefficient (LuGre, Elasto-Plastic, Gonthier)	σ_2	0.1 N·s/m	Dwell-time dynamics constant (Gonthier)	τ_{dw}	2
Stribeck velocity (LuGre, Elasto-Plastic, Gonthier)	v_s	0.001 m/s	Velocity threshold (Liang et al.)	v_0	0.001 m/s

The previously specified parameters were then introduced in the respective interface window of the MATLAB program shown in Figure 7.10.

Dynamic friction force models

For the previously selected dynamic friction force models, fill the following fields with the correspondent numerical value of each specific parameter

Initial bristle deflection, z_0 [m]

Dahl

Stiffness coefficient, σ_0 [N/m]

LuGre

Stribeck velocity, v_s [m/s]

Stiffness coefficient, σ_0 [N/m]

Damping coefficient, σ_1 [N · s/m]

Viscous coefficient, σ_2 [N · s/m]

Reset Integrator

Stiction-steady state ratio, a [-]

Stiffness coefficient, σ_0 [N/m]

Damping coefficient, σ_1 [N · s/m]

Liang Bristle Model

Threshold velocity, v_0 [m/s]

Stiffness coefficient, σ_0 [N/m]

Damping coefficient, σ_1 [N.s/m]

Elasto-plastic

Integer power, i [-]

Stiffness coefficient, σ_0 [N/m]

Damping coefficient, σ_1 [N · s/m]

Viscous coefficient, σ_2 [N · s/m]

Stribeck velocity, v_s [m/s]

Gonthier

Stribeck velocity, v_s [m/s]

Initial dwell-time, s_{dw} [-]

Dwell-time dynamics constant, τ_{dw} [s]

Stiffness coefficient, σ_0 [N/m]

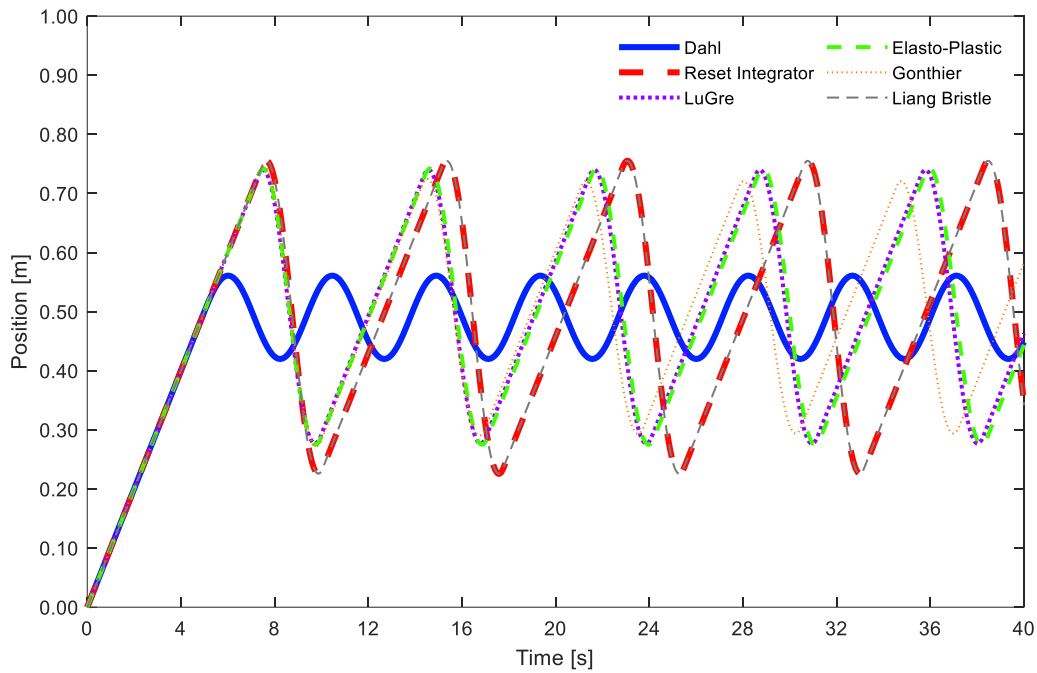
Damping coefficient, σ_1 [N · s/m]

Viscous coefficient, σ_2 [N · s/m]

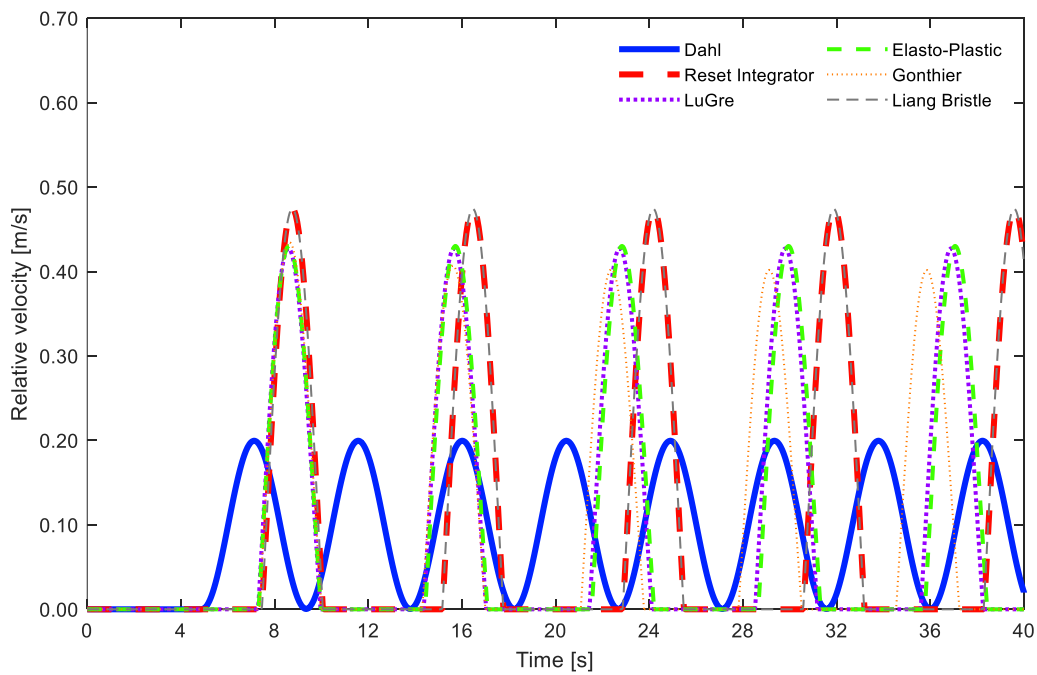
Solve

Figure 7.10 – Dynamic friction force models window from the MATLAB computational code to define the specific parameters of each model.

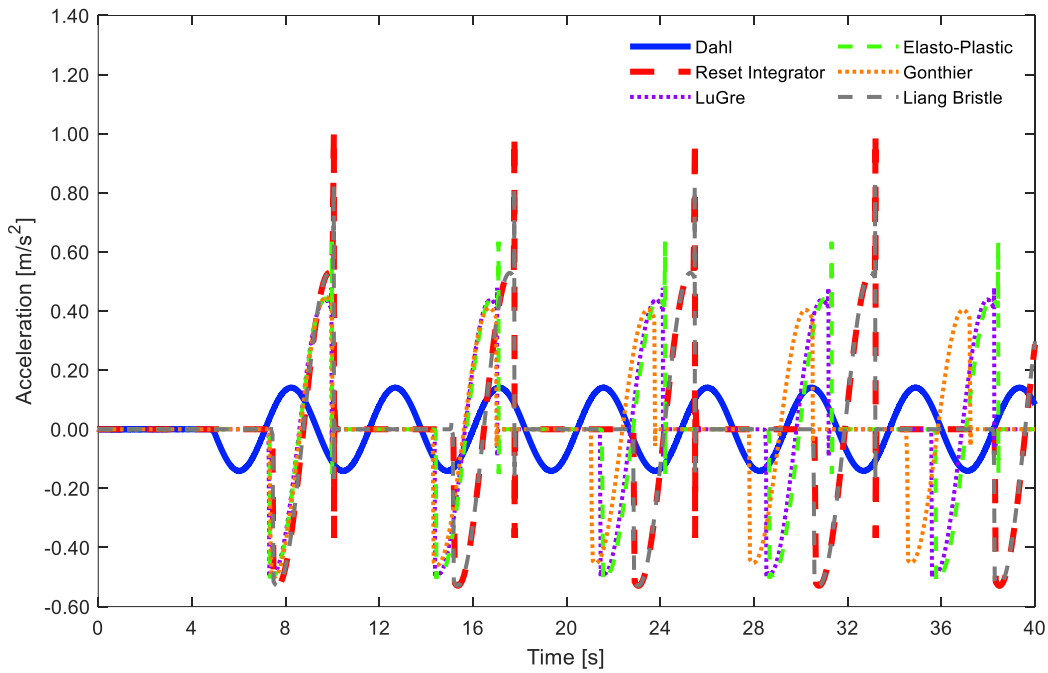
Similarly to what has been previously done for the two categories of static friction force models, after pressing the *Solve* button, it was possible to obtain the results of the simulation of the benchmark problem considering dynamic friction force models, which resulted in the plot of the different graphs that quantify the behavior of the system in analysis in terms of position, relative velocity, acceleration and friction as functions of the simulation time and which are represented in Figure 7.11 from a) to d).



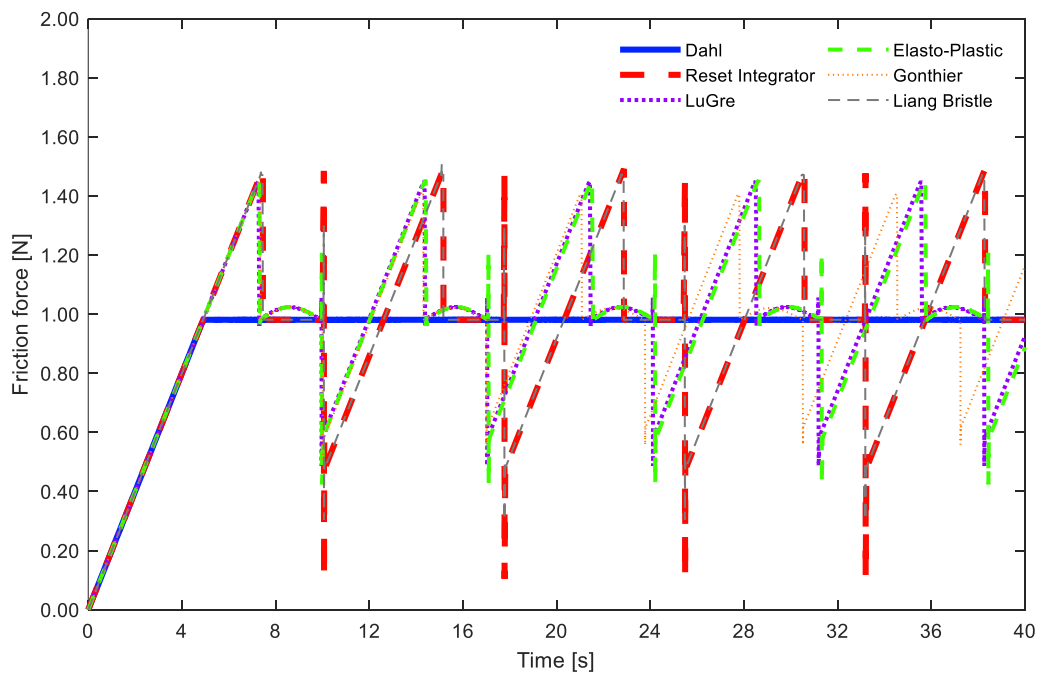
a)



b)



c)



d)

Figure 7.11 – Graphical results of the benchmark problem for the dynamic friction force models a) Position [m] vs Time [s] graph b) Relative velocity [m/s] vs Time [s] graph c) Acceleration [m/s^2] vs Time [s] graph d) Friction force [N] vs Time [s] graph using the ode45 solver.

A first look analysis of the position-time graph of Figure 7.11 a) enables to draw two main conclusions: (i) the Dahl (1968) friction model has a behavior very similar to the Coulomb friction force

model, since it exhibits an oscillatory sinusoidal-like motion over time and in what concerns the friction force vs time, it is not capable of reaching the maximum friction force value correspondent to the static friction level, being the maximum value correspondent to the Coulomb (kinetic) friction level; (ii) The other models have a very similar behavior to the static with stiction friction force models in terms of stick-slip values and time instants. The modeling of viscous and Stribeck effects by LuGre, Elasto-Plastic and Gonthier models, related to the viscous damping specific parameters, limits the amount of slip of the sliding, which results in smaller maximum relative velocities, acceleration and friction force values after the first stick-slip cycle.

Figure 7.12 shows a zoomed region of the friction force vs simulation time graph between the time instants $t = 4$ s and $t = 16$ s.

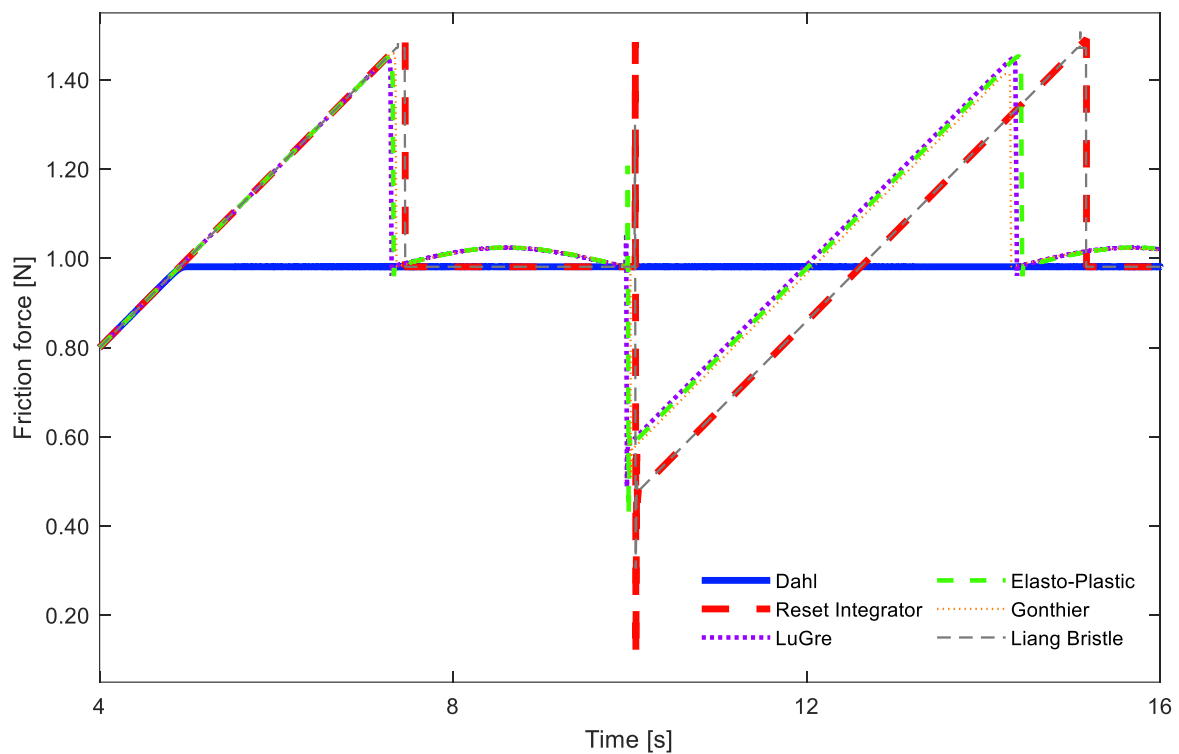


Figure 7.12 – Zoomed region of the friction force vs time graph.

The interest of analyzing this specific region of the aforementioned graph is related to the fact of that the models capable of modelling the viscous and Stribeck effects (LuGre, Elasto-Plastic and Gonthier) exhibit difficulties in reaching the maximum value of static friction force at the end of the initial sticking phase, which allows to conclude that they show an early breakaway force in comparison with Reset Integrator and Liang et al. models. This behavior can be an intrinsic response of these models or can just result from a poor choice of parameters that are somehow inadequate to this specific dynamic problem.

Another concerning problem is the fact that after the first sliding phase, which starts approximately at $t = 7$ s, the next sticking phase for the LuGre, Elasto-Plastic and Gonthier models show an incapacity of fully utilizing the stiction potential. This brings unfeasible simulation results since the subsequent sticking phases end with stiction values that are around 40% lower than the maximum static friction force ($f_s = 1.47$ N) for the LuGre and Gonthier models, and around 23% for the Elasto-Plastic model. This behavior can be explained by higher damping forces acting on the system for the LuGre, Gonthier and Elasto-Plastic models. It is worth of notice that the Reset Integrator and Liang Bristle models do not yield this type of problem, being capable of using the full static friction force potential.

To analyze the capacity of these more complex models of modelling frictional lag and hysteresis phenomena, the graph of friction force vs relative velocity was plotted as shown in Figure 7.13.

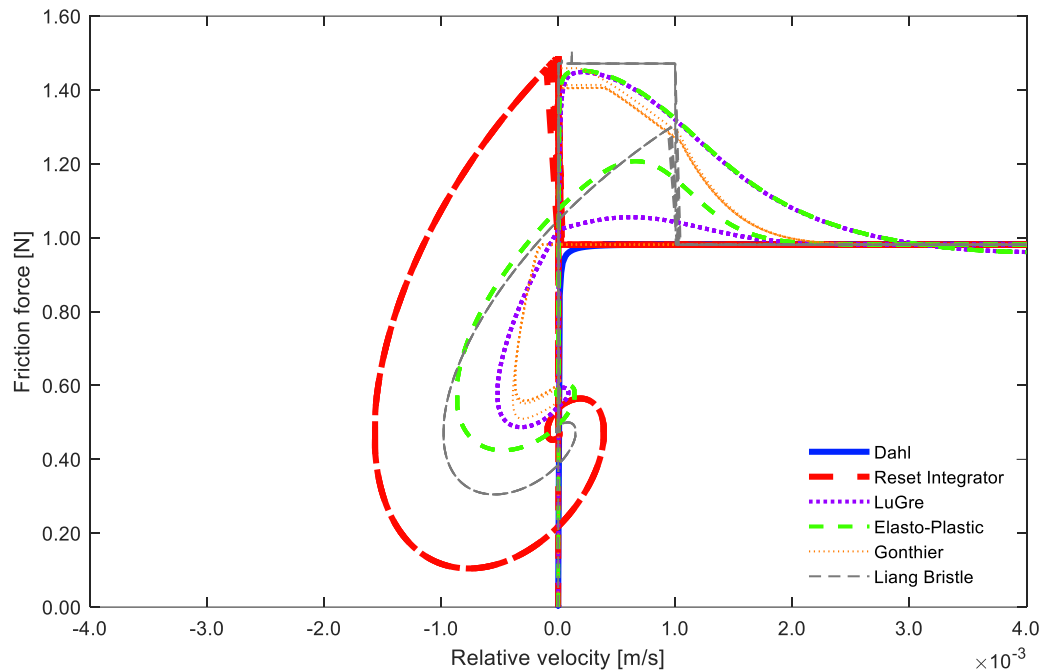


Figure 7.13 – Graph of friction force [N] as a function of relative velocity [m/s] for dynamic friction models.

It is possible to observe that both the LuGre and Elasto-Plastic are capable of modeling the frictional lag phenomenon due to the increased friction force for increasing relative velocity values, which is materialized by the top and bottom curve of each model.

Another important phenomenon that should be studied is the pre-sliding displacement for which the graph represented in Figure 7.14 was plotted.

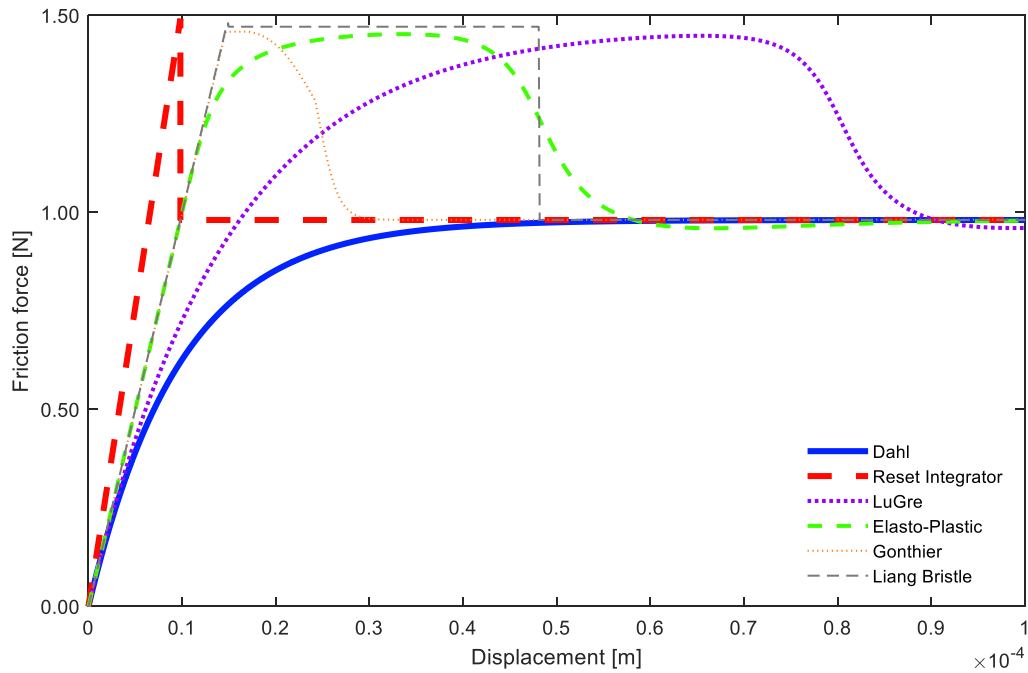


Figure 7.14 – Graph of friction force [N] as a function of displacement [m] for dynamic friction models.

Dahl (1968) states that the pre-sliding displacement range is between 1 – 50 μm which shows that almost all the analysed dynamic friction force models are capable of modelling this phenomenon, being the Dahl model right in the boundary of the aforementioned interval and the LuGre a bit over the normally accepted interval.

Similarly to static models, the dynamic friction force models were also object of study in terms of computational efficiency using the same solvers and the same equipment mentioned in the previous chapter, being the results presented in Table 7.2.

Analyzing the results from the simulations of the dynamic system using the different dynamic friction force models, it is possible to state that the most efficient model in terms of simulation time was the Liang et al. model with a mean time of 1.1576 s using the ode23s solver and the slower computational time was verified for the Elasto-Plastic model with a mean runtime of 6.1448 s using the ode78 solver. Although not represented in Table 7.2, the mean value of the simulation times of all the solvers yields the Reset Integrator friction model as the most efficient model in terms of computational elapsed time, while the Elasto-Plastic model is the least efficient. In comparison with the static friction force models, where the ode23s was considered an unreliable solver, since it was not capable of converging to a solution for two models, for the dynamic friction force models this behavior was not verified. A small note should be given to the ode78 solver that exhibited some numerical instabilities when applied to the Liang et al. model.

Table 7.2 – Solver and average simulation time for the dynamic friction force models.

Model	Solver	Average Simulation Time [s]	Model	Solver	Average Simulation Time [s]
Dahl	ode45	3.2448	Elasto-Plastic	ode45	3.8298
	ode15s	2.5948		ode15s	1.8776
	ode23s	1.5837		ode23s	1.3486
	ode78	4.4978		ode78	6.1448
LuGre	ode45	3.6386	Gonthier	ode45	1.6792
	ode15s	2.3357		ode15s	2.5156
	ode23s	1.3078		ode23s	1.9086
	ode78	5.7774		ode78	2.5347
Reset Integrator	ode45	1.5395	Liang	ode45	2.3524
	ode15s	1.4439		ode15s	1.4591
	ode23s	1.2175		ode23s	1.1576
	ode78	2.4610		ode78	3.6724 *NI

*NI – Numerical Instabilities

7.3 Summary and Discussion

Dynamic friction force models represent a significant progression in friction modeling by introducing a state variable, often referred to as bristle deflection, to characterize friction at the asperity level. This additional complexity allows these models to capture a variety of complex phenomena, such as hysteresis, frictional lag, and pre-sliding displacement, which are beyond the scope of the vast majority of static friction models. By relying on continuous mathematical functions that incorporate this state variable, dynamic models also eliminate numerical instabilities near null tangential velocity, enabling more realistic and robust simulations.

A total of 11 dynamic friction models were analyzed in detail, highlighting their ability to simulate the nuanced behaviors of frictional interactions. These models can effectively account for phenomena like

the viscous and Stribeck effects, which influence system dynamics during transitions between sticking and sliding phases. Additionally, many models demonstrate the ability to simulate time-dependent responses and micro-scale deformations, providing a comprehensive depiction of the frictional forces acting at the contact interface.

However, dynamic models come with substantial drawbacks. Their increased complexity requires the calibration of multiple parameters to accurately represent physical behaviors. Some models, such as Gonthier's, demand the definition of a large number of specific parameters, which often require experimental data for proper calibration. This need for detailed calibration can make these models challenging to implement in practical applications, particularly when the access of experimental data is limited.

In terms of computational performance, dynamic friction models generally required more time to solve the equations of motion for the 1 DoF benchmark problem than their static counterparts, despite the relative simplicity of the problem. This increased computational demand highlights a potential limitation of dynamic models, particularly for systems requiring real-time performance or extensive simulations. Notably, the Liang et al. model emerged as the most computationally efficient among the dynamic models analyzed, with an average runtime significantly lower than others. Conversely, the Elasto-Plastic model was the least efficient model.

This contrast in computational efficiency underscores the variability in runtime performance among dynamic models, which is influenced by the complexity of the mathematical formulation and the solver employed. Despite this variability, the dynamic models collectively demonstrated slower runtimes compared to static models when applied to the same problem. This suggests that while dynamic models offer superior precision in simulating friction phenomena, their computational cost makes them less suitable for scenarios requiring quick or iterative simulations.

It is also worth noting that the behavior of dynamic friction models in more complex dynamical systems cannot be precisely predicted based solely on the results of the benchmark problem. The increased complexity of these systems may amplify computational costs or introduce additional challenges related to parameter sensitivity and numerical stability.

In conclusion, dynamic friction models offer enhanced capabilities for capturing complex frictional phenomena but at the cost of greater computational demands and the need for extensive parameter calibration. While they are well-suited for high-precision applications, their practical use in real-world engineering problems must balance accuracy and computational feasibility, particularly when applied to more complex multibody dynamics systems.

8. EXPERIMENTAL APPARATUS DEVELOPMENT

A wide variety of friction force models are presented in literature. These models fall into two main categories – static and dynamic models – and are characterized by numeric parameters, from which some have physical meaning and others are simply empirical calibration parameters defined to achieve a better curve fit of the model in analysis to experimental data.

Due to technical challenges and the cost of designing and building specific experimental apparatus that can isolate and accurately measure the influence of friction forces in dynamical systems, it is often difficult to obtain data to conduct parametric analysis and determine the influence of each individual parameter in the friction force values.

The identification of parameters for advanced friction models is a complex and challenging task, requiring extensive experimental dynamic tests. While numerous experimental setups have been developed to explore various frictional characteristics, relatively little effort has been directed towards designing practical systems specifically tailored for parameter identification in commonly used models.

This issue leads several authors to consider the values presented in literature for the various parameters of a specific friction force model without adapting them to the physical and structural properties of the system in analysis. Ultimately, the inaccurate or coarse estimations of parameters in friction force models can lead to unrealistic predictions of dissipative phenomena when simulating the dynamic behavior of a prototype or product during its development phase.

The main focus of this chapter is to describe the methods and results obtained with an experimental setup that prioritizes simplicity, flexibility, and versatility for friction force data acquisition in order to estimate friction force model's parameters and compare them with those normally utilized in literature.

8.1 Design of the Experimental Workbench

The design of the experimental workbench presented in this work prioritized the simplicity of construction and data acquisition, allowing the study of different contact surfaces as well as a wide range of operating velocities and normal forces. Figure 8.1 exhibits the CAD model of the experimental apparatus built. The technical drawings of the tailored-made parts that compose the experimental apparatus are presented in Appendix I.

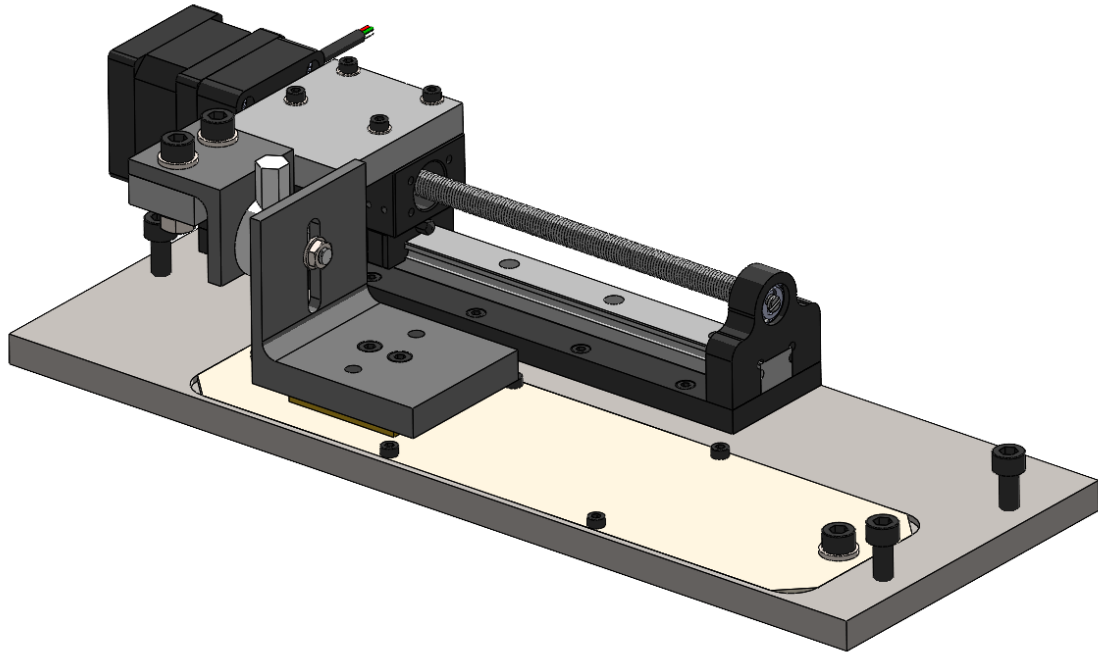


Figure 8.1 – CAD model of the experimental workbench designed in SolidWorks.

The experiment is composed of a L-shaped slider which has a bolted sliding material that can be easily and rapidly changed. The elliptical hole in the vertical face of the slider allows a variable and continuous adjustment of the slider height, where material samples of various thicknesses/heights can be tested, showing the versatility of the experimental apparatus. The workbench base sheet metal has three threaded holes with M6 bolts that are used to ensure that the whole apparatus is conveniently levelled, since the normal force exerted in the slider is sensible to the reference plane inclination angle.

To allow the study of multiple contact material pairs, a pocket was designed to receive test sheets of different materials. These test sheets are bolted to the workbench base by two M6 bolts in each end and four intermediate M3 bolts in the middle section of its contour to reduce the probability of the sheet exhibiting an upwards deflection, which would significantly affect the contact between surfaces and, consequently, the normal force.

The slider is connected to an L-shaped bracket by the load cell, which is used to measure the external force applied to the slider. This bracket is attached to the ball screw utilizing an auxiliary L-shaped support which is directly bolted to the linear guide rail block using four M3 bolts. The ball screw motion is controlled by a NEMA17 stepper motor and its corresponding driver. The code that controls the stepper motor was written in Arduino programming language and an Arduino UNO® microcontroller board was used to make the PC-Stepper motor driver communication.

The thicknesses of the different materials, especially those utilized in the connections that involve the load cell, were chosen in order to provide a sufficiently rigid structure, thus minimizing deflections and, consequently, errors in measuring the input force applied on the slider due to misalignments. Figure 8.2 shows the real-world manufactured experimental workbench.

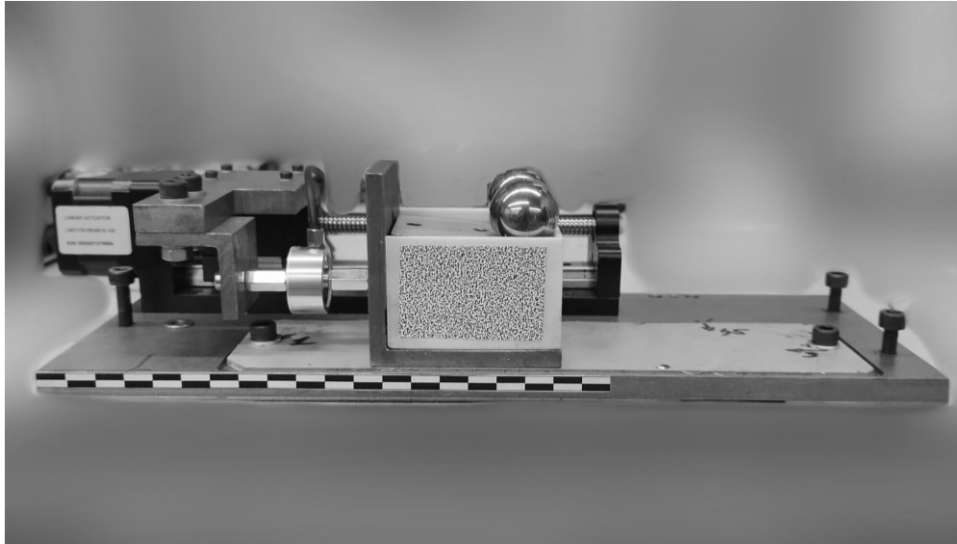


Figure 8.2 – Real-world manufactured workbench for friction force data acquisition.

In the next section a detailed explanation will be presented of the experimental apparatus procedure to obtain friction force data, as well as the different electronic equipment utilized to acquire the different required data in each dynamic test.

8.2 Working Principle and Data Acquisition

The designed experimental apparatus, in terms of dynamic system modeling, can be reduced to a one degree of freedom slider that can have forward and backward motions. Figure 8.3 shows a very simple free body diagram of the slider.

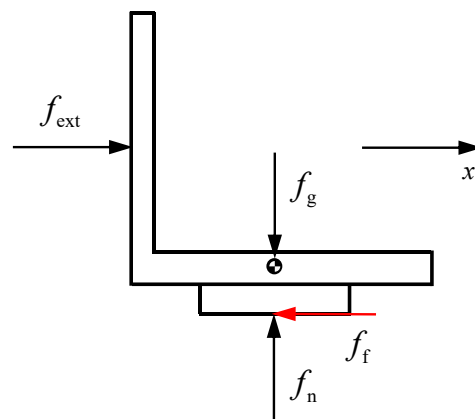


Figure 8.3 – Free body diagram of the slider during a forward motion cycle.

The external force exerted on the slider is continuously measured by the load cell, which is able to measure tension or compression forces, depending on the direction of the slider's motion. The load cell is connected to an amplifier that is responsible for enhancing the signal from the load cell to make it readable by a NI 9234 data acquisition module which is coupled to a NI cDAQ-9171 chassis. The load cell amplifier is powered by a DC-power supply with an output voltage of 20 V. The output voltage of the amplifier was defined to a range of -5V to +5V, because the maximum capacity of the load cell is 5 kg, which results in a direct measurement of the load. Figure 8.4 shows the electronic instrumentation used to measure the load cell data.

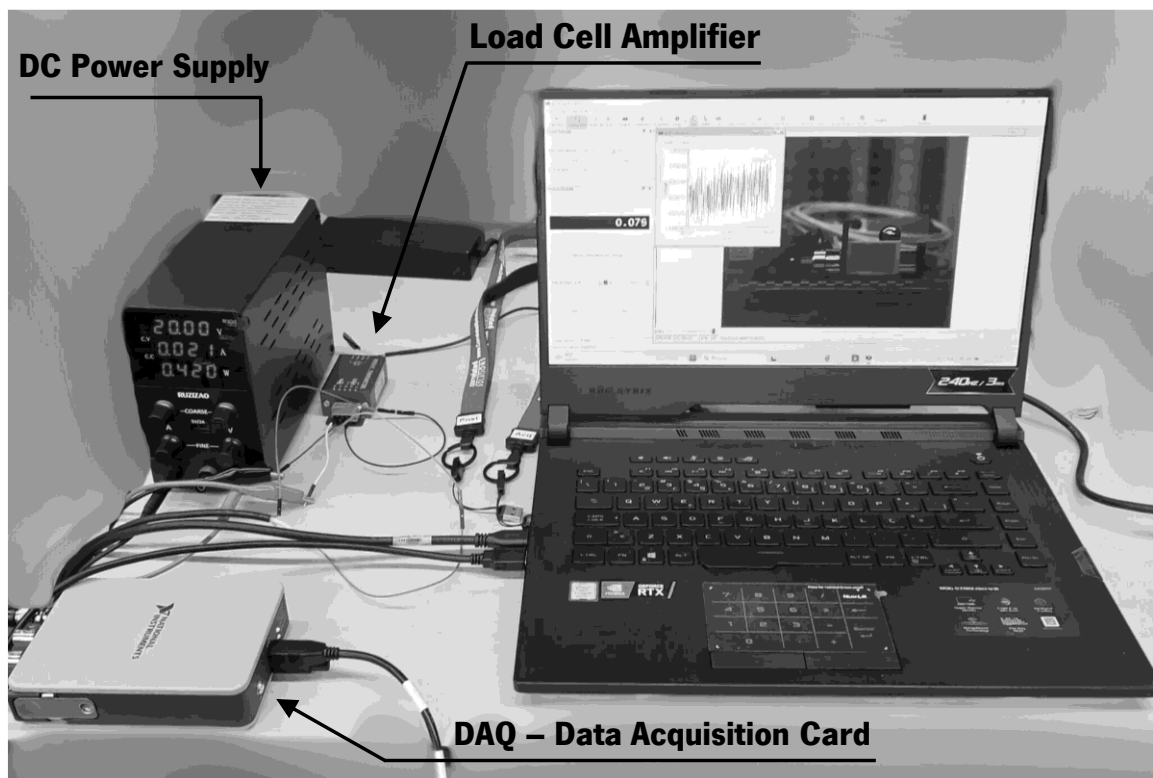


Figure 8.4 – Electronic equipment utilized to acquire analog force data from the load cell.

Friction force models exhibit a dependence of the friction force relatively to the tangential velocity between the contacting surfaces. In order to obtain the position and other kinematic informations about the sliding block, a tracking approach was implemented. To perform the tracking of the slider, a Basler ace U acA2440-75um camera was utilized, as well as a GS Vitec | MULTILED QX LED to provide an adequate illumination for the tests. Figure 8.5 shows the camera and the illumination LED utilized for the friction data acquisition tests.



Figure 8.5 – Illumination LED and camera utilized for the slider's tracking procedure.

After defining the instrumentation to acquire forces and determining the kinematic properties of the moving slider, it is necessary to implement a strategy to synchronize the data obtained from different sources. The simultaneous acquisition of force from the load cell and the position of the slider from the recorded images of the camera can be made in softwares such as MATLAB, for which specifically programs should be coded, or using other alternatives such as LabVIEW, which is a graphical system design and development platform distributed by National Instruments, capable of acquiring and synchronizing data obtained from different sensors using a block diagram type of programming (National Instruments, 2024). For this specific study, neither of the previously mentioned solutions was utilized. Instead, the VIC-2D software from Correlated Solutions was used to synchronize the data from the load cell and the captured images that are used to track the slider position.

Although the Basler camera has a maximum frequency of frames acquired per second of 75, in practice it was only possible to obtain a stabilized number of 40 frames per second. Thus, the frequency of data acquisition for the experiments conducted was 40 Hz.

The position, velocity and acceleration of the sliding block is not determined in real-time. The acquired frames need to be treated by tracking software to obtain the kinematic properties of the slider. There are many commercial solutions regarding motion tracking softwares. The industry and research standard for advanced motion analysis for both tracking of objects in 2D and 3D is the TEMA software by IMAGE systems (IMAGE systems, 2024). Due to the unavailability of a license for this specific software, an open-source alternative was used. The software utilized for the dynamic tests was Tracker, which is a

free video analysis and modeling tool built on the Open-Source Physics (OSP) Java framework (Physlets, 2024). Figure 8.6 shows a diagram that should be followed to obtain friction force data using the described approach.

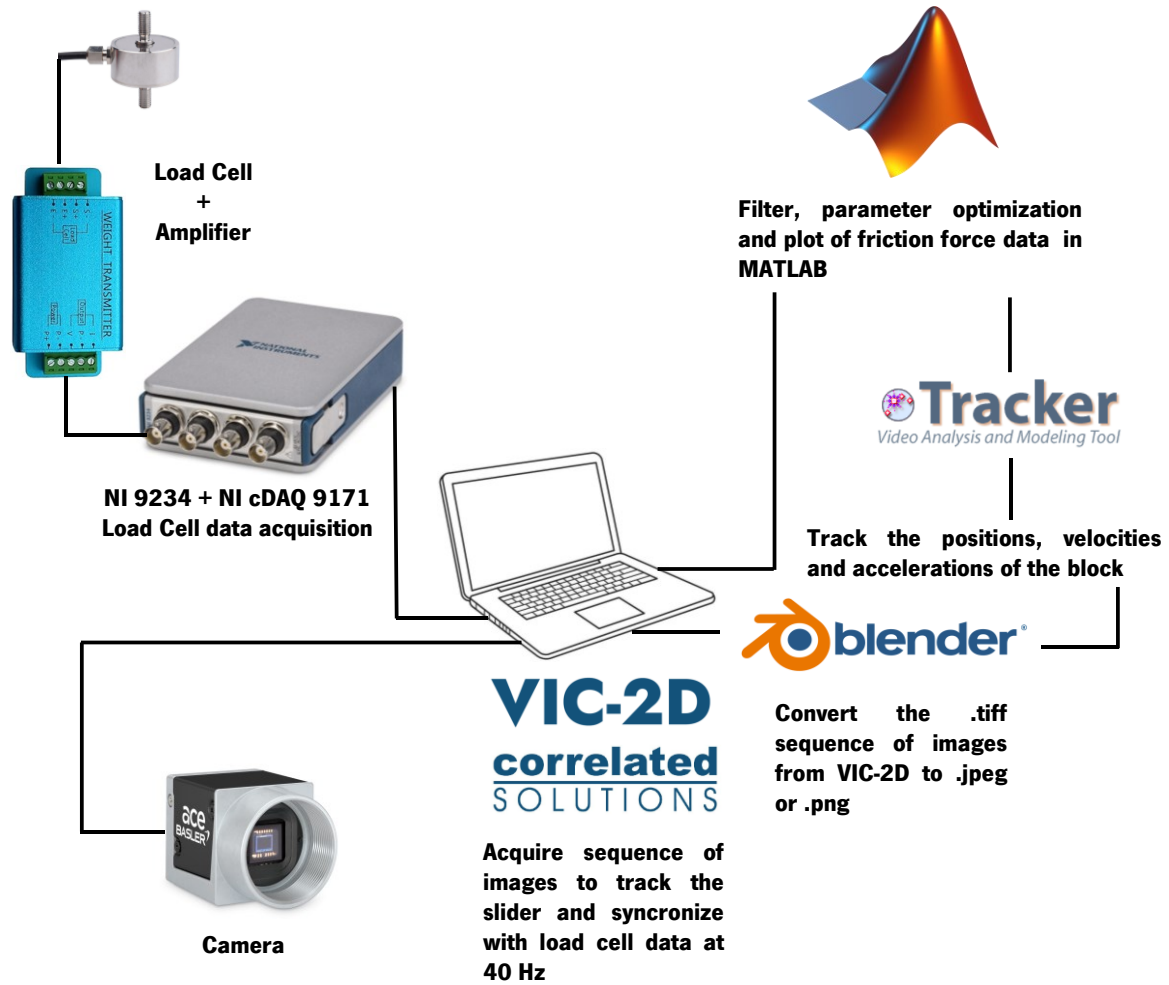


Figure 8.6 – Diagram of the different data sources, equipments and used softwares to obtain friction force data.

It is possible to verify that the last step of the data acquisition process is to filter, perform parameter optimization and plot friction force data within MATLAB. Although a load cell is used, this transducer does not provide a direct measurement of the friction force. By analyzing again the schematic from Figure 8.3 the equation of motion for the one-dimensional movement of the slider along the horizontal surface can be expressed as

$$f_{\text{ext}} - f_f = m\ddot{x} \quad (8.1)$$

where f_{ext} is the external force imposed to the slider, which corresponds directly to the force measured by the load cell, f_f is the friction force, m is the mass of the slider and $\ddot{x}$ is the acceleration of the slider. Rearranging Equation (8.1) in order to the f_f term, it is possible to obtain the friction force that

acts on the sliding block in each instant. For each dynamic test conducted it was ensured that both the force applied on the slider and its displacement are null in the initial instant.

In the next section the results obtained from the dynamic tests realized will be discussed, as well as the parameter optimization procedure and their comparison to the state-of-the-art parameters presented in literature.

8.3 Experimental Data and Parameter Optimization

The analysis of friction force data is usually correlated with other kinematic properties of the object in analysis, namely displacement or relative velocity. The theoretical friction force models, especially those associated with the static category, are usually accompanied by a graphical representation of the expected shape of the curve for positive and negative values of the relative velocity between the contacting bodies.

To obtain a friction force profile curve similar to the ones presented in literature, a reciprocating motion of the slider was defined with smooth accelerations and decelerations near the points of motion reversal. The force data from the load cell and the positions, velocities and accelerations determined from the tracking software were filtered by a second order low-pass filter implemented in MATLAB. Figure 8.7 shows the displacement profile of the slider during the tests.

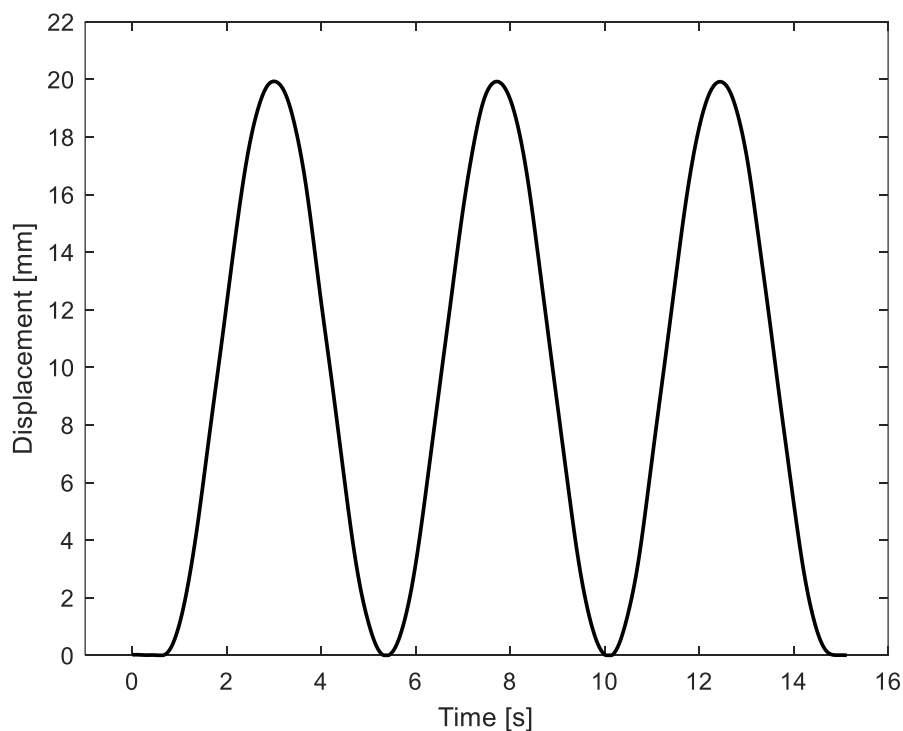


Figure 8.7 – Displacement profile of the slider for the conducted dynamic tests.

The reciprocating motion of the slider was defined by three forward and backward cycles for each trial conducted. In total, three trials for this dynamic test were executed. The contact material pair for the tests was anodized aluminum-brass, the first material being the one on the workbench pocket and the second bolted to the slider structure. The choice of the materials was made considering the available materials at the workshop during the manufacturing process of the experimental apparatus. The average results for the three trials in terms of friction force-time, friction force-displacement and friction force-relative velocity are depicted in Figure 8.8, Figure 8.9 and Figure 8.10, respectively.

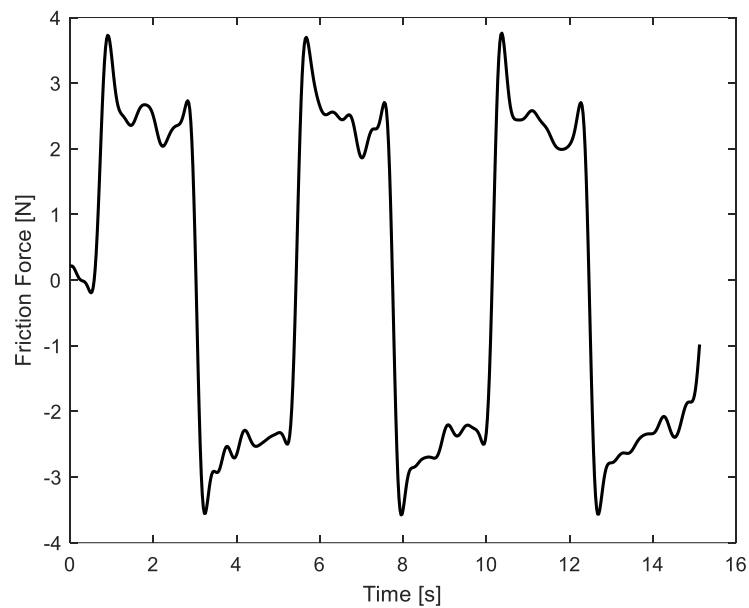


Figure 8.8 – Average friction force-time experimental data obtained from the reciprocating motion test.

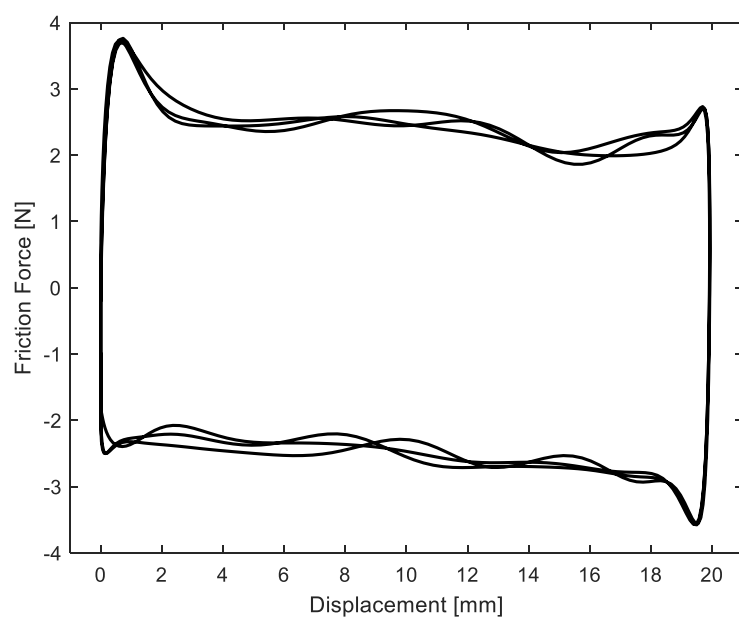


Figure 8.9 – Average friction force-displacement experimental data obtained from the reciprocating motion test.

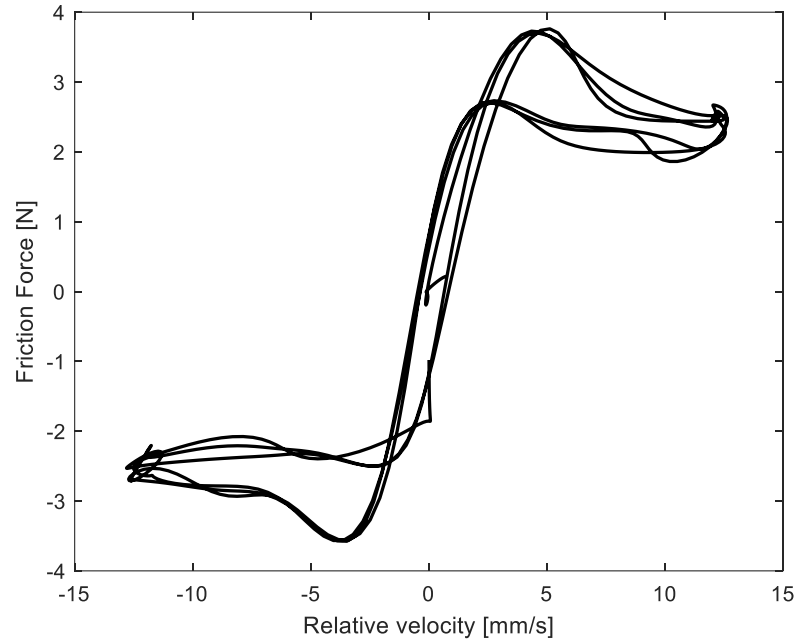


Figure 8.10 – Average friction force-relative velocity results obtained from the reciprocating motion test.

The experimental results obtained exhibit shapes very similar to other works found in literature. More specifically, in Figure 8.8 it is well represented the stiction phenomenon, where during the motion direction reversal, a peak friction force is verified. Figure 8.9 shows the frictional hysteresis loop and in Figure 8.10 is easily identifiable the non-reversibility characteristic of friction force.

After obtaining friction force data from a real-world physical experiment, it is important to assess if the theoretical friction force models described in literature, as well as the respective parameters, are capable of fully describing the dynamic behavior of the object in analysis. The following paragraphs of this section will be dedicated to the parameter identification and optimization procedure utilizing MATLAB.

A careful analysis of the results presented in the graphs of Figure 8.8 to Figure 8.10 allow the identification of several complex friction phenomena, normally associated with dynamic friction force models, which consider the existence of state variables to represent hysteretic and non-reversibility behaviors. For this reason, only dynamic friction force models were considered as candidates for the parameter identification and optimization process.

Moreover, by analyzing the mathematical formulations of the dynamic friction force models presented in section 7.1 of this work, it is possible to conclude that all the models proposed after the latter years of the 20th century, that is, all the models proposed after the LuGre model, share a very similar general formulation. Since this dynamic model is available in a wide variety of commercial multibody dynamic softwares such as Adams and RecurDyn, the parameter optimization process was especially implemented for the LuGre model.

The first step of the optimization process is related to the identification of the friction model parameters that should be fitted by the optimization algorithm to better describe the experimental data. The LuGre model is characterized by six parameters: (i) the static friction coefficient, μ_s , (ii) the kinetic friction coefficient, μ_k , (iii) the Stribeck characteristic velocity, v_s , (iv) the bristle stiffness coefficient, σ_0 , (v) the bristle damping coefficient, σ_1 , and (vi) the viscous friction coefficient, σ_2 .

The second step is related to the choice of the optimization algorithm. MATLAB possesses a wide variety of in-built algorithms within the *Optimization Toolbox*, such as *fmincon*, *fsolve*, *fminunc*, least square algorithms (*lsqlin*, *lsqcurvefit* and *lsqnonlin*), linear programming algorithms (*linprog*), mixed-integer linear programming algorithms (*intlinprog*), quadratic programming algorithms (*quadprog*), and multiobjective optimization (*fgoalattain* and *fminimax*) (MathWorks, 2024a). The previously mentioned algorithms often show issues during their implementation. Some of the disadvantages are related to a relatively high sensitivity to initial guesses, where the user must have a complete understanding of each optimization variable to provide a feasible approximation of the optimal solution, slow convergence and tendency to converge to a local, non-global optimum solution.

To overcome the potential problems that can arise from the use of the aforementioned algorithms, the optimization procedure was conducted using the *Global Optimization Toolbox* from MATLAB, which provides functions that search for global solutions to problems that contain multiple maxima or minima. The algorithms associated with this toolbox are the *surrogate*, *pattern search*, *genetic algorithm*, *particle swarm*, *simulated annealing*, *multistart*, and *global search* (MathWorks, 2024b).

For this study, the genetic algorithm was implemented. The genetic algorithm (ga) is an optimization technique inspired by the principles of natural selection and evolution. It iteratively improves a population of candidate solutions by simulating biological processes such as selection, crossover, and mutation. At each iteration, individuals from the current population are chosen as parents to generate the children for the next generation. Through these evolutionary steps, the algorithm gradually converges towards optimal or near-optimal solutions for both constrained and unconstrained optimization problems (MathWorks, 2024c).

The optimization techniques are usually governed by an objective function, which represents the goal of the optimization process. It is defined by a value or expression that needs to be minimized or maximized and allows to quantify the quality of the solution in each iteration based on the problem's requirements. For this specific study, the main goal is to minimize the error between the experimental data and the friction force value given by the LuGre model using the curve fitting optimized parameters.

The objective function of this study's optimization procedure is then to minimize the least squares differences of the between the experimental values and the LuGre's model predictions, formulated as

$$\text{objective function} = \sum_{i=1}^n \left(f_{\text{LuGre}}(\mu_s, \mu_k, v_s, \sigma_0, \sigma_1, \sigma_2) - f_{\text{exp}} \right)^2 \quad (8.2)$$

where $f_{\text{LuGre}}(\mu_s, \mu_k, v_s, \sigma_0, \sigma_1, \sigma_2)$ is the friction force predicted by the LuGre model and f_{exp} is the experimental friction force data.

The genetic algorithm does not require the definition of initial guesses for each optimization variable, but it is necessary to define the lower and upper bounds of each variable. By analyzing the graph from Figure 8.9 it is possible to obtain an estimate value for the static friction force, f_s , and for the kinetic/Coulomb friction force, f_c . The mass of the slider utilized for the dynamic tests is equal to 1.95 kg, which yields a normal force, f_n , of 19.13 N. Dividing the peak static friction force, which was determined as being equal to 3.73 N, by the normal force, f_n , results in an estimate static friction coefficient, μ_s , of 0.19. The mean value of the Coulomb friction force level was determined to be equal to approximately 2.35 N. Performing a division of this friction force magnitude by the normal force exerted in the slider yields an estimate kinetic friction coefficient, μ_k , of 0.12. The lower and upper bounds of the static and kinetic friction coefficients were defined by two distinct intervals, where the minimum value corresponds to half of the estimate value and the maximum value corresponds to double the estimate value, given as

$$\begin{aligned} 0.09 &\leq \mu_s \leq 0.38 \\ 0.06 &\leq \mu_k \leq 0.24 \end{aligned} \quad (8.3)$$

The definition of the boundary values of the Stribeck characteristic velocity, v_s , and the other specific parameters of the LuGre friction force model was made taking the commonly utilized values in literature as references. The lower and upper bound values of these parameters were defined as

$$\begin{aligned} 0.0005 &\leq v_s \leq 0.002 \\ 10^4 &\leq \sigma_0 \leq 10^6 \\ \sqrt{10^4} &\leq \sigma_1 \leq \sqrt{10^6} \\ 0.05 &\leq \sigma_2 \leq 0.2 \end{aligned} \quad (8.4)$$

where the criteria utilized for the Stribeck velocity, v_s , and the viscous friction macrodamping, σ_2 , was the same defined for the static and kinetic friction coefficients. For the bristle's stiffness, σ_0 , and damping, σ_1 , the lower bounds were considered to be an order of magnitude lower than the commonly found parameters in literature, 10^5 and $\sqrt{10^5}$, respectively, and the upper bounds to be an order of magnitude higher.

To improve the speed of convergence and the accuracy of the optimization results, the input variables were normalized, being their respective lower and upper bounds defined as

$$\begin{aligned}
0 &\leq \mu_{s_{norm}} \leq 1 \\
0 &\leq \mu_{k_{norm}} \leq 1 \\
0 &\leq v_{s_{norm}} \leq 1 \\
0 &\leq \sigma_{0_{norm}} \leq 1 \\
0 &\leq \sigma_{1_{norm}} \leq 1 \\
0 &\leq \sigma_{2_{norm}} \leq 1
\end{aligned} \tag{8.5}$$

To obtain the non-normalized or non-scaled value of each parameter, after the optimization process is over, the following expression must be implemented

$$x = LB + \frac{(x_{norm} - LB_{norm})}{UB_{norm} - LB_{norm}} \cdot (UB - LB) \tag{8.6}$$

where x is a non-normalized parameter, x_{norm} is the normalized parameter, LB and UB are the non-normalized lower and upper bounds, respectively, and LB_{norm} and UB_{norm} are the normalized lower and upper bounds, respectively.

Regarding the genetic algorithm options, there were some modifications performed for this study relatively to the default options. The population size, which for MATLAB ga algorithm is set to 50 by default for problems that have less than five optimization variables and 200 for those that have more than five, was set to 600 for this case study. This means that the problem has an initial population of 600 candidate solutions, known as individuals, and that each one of these 600 individuals contains a random value, defined between the lower and upper bounds, for each one of the 6 optimization variables considered for this problem. The maximum generations for the problem was set to 1000 to ensure that the stopping criteria of the optimization process is governed by the function tolerance, which was set as 10^{-10} .

In order to guarantee that the optimal fitted solution for the static friction coefficient is always higher than the kinetic friction coefficient, a linear inequality constraint was defined being formulated as

$$\mu_k - \mu_s < -0.05 \quad (8.7)$$

which states that the minimum difference between μ_k and μ_s should never be less than -0.05.

For the experimental data of the three dynamic tests conducted, three genetic algorithm optimization processes were performed. Table 8.1 presents the results for the best individual from each of the three optimization test runs, along with the average values of all optimization variables.

Table 8.1 – Optimized parameters fitted to the LuGre model by the MATLAB genetic algorithm.

Run	Parameters						Elapsed
	μ_k [–]	μ_s [–]	v_s [m/s]	σ_0 [N/m]	σ_1 [N·s/m]	σ_2 [N·s/m]	Time [s]
1	0.12747	0.19030	0.00050	95831.8559	997.848	0.1932	1029.97
2	0.12652	0.19100	0.00050	95843.3883	1000.000	0.2000	1234.96
3	0.12679	0.19070	0.00052	96332.4741	996.484	0.1880	1102.21
Average	0.12693	0.19067	0.00051	96002.5728	998.111	0.1937	1122.38

The optimization results across the different optimization runs show very similar values for each parameter, which means that despite the randomness and stochastic nature of the genetic algorithm optimization procedure, the boundary conditions of the problem guarantee a certain repeatability of solutions for each variable. The static and kinetic friction coefficients have results that near-perfectly match the predictions made earlier to define the lower and upper bounds of these variables for the optimization procedure. Some of the remaining specific parameters are near the usually utilized parameters in literature for the LuGre friction force model in multibody dynamics problems, while others have considerable different values. Table 8.2 presents an analysis of the percent differences between the literature parameters for the LuGre model and the ones fitted within the context of this work.

Table 8.2 – Comparison of the percent difference between literature values and optimization fitted values for the LuGre friction force model.

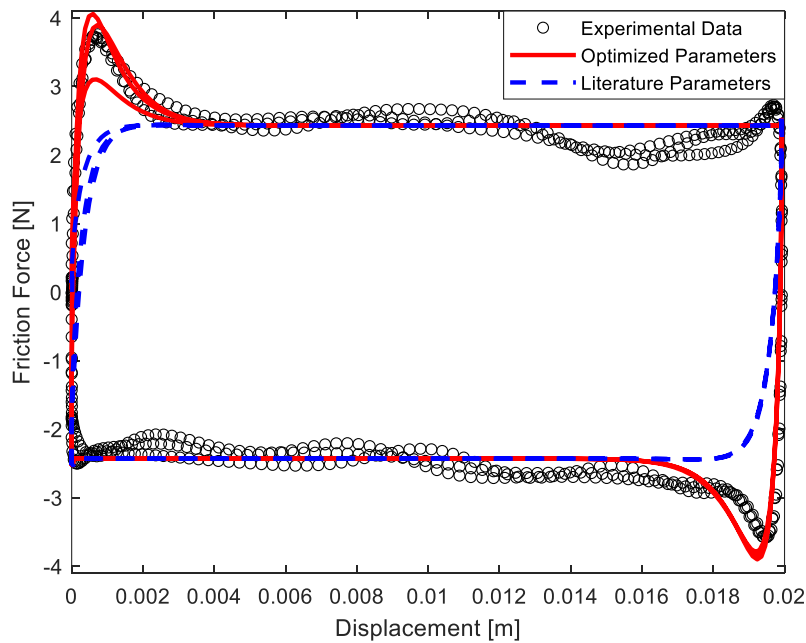
Parameter	Literature Value	Optimization Fitted Value	Percent Difference [%]
v_s [m/s]	0.001	0.0005	67
σ_0 [N/m]	100000	96002.5728	4
σ_1 [N·s/m]	316.2278	998.111	104
σ_2 [N·s/m]	0.1	0.1937	64

The percent differences between the literature values and the optimization fitted values have been calculated using Equation (8.8)

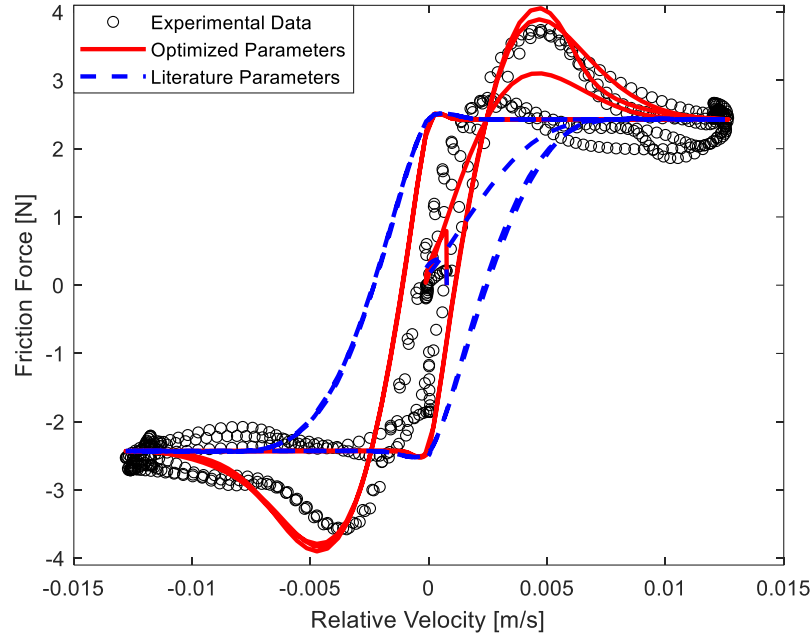
$$pd[\%] = \frac{|x_{lit} - x_{opt}|}{\left[\frac{(x_{lit} + x_{opt})}{2} \right]} \cdot 100 \quad (8.8)$$

where x_{lit} is the friction force model parameter numerical value usually utilized in literature and x_{opt} is the friction force model specific parameter numerical value obtained from the optimization procedure described in this work.

The analysis of the percent differences between the literature and the optimized parameters show that there are significant differences between the values obtained for the Stribeck velocity, v_s , bristle microdamping, σ_1 , and viscous friction macrodamping, σ_2 , being the biggest difference verified for the bristle's microdamping. In what concerns the bristle's stiffness, σ_0 , a percent difference of just 4% was verified. After analyzing the numerical differences between the parameters from the optimization procedure and those normally utilized in numerical applications, it is important to verify the influence of these parameters in the LuGre friction force model ability to describe the experimental data acquired from the workbench. Figure 8.11 a) shows the friction force-displacement behavior of the LuGre model utilizing the optimized parameters and the literature parameters and their curve fitting ability against experimental data, while Figure 8.11 b) shows the same friction force data plotted as a function of relative velocity.



a)



b)

Figure 8.11 – Comparison of the literature and optimized parameters of the LuGre friction force model relatively to the experimental data as a function of a) displacement b) relative velocity.

The graph of Figure 8.11 a) shows that both the optimized parameters and the literature ones can model the hysteresis loop associated with the friction force-displacement relationship. In terms of capturing the peak friction forces associated with the static state during direction reversals of the slider's motion, it is possible to notice that the literature parameters do not correctly fit the experimental data concerning the stiction phenomena.

The analysis of Figure 8.11 b) graph shows an enhanced view of the differences between the use of the optimized and the literature parameters for the LuGre friction force model. The literature parameters yield a central hysteresis loop with a width that is considerably higher than the one modeled by the optimized parameters. The optimized model, in turn, has a width that fits more accurately with the experimental data. The peak friction forces verified during motion reversals are only correctly modeled by the LuGre model utilizing the optimized parameters, although the peak friction force is slightly higher than the one verified for the experimental data.

Another friction phenomenon that is visually identifiable in Figure 8.11 b) is the non-reversibility. The experimental data shows quite clearly a difference between the acceleration and deceleration phases of motion for both directions of the slider. Although the LuGre model with optimized parameters can model this phenomenon for both acceleration and deceleration phases in both directions of the slider's motion, the literature parameters cannot properly model this phenomenon in this case, since there is not a differentiation in terms of an acceleration and a deceleration curve.

8.4 Summary and Discussion

This chapter was dedicated to the description of a simple and versatile experimental apparatus proposed to acquire friction force data. It is of capital importance to develop such experiments to accurately predict and fit numerical values to the friction force models available in literature, in order to obtain better numerical results for the dynamic simulations of complex multibody models and control applications that require the consideration of these dissipative effects in their behavior.

The tailored-made parts designed for the experimental apparatus were entirely manufactured by the author of this work in-house at the University of Minho workshops of the Mechanical Engineering Department, using a conventional milling machine. All the components were mainly obtained from used parts. The lack of experience of the machinist combined with excessive clearances in the table screw of the milling machine and parts that showed some warp, culminated in an assembly of the workbench that can be structurally improved to achieve more accurate results of the forces measured by the load cell, using parts that have more precise geometric and dimensional tolerances.

For the determination of the kinematic properties of the slider, namely the displacement, velocity and acceleration, they were not determined by real-time measurements. Instead, a tracking approach was considered, by using a camera setup and later obtain the displacement and its derivatives (velocities and accelerations) using the Tracker software. The relatively low frequency of 40 Hz for the data acquisition during the tests was mainly restricted by the stable frame rate capacity of the utilized camera. By using a higher frequency of data acquisition for both the load cell and image acquisition to track the slider, it is expected that the friction force results obtained are even more precise than the ones obtained.

The tracking of the slider to obtain data of its position, velocity and acceleration could be replaced by other contactless measurement technologies. For example, to determine the velocity of the slider, a laser Doppler velocimeter could be used and accelerometers are a proposed solution to measure the slider's acceleration. In the measurement of friction force data, it is of capital importance to choose contactless technologies to determine the kinematic properties of the moving object, in order to minimize the interference of the measurement technology in the dynamic behavior of the analyzed system. Moreover, the use of different transducers to measure different kinematic properties is recommended, to minimize the errors that arise from indirect measurements, that is, integrating velocity data to obtain displacement data or perform derivation of velocity data to obtain accelerations, for example.

Although some inaccuracies can be associated with the obtained experimental data, the friction force results exhibited the expected behavior when plotted as functions of time, displacement and relative velocity and considering the friction phenomena described in literature. It was possible to verify the stiction

behavior when the slider changed the direction of motion, in the vicinity of the null value of relative velocity, the friction hysteresis loop, as well as the non-reversibility of friction during the acceleration and deceleration phases of motion.

The optimization procedure could be performed with a wide variety of algorithms in-built in the *Optimization Toolbox* and *Global Optimization Toolbox* of MATLAB. The genetic algorithm was chosen due to the higher accuracy and efficiency in determining a global optimum solution. The objective function was defined considering the least square difference between the numerical LuGre friction force using optimized parameters and the experimental data obtained from the workbench. Only one dynamic friction force model was considered for the optimization process, since the most recent models have a very similar fundamental mathematical formulation and the static models do not have the ability of capturing complex friction phenomena, which would result in a non-acceptable fitting of these models against experimental data.

The lower and upper bounds of each parameter of the LuGre model were defined by analyzing the experimental data and also by taking the literature parameters as references. To obtain better results and improve the efficiency of the optimization process, the parameters that entered the optimization process as inputs were normalized. An initial population of 600 candidate individuals was considered for the genetic algorithm process. Additionally, a linear inequality constraint was defined to ensure that all the sets of parameters considered for the optimization process had a static friction coefficient higher than the kinetic friction coefficient. The results from three different optimization runs yielded very similar results, which show that the boundary conditions set to the optimization process can provide similar results, despite the randomness and stochastic nature of the genetic algorithm.

The final task involved comparing the LuGre model using optimized parameters to the same model employing parameters from literature, with respect to experimental data. The analysis revealed that the literature-based parameters were unable to capture several key friction phenomena, such as stiction, hysteresis, and non-reversibility. This inadequacy underscores the limitations of general-use parameters in accurately representing the friction dynamics of specific systems.

To better predict real-world behavior of dynamic systems, it is crucial to determine parameters that are precisely tailored to the physical properties of the system under analysis. Such system-specific parameters significantly enhance the accuracy of multibody dynamics simulations and improve the effectiveness of control applications that require accurate modeling of dissipative effects. This approach ensures that the simulated dynamic behavior aligns closely with the physical system performance, addressing the complexities inherent to the real-world dynamical behavior of mechanical systems.

9. MULTIBODY EXAMPLE OF APPLICATION

Friction is a highly nonlinear and complex phenomenon that appears as a tangential resistive force acting between contacting surfaces, which have relative motion. By and large, friction has a nefarious effect on the dynamic performance of a large variety of mechanical systems, namely in terms of energy dissipation, noise produced, reliability, just to mention a few. Therefore, it is important to develop approaches that allow a detailed identification and prediction of friction influence on the dynamic behavior of multibody systems.

A vehicle dynamics case study within the framework of multibody dynamics systems is presented in this section. The scope of this work is to study the influence of friction force on the performance of a vehicle's suspension system. For this purpose, a rigid multibody model of the University of Minho Formula Student (FSUM) car is utilized as an example of application.

The requirements of the suspension systems are greatly dependent on the practical application, where two main categories can be distinguished, namely passenger vehicles and competition vehicles. For passenger vehicles, the design process focuses on gathering simulation data to create control algorithms for active suspension systems (Herzog & Augsburg, 2021) and to define NVH (Noise, Vibration and Harshness) criteria. Passenger comfort metrics are also defined, which will serve as target values to reduce the transmissibility of unwanted vibrations from the suspension and chassis systems to the vehicle's body (Herzog & Augsburg, 2021). In turn, as far as competition vehicles is concerned, the prerequisites are quite different, since the main goal is to extract the maximum handling performance from the car, for which a deep knowledge of the suspension system dynamics is crucial to correctly define the setup of the shock absorbers in the presence of friction phenomena (Benini et al., 2017).

The main focus of this study is to develop a complete multibody model of the Formula Student of University of Minho car, which allow to examine the dynamic response of the suspension subsystem taking into account the friction phenomena. The car is subjected to a series of dynamic tests that include a fast obstacle avoidance maneuver, that was defined in accordance with the ISO 3888-2 standard and a phase delayed excitation of the front and rear suspension subsystem performed in a Four-Poster Test Rig. These tests were chosen in order to study the compression and rebound motions of the suspension subsystem in situations similar to those found in real world scenarios.

9.1 Vehicle Multibody Model

The vehicle chosen as the complex multibody model is the single-seater prototype of FSUM team from the year 2023. Figure 9.1 shows the complete CAD model of the analyzed car and the corresponding global coordinate system xyz .



Figure 9.1 – Complete CAD model of the FSUM prototype car of 2023.

The vehicle model was divided into four main subsystems: (i) the chassis; (ii) the front wheels suspension; (iii) the rear wheels suspension; (iv) the steering subsystem. Figure 9.2 a) exhibits the chassis of the car that comprises the chassis itself, the driver and all the powertrain and electronic components and Figure 9.2 b) shows the steering subsystem where only the steering box (2) and the steering tie-rods (3) have been modeled.

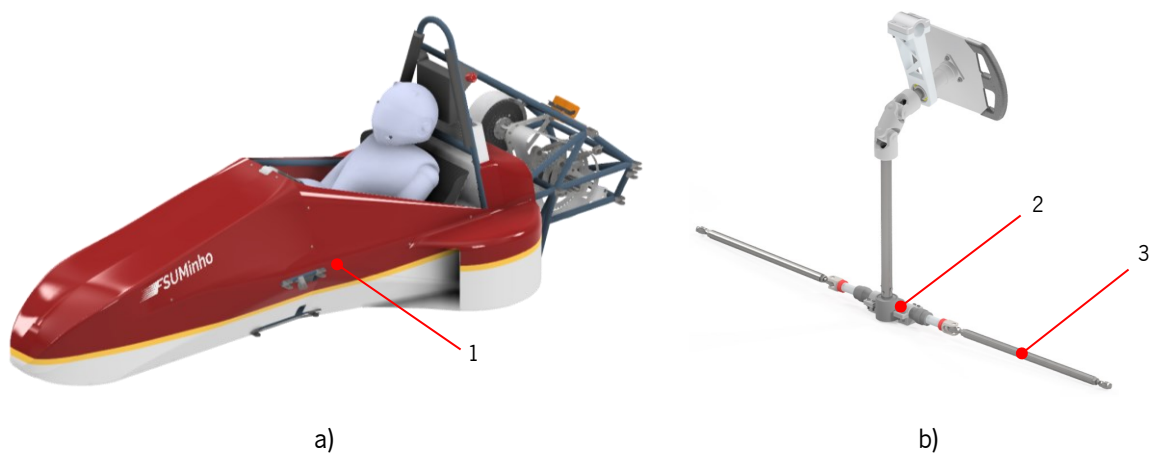


Figure 9.2 – Subsystems of the FSUM car multibody model a) chassis b) steering system.

The suspension subsystem for both front and rear is of double wishbone (double A-arms) type, being the most common type of suspension utilized in race cars, since in these vehicles the cornering

performance is of capital importance, while riding comfort is a secondary concern. Figure 9.3 a) shows the front left suspension subsystem and Figure 9.3 b) the rear left suspension subsystem of the car. All the components of the car model are numbered and their description and properties are presented in Appendix II and Appendix III.

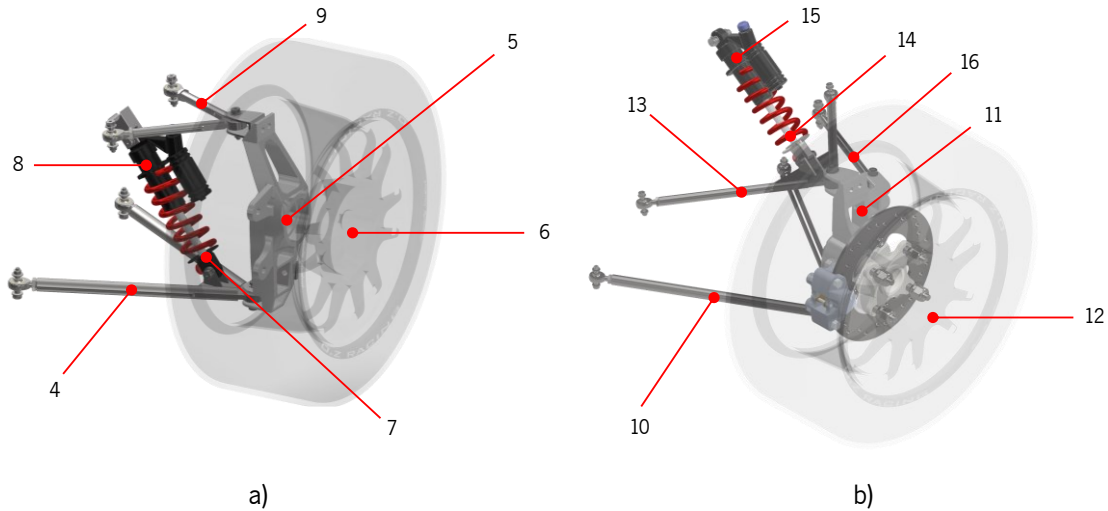


Figure 9.3 – Suspension subsystems of the FSUM car multibody model a) front left b) rear left.

The complete model of the car is constituted by 30 rigid bodies and 41 mechanical joints have been defined between spherical, relative translational, revolute, universal and cylindrical joints, which results in a vehicle model with 18 DoF, from which 14 are active degrees of freedom. In the following sections a more detailed analysis of the multibody model components inertial properties and positions, kinematic joints, DoF definition, and tire and suspension parameters will be conducted.

9.1.1 MBD Model – Center of Gravity Position, Inertial Properties and Euler Parameters

The definition of a multibody model starts with the specification of the mass of each component, the inertia matrix in local coordinates, $\xi\eta\zeta$, the center of gravity position in the global reference frame, xyz , and the Euler parameters. The inertial properties of all the vehicle components are presented in Appendix II and were obtained by defining the material properties in the CAD system for the designed components and by consulting the manufacturer's datasheet for the off-the-shelf standard components.

Considering that each component has its inertial properties defined in the CAD system, it is intuitive to determine the center of gravity coordinates on the global reference frame, since most CAD systems perform the CG determination automatically through an implemented command. In contrast, the determination of the Euler parameters is not straightforward, because the local reference frame

orientation of each individual component, most of the times, is not coincidental with the global reference frame.

An unconstrained body in space requires six independent coordinates to fully define its configuration: three coordinates for its translational position and three for its rotational orientation. The rotational orientation of a rigid body is often described using three successive rotations, such as Euler angles or Bryant angles. However, this approach can introduce singularities in certain configurations, leading to computational difficulties (Nikravesh, 1988). To circumvent these issues, the angular orientation of the rigid body must be described using Euler parameters, which provide a more robust representation of the spatial orientation of a rigid body.

The vector of Euler parameters for a body i in global coordinates is given as

$$\mathbf{p}_i = \{e_0 \quad e_1 \quad e_2 \quad e_3\}^T = \left\{ \cos \frac{\phi}{2} \quad \mathbf{u}_e \quad \cos \frac{\phi}{2} \right\}^T \quad (9.1)$$

where ϕ is the angle that the body must be rotated about the unique axis represented by the unit vector $\mathbf{u}_e$ to make its reference parallel to the global reference frame. At any instant in time, the orientation of a body can be specified by a transformation matrix, where the elements may be expressed in terms of suitable sets of coordinates, such as Euler angles, Bryant or Euler parameters (Nikravesh 1988). A transformation matrix $\mathbf{A}$ has the following general form

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad (9.2)$$

that can be defined in terms of Euler parameters as

$$\mathbf{A} = 2 \begin{bmatrix} e_0^2 + e_1^2 - \frac{1}{2} & e_1 e_2 - e_0 e_3 & e_1 e_3 + e_0 e_2 \\ e_1 e_2 + e_0 e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2 e_3 - e_0 e_1 \\ e_1 e_3 - e_0 e_2 & e_2 e_3 + e_0 e_1 & e_0^2 + e_3^2 - \frac{1}{2} \end{bmatrix} \quad (9.3)$$

Although the full deduction of the transformation matrix in terms of Euler angles is not presented in this work, the interested reader can consult the Chapter 6 of Nikravesh textbook to understand the algebraic transformation from the direction cosines of the orientational axis of rotation and the angles of rotation to Euler parameters (Nikravesh 1988).

The angular orientation of a given body fixed coordinate system can be envisioned to be the result of three successive rotations (Flores, 2015). For the Euler angles, twelve possible conventions can be considered to define three consecutive rotations about three different axes, being the zxz convention the most utilized.

The Bryant angle convention considers three consecutive rotations about axes that are different from the ones used for the Euler angles. The first rotation must be executed counterclockwise about the x axis through an angle ϕ_1 . The resultant coordinate system will be labeled $\xi''\eta''\zeta''$ as shown in Figure 9.4 b). The second rotation, produced by a counterclockwise rotation about the η'' with an amplitude defined by ϕ_2 yields the $\xi'\eta'\zeta'$ coordinate system as denoted in Figure 9.4 c). The third and last rotation may be carried out about the ζ' axis through an angle ϕ_3 , resulting in the $\xi\eta\zeta$ coordinate system as observable in Figure 9.4 d) (Nikraves 1988).

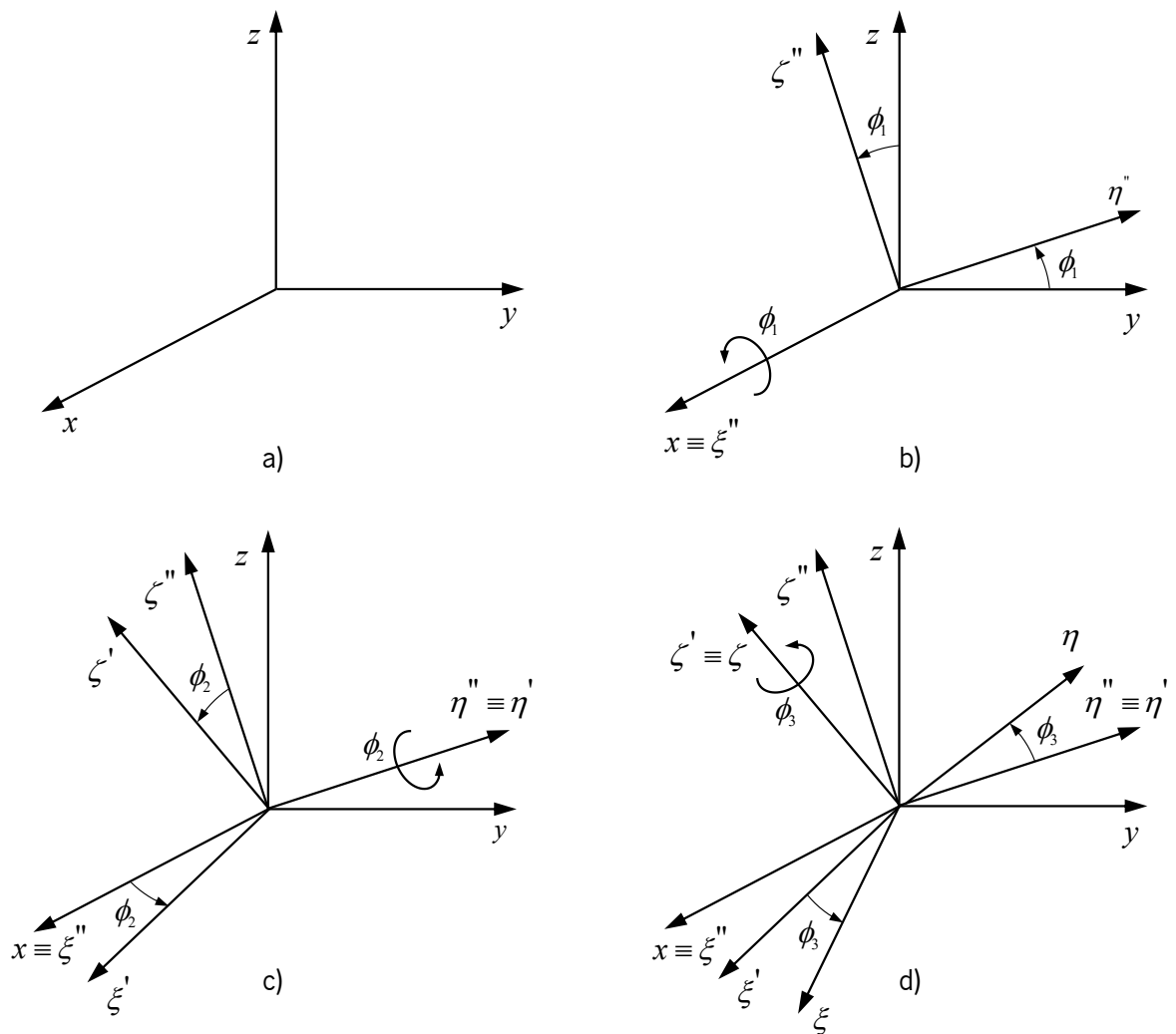


Figure 9.4 – Different stages of rotation for the Bryant angles convention. a) Reference global system; b) First rotation c) Second rotation d) Third rotation.

The transformation matrices for the individual rotations are as follows

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi_1 & -s\phi_1 \\ 0 & s\phi_1 & c\phi_1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} c\phi_2 & 0 & s\phi_2 \\ 0 & 1 & 0 \\ -s\phi_2 & 0 & c\phi_2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} c\phi_3 & -s\phi_3 & 0 \\ s\phi_3 & c\phi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (9.4)$$

where $c \equiv \cos$ and $s \equiv \sin$. The complete matrix of transformation, $\mathbf{A} = \mathbf{DCB}$, is given by

$$\mathbf{A} = \begin{bmatrix} c\phi_2 c\phi_3 & -c\phi_2 s\phi_3 & s\phi_2 \\ c\phi_1 s\phi_3 + s\phi_1 s\phi_2 c\phi_3 & c\phi_1 c\phi_3 - s\phi_1 s\phi_2 s\phi_3 & -s\phi_1 c\phi_2 \\ s\phi_1 s\phi_3 - c\phi_1 s\phi_2 c\phi_3 & s\phi_1 c\phi_3 + c\phi_1 s\phi_2 s\phi_3 & c\phi_1 c\phi_2 \end{bmatrix} \quad (9.5)$$

The majority of CAD systems is able to give the rotational orientation of a part in the global reference frame correspondent to an assembly in terms of x , y and z axes, relatively to the rotational orientation of the part in local rotational coordinates. In this study, the Bryant angles convention was considered over the Euler angles since it is easier to track the rotations of the local reference frame of a component in a xyz format to obtain the rotational orientation of the component in global coordinates.

Considering that the trace of the transformation matrix $\mathbf{A}$ is denoted as

$$\text{tr } \mathbf{A} = a_{11} + a_{22} + a_{33} \quad (9.6)$$

and associating the corresponding terms of the transformation matrices of Equation (9.3) and Equation (9.5) it is possible to determine the Euler parameters from a transformation matrix defined in terms of Bryant angles using the following expressions

$$\begin{aligned} e_0 &= \sqrt{\frac{\text{tr } \mathbf{A} + 1}{4}} \\ e_1 &= \sqrt{\frac{1 + 2a_{11} - \text{tr } \mathbf{A}}{4}} \\ e_2 &= \sqrt{\frac{1 + 2a_{22} - \text{tr } \mathbf{A}}{4}} \\ e_3 &= \sqrt{\frac{1 + 2a_{33} - \text{tr } \mathbf{A}}{4}} \end{aligned} \quad (9.7)$$

Figure 9.5 exhibits a flowchart that describes the sequential procedure to obtain the Euler parameters that define the rotational orientation of a body in a global reference frame knowing the three consecutive rotations of the local reference frame axes around the x , y and z global axes.

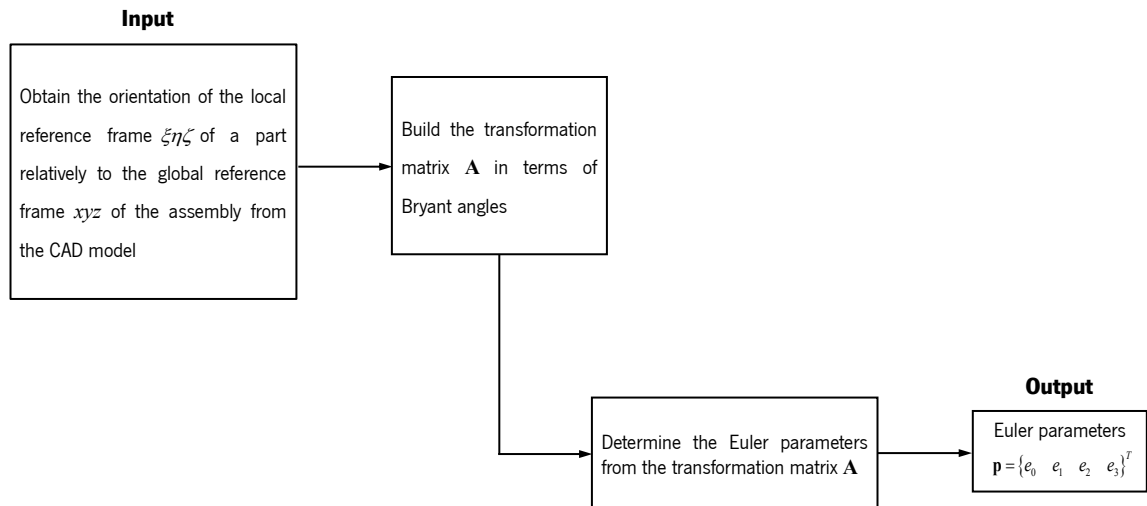


Figure 9.5 – Flowchart of the determination of Euler parameters from Bryant angles convention.

To accurately determine the Euler parameters of each component of the FSUM car multibody model, the angular orientation of each one of the local reference frame axes ξ , η and ζ relatively to the correspondent x , y and z axes of the global reference frame of the car's assembly was determined. This step was executed automatically through the embedded functionalities of the utilized CAD system and the Euler parameters were directly obtained from the transformation matrix $\mathbf{A}$ defined in terms of Bryant angles. The position of the center of gravity of the individual parts of the multibody car model were also obtained directly from the CAD model after defining the corresponding inertial properties. Appendix III presents a table containing the CG and the Euler parameters information about the FSUM car multibody model left-hand side components, knowing that the right-hand side of the vehicle was defined considering the symmetry of the model.

9.1.2 Kinematic Joints and Degrees of Freedom Definition

A multibody system is a group of mechanical components that describe large translational and rotational displacements, in which the relative motion of the bodies is constrained by kinematic joints and force elements (Nikravesh 1988). After defining the inertial properties, the position of the components' CG and the Euler parameters, it is mandatory to constrain the relative movement between two connecting bodies. Throughout this section, a detailed explanation of the chosen kinematic joints for each subsystem of the car will be carried out as well as the resultant DoF of the multibody model.

The first subsystem to be analyzed is the front left suspension which is composed of six bodies and ten joints: six spherical joints, three revolute joints and one cylindrical joint. Figure 9.6 shows the real-world joints used to connect the different bodies of the left front suspension subsystem.

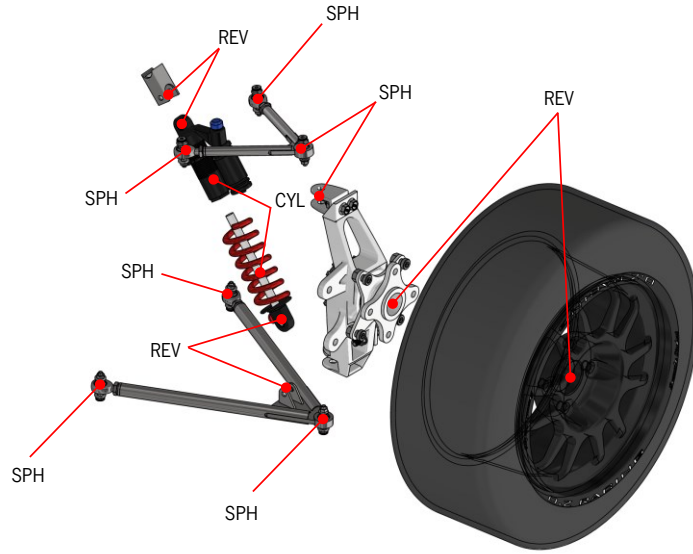


Figure 9.6 – Real-world kinematic joints used to connect the different bodies of the front left suspension subsystem.

The left front suspension subsystem must have three independent DoF, being one associated with the wheel rotation, another associated with the vertical motion of the suspension when subjected to compression and rebound, while the third one corresponds to the steering of the wheels. Utilizing the Chebychev-Grübler-Kutzbach criterion given by the following equation (Flores & Claro, 2007)

$$\text{DoF} = 6 \cdot (n - 1) - 5j_5 - 4j_4 - 3j_3 - 2j_2 - j_1 \quad (9.8)$$

where n is the number of bodies of the system considering the ground and j_k is the number of joints that constrain k degrees of freedom. Applying the formula of Equation (9.8) to the subsystem represented in Figure 9.6 yields a total of -1 DoF, which means that the considered subsystem is not a mechanism, but a statically indetermined structure. This is a classic problem within the framework of multibody dynamics models where redundant constraints have been introduced to the model making it over constrained.

Although modern versions of multibody dynamics commercial programs are capable of identify and removing redundant constraints, these algorithms can be viewed as a black box that does not help the user to comprehend the deficiencies of the created model in the light of multibody dynamics theory (Blundell & Harty, 2015). For this specific case study, the problems are associated with the choice of joints of the upper and lower A-arms and with the shock absorber.

There are two viable approaches to modeling the two spherical wishbone mounting connections to the car's chassis. The first is to consider two bushing force elements instead of the two spherical joints that appear in the real-world model. This alternative transforms the kinematic constraints of the spherical joints into dynamic constraints associated with reaction forces. Although this is the most realistic approach, it is necessary to have knowledge of the properties of the materials that compose the bushing

to define a constitutive model that establishes a relation between the reaction force and the overlap behavior of the contacting parts. The contact stiffness constitutive model could be obtained through a finite element analysis, for example.

Since this procedure is computationally expensive, these two joints were replaced by a kinematically equivalent revolute joint where its main axis is defined along the center points of both spherical joints as represented in Figure 9.7.

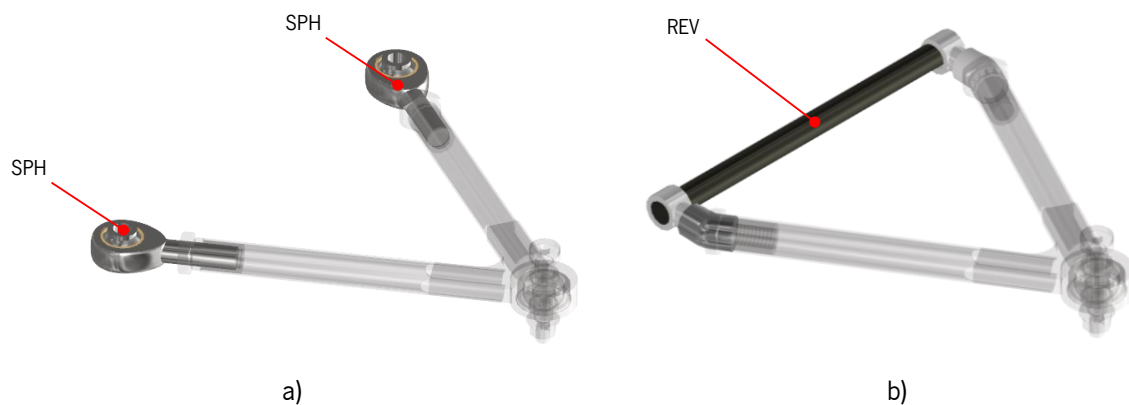


Figure 9.7 – Representation of the multibody dynamics computational approach to solve the over constrained problem in a double wishbone A-arm type suspension a) Spherical joints in both A-arm mounting points to the chassis that generate an over constrained model b) Revolute joint that solves the over constrained problem while being kinematically equivalent.

This approach is rather simple to implement but has the disadvantage of not being suitable to predict the mounting reaction forces (Blundell & Harty, 2015). Since the multibody model of this case study is rigid, the compliance of the components is not considered for the dynamic behavior of the car, so the use of a revolute joint instead of the bushing elements will not drastically simplify the dynamics of the vehicle model. By replacing the spherical joints mounts of both upper and lower A-arms with revolute joints and applying Equation (9.8), a model with 1 DoF is obtained. There are still two additional DoF that the model should have and the problem with redundant constraints is located within the shock absorber joints.

Analyzing the shock absorber joints presented in Figure 9.6, it is possible to verify that there is a cylindrical joint that connects both the lower end and the upper end of the shock absorber, a revolute joint that connects the upper end of the shock absorber to the chassis and a revolute joint that connects the lower end of the SA to the lower A-arm. Having two bodies connected by a cylindrical joint and each end constrained by a revolute joint, a total mobility of -2 DoF is obtained, which means that the shock absorber is a statically indetermined structure.

The strategy employed to overcome the over constrained modeling of the shock absorber was to replace the two revolute joints of both SA ends by universal joints as depicted in

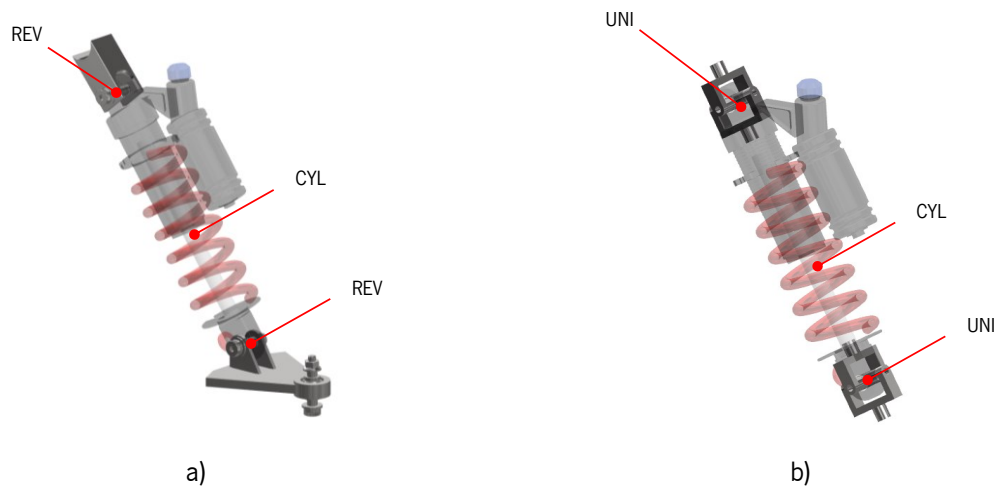


Figure 9.8 – Replacement of real-world joints with kinematically equivalent joints in the shock absorber a) Real world revolute joints on both ends of the SA b) Universal joints replacement on both ends of the SA.

Applying Equation (9.8) to the new shock absorber modeling gives a total of 0 DoF. The final modeling of the left front suspension subsystem contains six bodies and eight joints, being modeled considering three revolute joints, one cylindrical joint, two spherical joints and two universal joints. The application of Equation (9.8) to the alternative model of the left front suspension gives a total mobility of 3 DoF, being one associated with the rotation of the wheel, one is responsible for the vertical motion of all the suspension subsystem and the last degree of freedom corresponds to the rotation of the wheel around its own vertical axis, which will be controlled by the steering subsystem.

The rear left suspension subsystem is composed of seven bodies and twelve joints: three revolute joints, one cylindrical joint, and eight spherical joints. Figure 9.9 shows the real-world joints considered to connect the different bodies of the rear left suspension system.

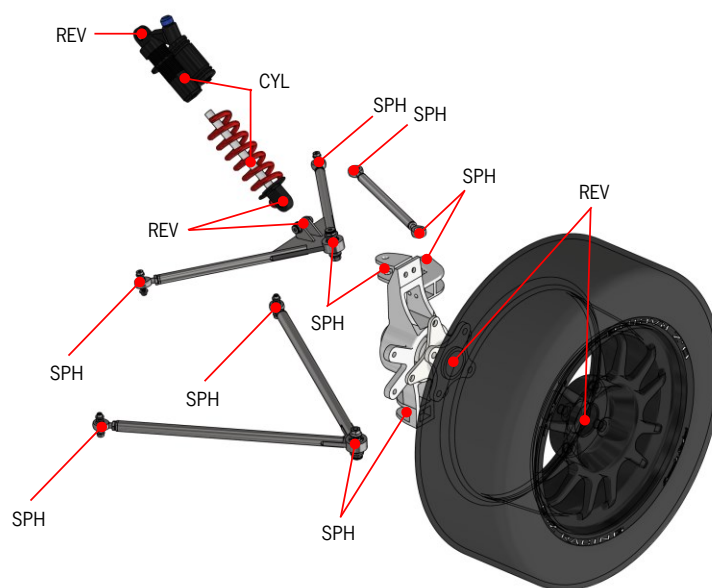


Figure 9.9 – Real-world kinematic joints used to connect the different bodies of the left rear suspension.

Applying the Chebychev-Grübler-Kutzbach criterion given by Equation (9.8) to the left rear suspension subsystem results in a total of -1 DoF which means that this subsystem is also a statically indetermined structure if the real-world joints are not replaced by kinematically equivalent ones.

The spherical joints associated with the mounting points of the upper and lower A-arms to the chassis were replaced by revolute joints and the revolute joints related to both ends of the shock absorber were replaced by universal joints. These joints replacements are equal to what has been done for the front suspension subsystem. The new multibody model for the left rear suspension system has seven bodies and ten joints where three joints are revolute joints, one joint is a cylindrical joint, four joints are from the spherical type and two universal joints. The total mobility of the new multibody model of the rear left suspension subsystem in terms of number of degrees of freedom is equal to 3. One degree of freedom is associated with the wheel rotation, one is responsible for the vertical motion of the whole suspension subsystem and the third degree of freedom is a passive DoF attributed to the rotation of the toe tie-rod around its own longitudinal axis. This third degree of freedom does not affect the dynamics of the suspension subsystem and could be easily removed by simply replacing one of the spherical joints on the ends by one universal joint.

The steering system is composed by three bodies which are the steering box, the left steering tie-rod and the right steering tie-rod. The left steering tie-rod is connected to the left front wheel knuckle by a spherical joint and connected to the steering box by also a spherical joint. The same joints are used to connect the right steering tie-rod to the front right wheel knuckle and to the steering box. A relative-translation joint has been defined to connect the steering box to the chassis, allowing the control of the steering system to steer both front wheels. Figure 9.10 shows the steering system, and the joints used to model this subsystem.

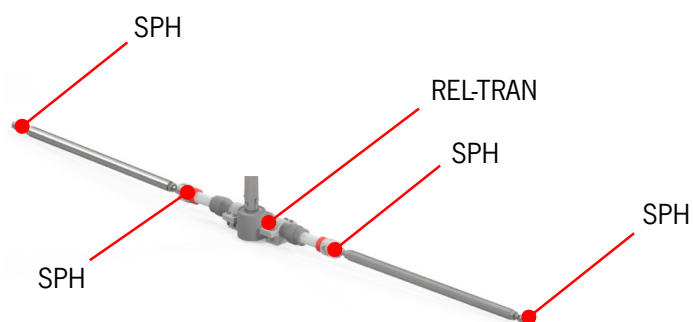


Figure 9.10 – Real-world kinematic joints connecting the different bodies that compose the steering subsystem.

The total number of degrees of freedom of the steering system was calculated using Equation (9.8) and resulted in 1 DoF which corresponds to the actuation of the steering system associated with the translation of the steering box relatively to the car's chassis. It must be noted that this degree of

freedom will be controlled by defining the steering wheel angle input of the car and the steering degree of freedom of which one of the front wheels is controlled by this subsystem.

The table presented in Appendix III summarizes the kinematic joints defined to constrain the left-hand side of the multibody model of the FSUM car, being the right-hand side modeled taking into account the symmetry of the model. All the points and vectors were defined from the CAD models of the car's components.

The chassis of the car is connected to the different suspension components, but it is considered to be a free spatial rigid body, thus having a total of six degrees of freedom – three translations and three rotations. It has translational longitudinal movement along the x axis, lateral translational movement in the y axis direction and translational vertical motion in the z axis direction. The rotations are defined about the x axis, which is called roll, around the y axis known as pitch, and about the z axis commonly referred as yaw. Figure 9.11 shows the six degrees of freedom associated with the car's chassis.

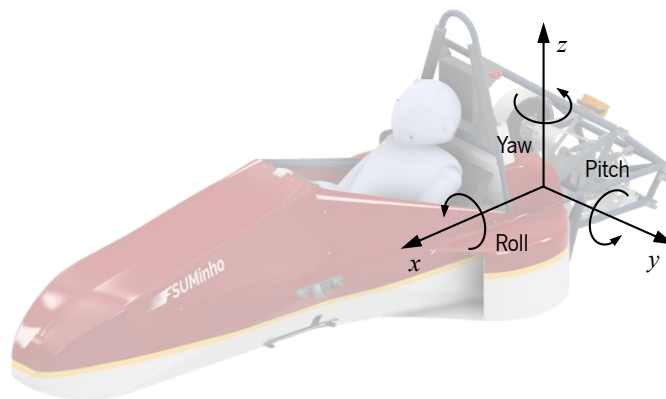


Figure 9.11 – Degrees of freedom associated to the car's chassis – 3 translations and 3 rotations.

Table 9.1 exhibits the calculation of the total number of degrees of freedom of the car.

Table 9.1 – Calculation of the total DoF of the FSUM car multibody model.

Joint Type	Number of Constrained DoF	Number of Joints	Total
Cylindrical	4	4	162
Revolute	5	12	
Spherical	3	16	
Relative-Translation (controlled by input)	6	1	
Universal	4	8	

Considering that the multibody model of the vehicle without kinematic joints has a total of 180 degrees of freedom, since it is constituted by 30 different bodies, by subtracting the 162 constrained

DoF, the FSUM car possesses a total of 18 degrees of freedom, from which 4 are passive and the other 14 are active. The 4 passive DoF are associated with the rotation of the steering tie-rods and the rear toe tie-rods about their own longitudinal axis.

9.1.3 Tire Model – Magic Formula

The tire represents one of the most critical and complex components in the analysis of vehicle dynamics and road loads, as it constitutes the sole point of contact between the vehicle and the road surface. The symbiosis between vehicle dynamics and tire models has been the subject of extensive research and development in the past decades where enormous progress has been made in the drivability of recent road and competition vehicles.

A pneumatic tire has three fundamental functions: (i) supporting the vehicle weight on various terrains and other vertical forces resultant from aerodynamic loads or road banking, (ii) generate proper longitudinal, lateral and vertical forces during vehicle cornering and/or cornering/braking for maneuvering, and (iii) absorbing the shock and vibration caused by road irregularities (Milliken & Milliken, 1995). The tire design process must result in a final product that is able to fulfill the previously mentioned functions, so it is of capital importance to understand the tire dynamic behavior in numerical and experimental terms (Li et al., 2014). The experimental approach is made by testing the tires in a specifically designed testing machine that is able to cover a variety of running conditions by varying the applied torque to the wheel, the wheel camber angle, the slip angle, tire pressure, vertical load, temperature, etc. Figure 9.12 shows a tire test machine from Calspan Research Facility (Calspan, 2019).

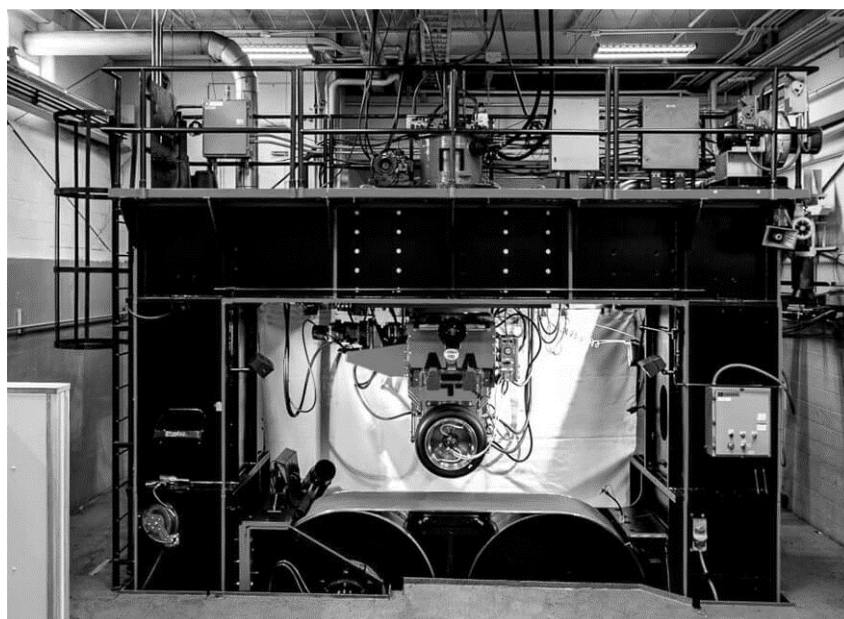


Figure 9.12 – Tire testing machine (Adapted from (Calspan, 2019)).

Despite this method being the most direct and reliable way of understanding the dynamic behavior of a tire, it is very expensive, time consuming and although it allows to test a variety of conditions, it is somehow limited to the number of variables that it is possible to test simultaneously in a single run. This is where the numerical methods play a crucial role in modeling and simulating the tire behavior in a virtual environment where multiple geometric and environmental variables can be studied simultaneously in a less time-consuming process.

The predictability and accuracy of a tire model is important. Ideally a single tire model should be capable of predicting accurately the tire-ground interactions for all operating conditions, but unfortunately this approach is not viable in engineering practices. Throughout the years, many tire models have been introduced in literature considering different applications such as handling and stability analysis, ride and comfort analysis, as well as road load analysis, among others (Li et al., 2014).

The main focus of this case study is to determine the dynamic impact of friction in the suspension system behavior, so a tire-road model suitable for handling and stability analysis must be chosen. The handling tire models focus in determining the longitudinal and lateral forces as well as the moments generated by these forces and relate them with the longitudinal slip as well as with the slip angles to assess the peak forces and moments after which the tires lose grip and sliding between tire and road occurs. This category of models can be used for pure slip conditions and combined slip cases in both steady and non-steady situations. In the literature review paper authored by Li, Yang, and Yang (2014) a comparison of 16 handling and stability tire models is made, such as the TMeasy tire model (Rill 2013), the analytical tire model (Gim and Nikraves 1990; Gim and Nikraves 1991) and The Magic Formula Tire Model (Pacejka, 2012).

The Magic Formula Tire Model, developed by Pacejka (2012), is one of the most significant and widely utilized models for tire behavior analysis, especially for race car vehicle dynamics tire-road interaction, reason that explains the choice of this model to simulate the tire-road contact for the FSUM car multibody model. Over the years, substantial improvements have been made to this model through the collaborative efforts of researchers from Delft University of Technology, Volvo, Michelin, TNO Automotive, and Eindhoven University of Technology (Uil, 2007).

Some advantages of this tire model are the following ones: (i) easy to program and apply, (ii) high accuracy and computational efficiency, (iii) one of the most widely used empirical tire models used in automotive industry and (iv) capability of capturing a variety of non-linear phenomena in a wide range of longitudinal slip and slip angle conditions. The disadvantages of the model are its empirical background, which makes it difficult to include other non-linear effects such as temperature and tire wear, the need of

experimental data to calibrate the model as well as the rather large number of parameters that need to be defined to use the model (Uil 2007).

The Magic Formula tire model longitudinal force (f_x), lateral force (f_y), aligning torque (M_z), rolling resistance moment (M_y), and overturning moment (M_x) are calculated using the formulas presented in Appendix V. Figure 9.13 shows the SAE tire force and moment axis system where all the geometric parameters as well as forces and moments that act on a tire are represented

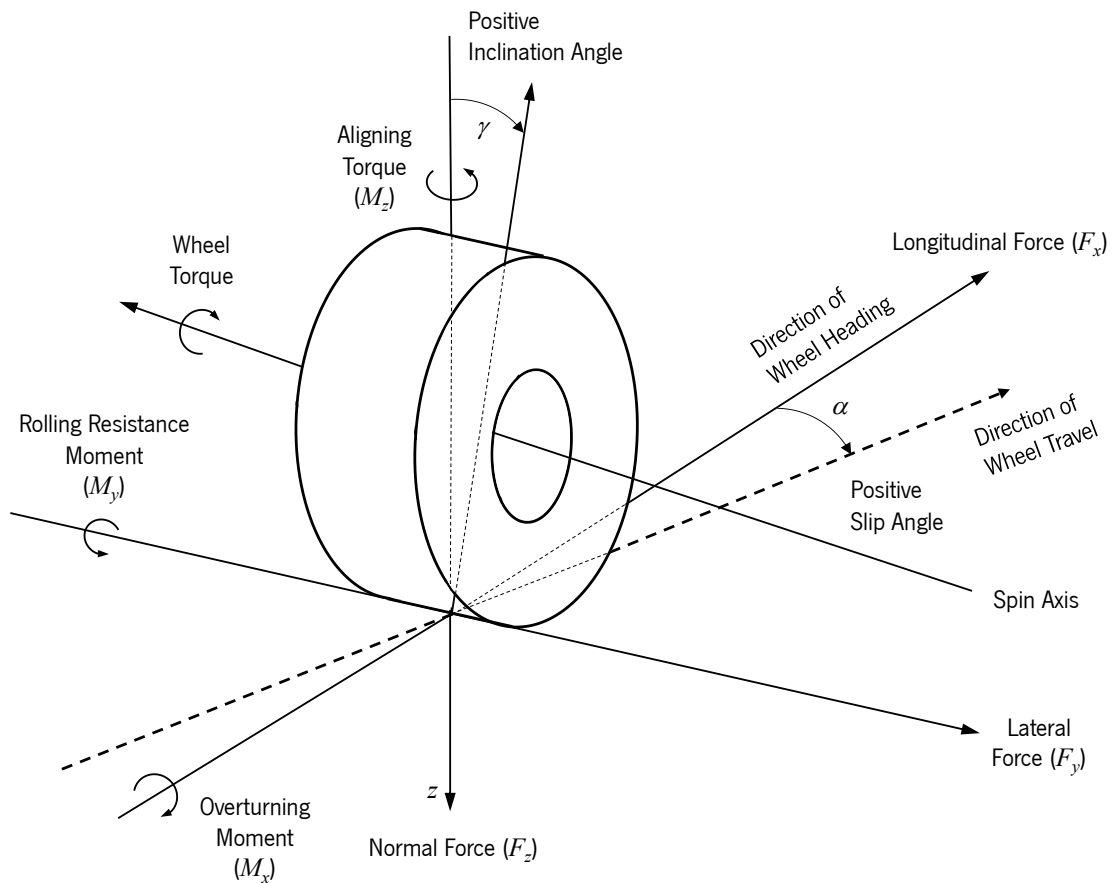


Figure 9.13 – SAE tire forces and moments axis system.

One of the major difficulties in vehicle dynamics simulations is the lack of tire data availability, due to the expensive cost and time-consuming aspects related to tire tests. The FSUM car was designed considering the use of Hoosier FSAE 20.5X7.0-13 R60A tires (Hoosier Racing Tire, 2024). These tires are a common choice among Formula Students teams since they are specifically designed for FS cars and are tested by the Formula SAE Tire Test Consortium (FSAE TTC) in Calspan's Tire Research Facility (Kasprzak & Gentz, 2006). The tire data was used in a MATLAB curve fitting process to obtain the parameters utilized in the different formulas that describe the Magic Formula tire model. The different tire forces and moments graphs fitted to the experimental data are presented in Appendix VI which allowed to obtain the numerical values of each parameter of the MF tire model.

Figure 9.14 shows the principal parameters that are needed as inputs for the nonlinear tire model.

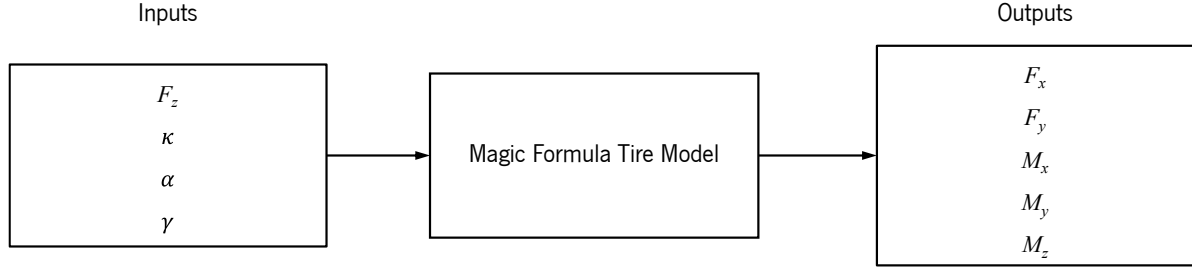


Figure 9.14 – Inputs and outputs of the nonlinear empirical Magic Formula Tire Model.

It is possible to observe that one of the inputs needed for the Magic Formula Tire model is the normal force (f_z). This data is normally defined and known as a control variable during experimental data acquisition in tire testing machines. For the MF tire model, it is possible to determine the normal force using the expression

$$f_z = \left\{ 1 + q_{V2} |\Omega| \frac{r_0}{v_0} - \left(q_{Fcx} \frac{f_x}{f_{z0}} \right)^2 - \left(q_{Fcy} \frac{f_y}{f_{z0}} \right)^2 \right\} \cdot \left\{ (q_{Fz1} + q_{Fz3} \gamma^2) \frac{\rho_z}{r_0} + q_{Fz2} \cdot \frac{\rho_z^2}{r_0^2} \right\} \cdot (1 + p_{pFz1} dp_i) \cdot F_{z0} \quad (9.9)$$

It is noticeable that, along other parameters that will not be analyzed in this work, Equation (9.9) requires the knowledge of the unknown outputs of longitudinal force (f_x) and lateral force (f_y) to determine the normal force (f_z). Pacejka (2012) also indicates that a simpler formula can be used to determine the contact normal force of the tire-road interaction formulated as

$$f_z = C_z \rho_z \quad (9.10)$$

where C_z represents the tire radial stiffness, obtained directly from the tire manufacturer for a pressure of 1.10 bar (Hoosier Racing Tire, 2024) and ρ_z is the vertical tire compression determined as

$$\rho_z = \max((r_0 - r_i - b + \zeta) \cos \gamma + \eta \sin \gamma, 0) \quad (9.11)$$

where for an elliptical shaped cross-section tire ζ and η are computed as

$$\zeta = \frac{b}{\sqrt{1 + \left(\frac{a}{b} \right)^2 \tan^2 \gamma}} \quad (9.12)$$

$$\eta = \frac{a \left(\frac{a}{b} \right) \tan \gamma}{\sqrt{1 + \left(\frac{a}{b} \right)^2 \tan^2 \gamma}} \quad (9.13)$$

being a equal to the unloaded radius of the tire and b equal to half of the tire cross-sectional width. Although the Pacejka alternative normal force formula from Equation (9.10) does not take into account the tire damping phenomenon, the term $d_z \dot{\rho}_z$ was added to account for it. The addition of the tire damping term makes the normal force (f_z) calculation very similar to the TMeasy (Rill, 2013) and UA tire model (Gim and Nikravesh 1990) where the final normal force formula is given by

$$f_z = C_z \rho_z + d_z \dot{\rho}_z \quad (9.14)$$

in which d_z represents the tire damping and $\dot{\rho}_z$ is the tire vertical compression rate.

Since it was not possible to obtain the tire damping coefficient from the tire manufacturer, this coefficient was estimated considering that for vehicle simulations the damping coefficient of a tire is typically equal to 10% of the critical damping coefficient (Popov & Geng, 2005; Stutts & Soedel, 1992). Considering that critical damping is given by the following formula

$$c_{crit} = 2\sqrt{C_z m_t} \quad (9.15)$$

where C_z is the tire radial stiffness and m_t is the tire mass, then the damping coefficient of the tire can be simply calculated as

$$c = 0.1 \cdot c_{crit} \quad (9.16)$$

Table 9.2 summarizes all the structural and geometric tire parameters needed as inputs to calculate the forces and moments generated by the tire-road interaction.

Table 9.2 – Tire structural and geometric input data

Parameter	Value
Unloaded radius, a	0.2540 m
Width, $2b$	0.1822 m
Mass, m_t	4.99 kg
Radial stiffness, C_z	137650 N/m
Radial damping, d_z	166 N·s/m

9.2 Friction and Shock Absorbers

The following section is dedicated to the description of how friction affects the dynamic behavior of shock absorbers. Firstly, an analysis of the different zones of a shock absorber where friction can be generated is made. After identifying the critical zones, it is explained how the friction forces are calculated from the normal contact forces within the joints that represent the shock absorbers and the way that these dissipative forces are considered in the equations of motion that characterize the multibody dynamics mathematical formulations.

9.2.1 Critical Friction Forces Generation Zones

Road vehicle shock absorbers (SA) are mechanical components responsible for working together with springs to keep the tire contact patch on the contacting surface by suppressing oscillation and controlling the motion of the sprung mass (Milliken & Milliken, 1995). The compression motion associated with suspension systems, which rise the wheel and tire, is typically known as jounce, bump or compression and the downward travel of the tire and wheel that causes the extension of the shock absorber and spring set is called rebound. When SA are subjected to bump motion, the damper compresses and helps to control the wheel travel and prevent an overshoot response, typical of undamped systems, while on rebound, the damper is responsible for controlling the elastic potential energy stored in the spring.

The vast majority of Formula Student (FS) teams utilize downhill bicycle rear shock absorbers for their suspension systems, since these SA combine both damper and spring in a parallel arrangement, which facilitates the design and assembly processes. They also have a wide range of adjustments for both jounce and rebound motions, allowing for an asymmetrical setup of the mentioned motions to better suit the intended vehicle handling behavior and are also pretty suitable for the typical loads associated to a FS type car. Figure 9.15 shows the DNM RCP-2S shock absorber utilized in the suspension subsystem of each one of the four wheels of the FSUM car.

The two most common types of friction present in shock absorbers of vehicles are the viscous friction, which is the governing and intended working phenomena of a suspension system, as dissipates vibrations transmitted to the vehicle's chassis, and the solid dry friction, that is generated between the rubbing of damper's seals and guides against their counterparts during actuation.



Figure 9.15 – DNM RCP-2S shock absorber utilized for the suspension subsystem of the FSUM car.

Figure 9.16 allows the visualization of a shock absorber's section view where the different seals are represented.

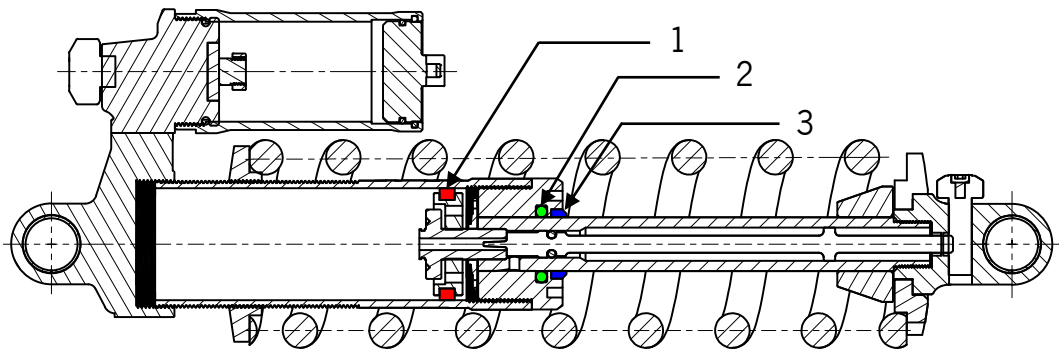


Figure 9.16 – Section view of a downhill bicycle shock absorber. The arrows indicate the critical contact zones where the relative motion between surfaces generates friction.

The different contact zones where relative tangential motion occur and, consequently, friction can be generated, are numbered in Figure 9.16 and can be identified as follows:

1. Piston outer seal and inner walls of the shock absorber's body.
2. Piston rod and sealing O-ring.
3. Piston rod and scraper seal.

The static friction verified in the seals requires a breakaway force to start movement, thus leading to a rigid performance of the damper until the applied force exceeds the stiction (static friction) level. Since shock absorbers are mounted parallel to suspension springs, the breakaway effect makes the suspension system inactive, causing wheel displacements to directly affect the vehicle's body, resulting in high accelerations that can affect the driver's steering inputs and overall control of the vehicle.

The frictional behavior, influenced by shock absorber and seal design, is a complex phenomenon, making it hard to predict. At high velocities, static friction's impact on the overall damping force is minimal,

so it is mainly relevant at low actuation speeds (Herzog & Augsburg, 2021a). The focus of this study is centered on shock absorber's working conditions where dry friction phenomena are preponderant.

9.2.2 Modeling of Friction Force in Cylindrical Kinematic Joints

A multibody model can be characterized as a mathematical and physical representation of a real-world system where the different bodies that compose it are connected by various kinematic joints and/or force elements. The modeling of suspension systems using kinematic joints, more specifically of shock absorbers, is made considering cylindrical joints (J. Ambrósio & Verissimo, 2009; Harsh, 2018).

A cylindrical joint is a kinematic constraint that allows two bodies i and j to move along a common axis while also allowing relative rotation about that same axis (Nikravesh 1988). Figure 9.17 shows a generic schematic of a cylindrical joint with the allowed translation and revolute degrees of freedom.

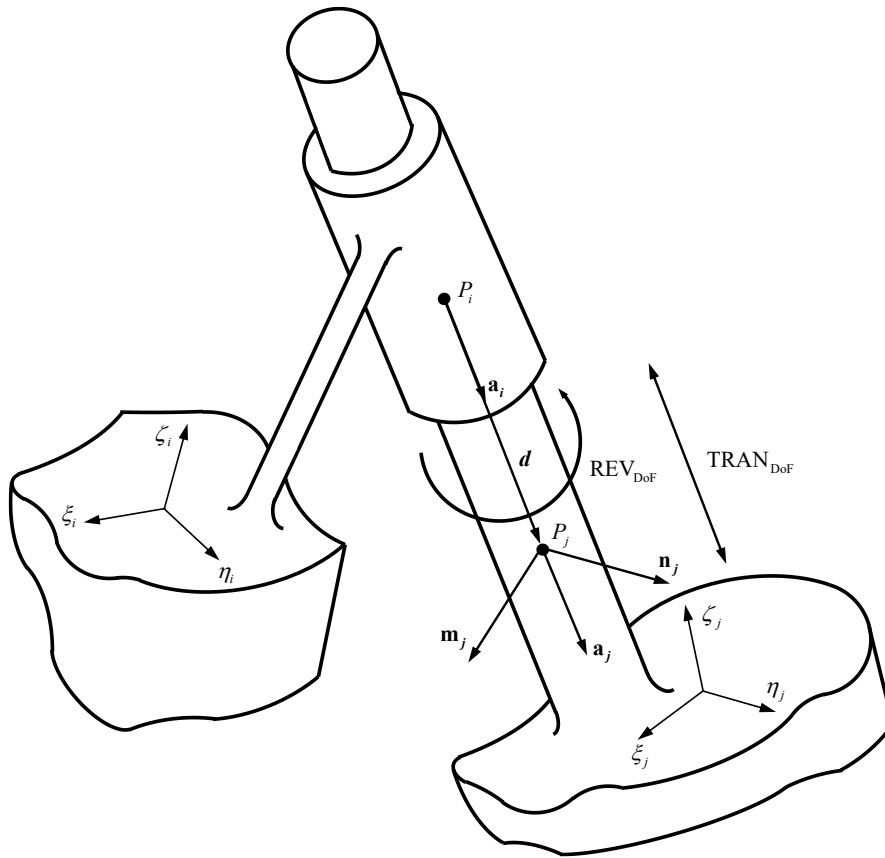


Figure 9.17 – Generic schematic of a kinematic cylindrical joint and allowed degrees of freedom.

The constraints equations for this joint are formulated using a unit vector $\mathbf{a}_i$, which is defined along the longitudinal axis of body i , and two other unit vectors, $\mathbf{m}_j$ and $\mathbf{n}_j$, defined in body j , that are mutually orthogonal between themselves, forming a set of three mutually orthogonal vectors. Vector $\mathbf{d}$ is

responsible for connecting the points P_i and P_j of bodies i and j , respectively. The constraint equations for the cylindrical joint can then be deduced as

$$\Phi \equiv \begin{cases} \mathbf{a}_i^T \mathbf{n}_j = 0 \\ \mathbf{a}_i^T \mathbf{m}_j = 0 \\ \mathbf{d}^T \mathbf{n}_j = 0 \\ \mathbf{d}^T \mathbf{m}_j = 0 \end{cases} \quad (9.17)$$

where these four equations are responsible for constraining the two translations and the two rotations that this type of joint does not allow. According to the dynamic equations of a constrained multibody system, the vector of external generalized forces and gyroscopic moments $\mathbf{g}$ is given by

$$\mathbf{M}\dot{\mathbf{v}} + \mathbf{D}^T \boldsymbol{\lambda} = \mathbf{g} \quad (9.18)$$

where $\mathbf{D}^T$ is the transpose of the Jacobian matrix that for the cylindrical joint case is given by

$$\mathbf{D}^T = \begin{bmatrix} 0 & 0 & -\mathbf{n}_j & -\mathbf{m}_j \\ \tilde{\mathbf{a}}_i \mathbf{n}_j & -\mathbf{m}_j \mathbf{a}_i & \mathbf{n}_j \tilde{\mathbf{s}}_i^P & \mathbf{m}_j \tilde{\mathbf{s}}_i^P \\ 0 & 0 & \mathbf{n}_j & \mathbf{m}_j \\ -\tilde{\mathbf{a}}_i \mathbf{n}_j & \mathbf{m}_j \mathbf{a}_i & (\tilde{\mathbf{s}}_j^P - \mathbf{d}) \mathbf{n}_j & (\tilde{\mathbf{s}}_j^P - \mathbf{d}) \mathbf{m}_j \end{bmatrix} \quad (9.19)$$

in which $\tilde{\mathbf{a}}_i$, $\mathbf{m}_j$, $\tilde{\mathbf{s}}_i^P$, $\tilde{\mathbf{s}}_j^P$ and $\mathbf{d}$ are the skew-symmetric matrices of the respective vectors. From Equation (9.18) it is possible to understand that $\mathbf{D}^T$ is related to the vector of the Lagrange multipliers, $\boldsymbol{\lambda}$, that contains the reaction forces/moments imposed by each kinematic constraint of the multibody in analysis, where for a cylindrical is represented as

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} \quad (9.20)$$

For a cylindrical joint, the matrix product $\mathbf{D}^T \boldsymbol{\lambda}$ is denoted as

$$\mathbf{D}^T \boldsymbol{\lambda} = \begin{bmatrix} 0 & 0 & -\mathbf{n}_j & -\mathbf{m}_j \\ \tilde{\mathbf{a}}_i \mathbf{n}_j & -\mathbf{m}_j \mathbf{a}_i & \mathbf{n}_j \tilde{\mathbf{s}}_i^P & \mathbf{m}_j \tilde{\mathbf{s}}_i^P \\ 0 & 0 & \mathbf{n}_j & \mathbf{m}_j \\ -\tilde{\mathbf{a}}_i \mathbf{n}_j & \mathbf{m}_j \mathbf{a}_i & (\tilde{\mathbf{s}}_j^P - \mathbf{d}) \mathbf{n}_j & (\tilde{\mathbf{s}}_j^P - \mathbf{d}) \mathbf{m}_j \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} \quad (9.21)$$

being of capital importance to make the correct correspondence between the terms of the matrix $\mathbf{D}^T$ and the terms of the $\boldsymbol{\lambda}$ vector, since it allows to understand which terms of the Lagrange multipliers vector are reaction forces and which ones are reaction moments. Taking body i as reference, from the analysis of Equation (9.21) it is concluded that

- λ_1 is a reaction moment around the $\tilde{\mathbf{a}}_i \mathbf{n}_j$ vector direction;
- λ_2 is a reaction moment around the $-\mathbf{m}_j \mathbf{a}_i$ vector direction;
- λ_3 is a reaction force in the $\mathbf{n}_j$ vector direction;
- λ_4 is a reaction force in the $\mathbf{m}_j$ vector direction

Representing the reaction forces and moments acting on the cylindrical joint in the different directions results in the schematic of Figure 9.18.

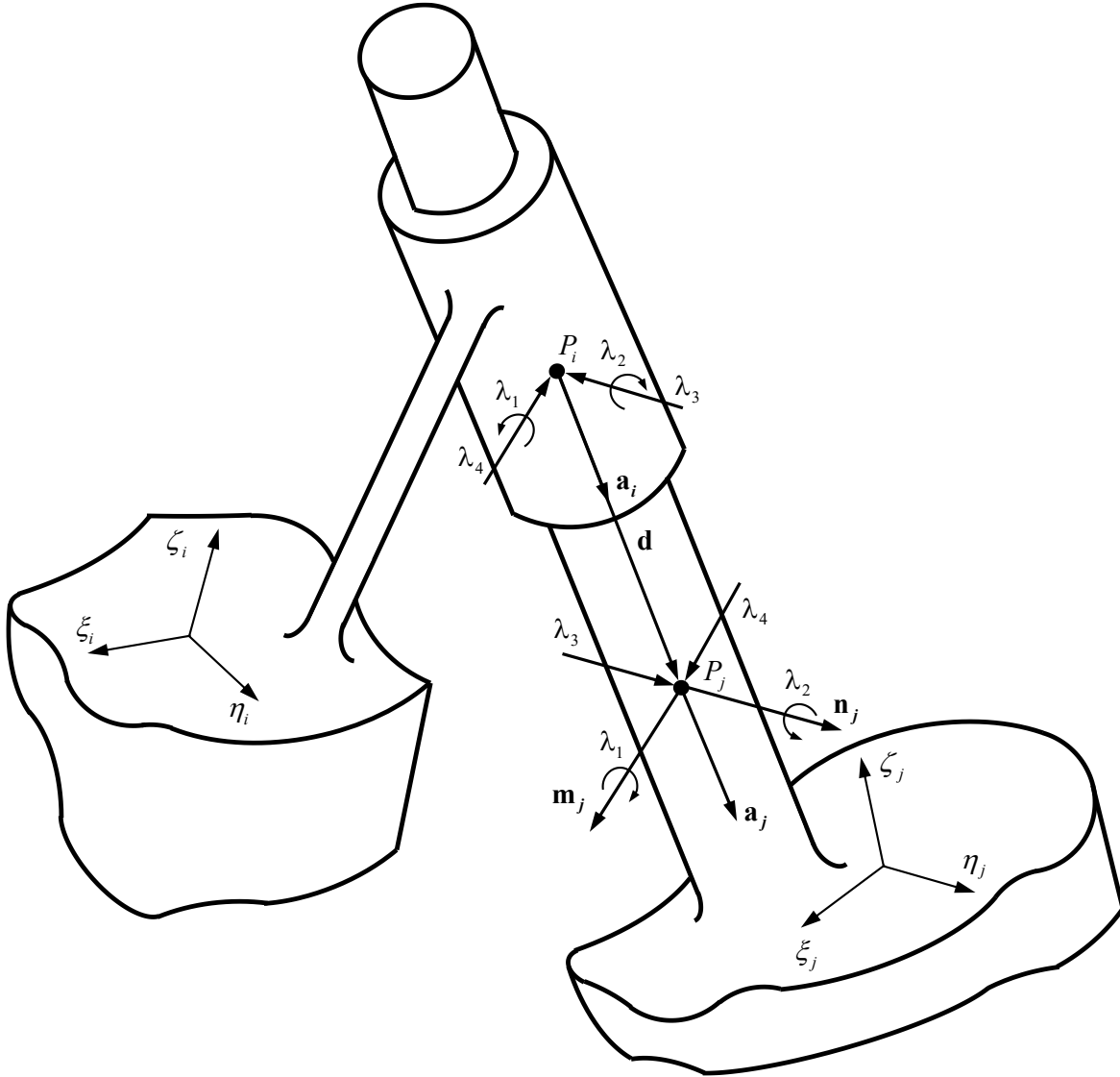


Figure 9.18 – Cylindrical joint with reaction forces and moments determined from the Lagrange multipliers vector.

The reaction forces and moments associated with the kinematic constraints of the cylindrical joint are extremely important to model the friction force, since the friction force is a function of the normal force and, consequently, of joint reactions. In this work, the friction forces are calculated as external generalized forces. The equations of motion within the multibody dynamics' framework can be written as

$$\begin{bmatrix} \mathbf{M} & \mathbf{D}^T \\ \mathbf{D} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{v}} \\ \boldsymbol{\lambda}_{i+1} \end{Bmatrix} = \begin{Bmatrix} \mathbf{g} + \mathbf{f}_f(\boldsymbol{\lambda}_i) \\ \boldsymbol{\gamma} \end{Bmatrix} \quad (9.22)$$

where $\mathbf{f}_f(\boldsymbol{\lambda}_i)$ is the generalized friction forces vector, which is calculated utilizing the reaction forces obtained in the previous time-step. For the initial approximation, a null vector has been considered for all the cylindrical joints associated with the shock absorbers.

One drawback of this strategy is that the precision of the friction force calculations is highly dependent on the stiffness of the problem and the chosen integration time-step. For low to moderately stiff problems, the reaction forces and moments obtained between two consecutive time-steps do not differ much, as long as an adequate integration time-step is used. For highly stiff problems, such as crashworthiness applications, this formulation will be unacceptable, since between two time-steps the reaction forces/moments will vary greatly, requiring very small integration time-steps, which would result in a computationally inefficient approach to solve the problem.

Since friction force is a function of the normal contact force, it is necessary to first determine the normal reaction forces acting on the shock absorbers. The analysis of the Lagrange multipliers vector of a cylindrical joint, $\boldsymbol{\lambda}$, indicates that there are two reaction moments and two reaction forces acting on the joint that connect the upper and lower parts of a shock absorber. For the analysis conducted in this study, two contact regions have been considered: (i) the piston outer seal and inner walls of the shock absorber's body and (ii) the piston rod and the scraper seal, placed at the bottom of the top part of the SA. In the two contact regions, it was assumed that the contact is a point contact, opposed to a surface or line type of contact.

Figure 9.19 shows a simplified schematic of the section view of a shock absorber with the possible contact points, P_{C1} and P_{C2} , in the two contact zones considered.

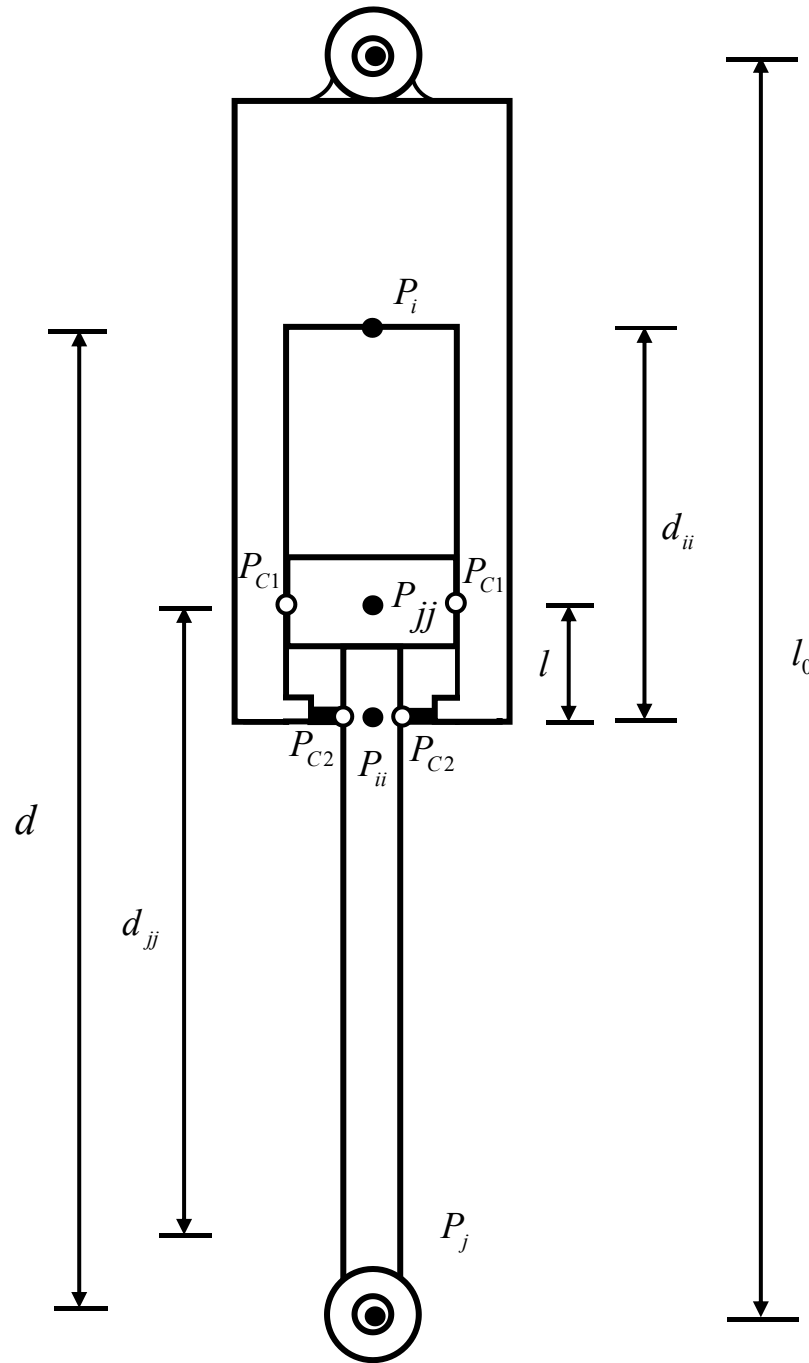


Figure 9.19 – Schematic section view of a shock absorber with the possible contact points considered.

To determine the normal contact forces in contact points P_{C1} and P_{C2} due to the reaction forces and moments, the cylindrical joint associated with the shock absorber was sectioned in the $\mathbf{a}_j\mathbf{n}_j$ plane, normal to the $\mathbf{m}_j$ vector, and in the $\mathbf{m}_j\mathbf{a}_j$ plane, normal to the $\mathbf{n}_j$ vector. Figure 9.20 shows the section view of the shock absorber projected in the $\mathbf{a}_j\mathbf{n}_j$ plane.

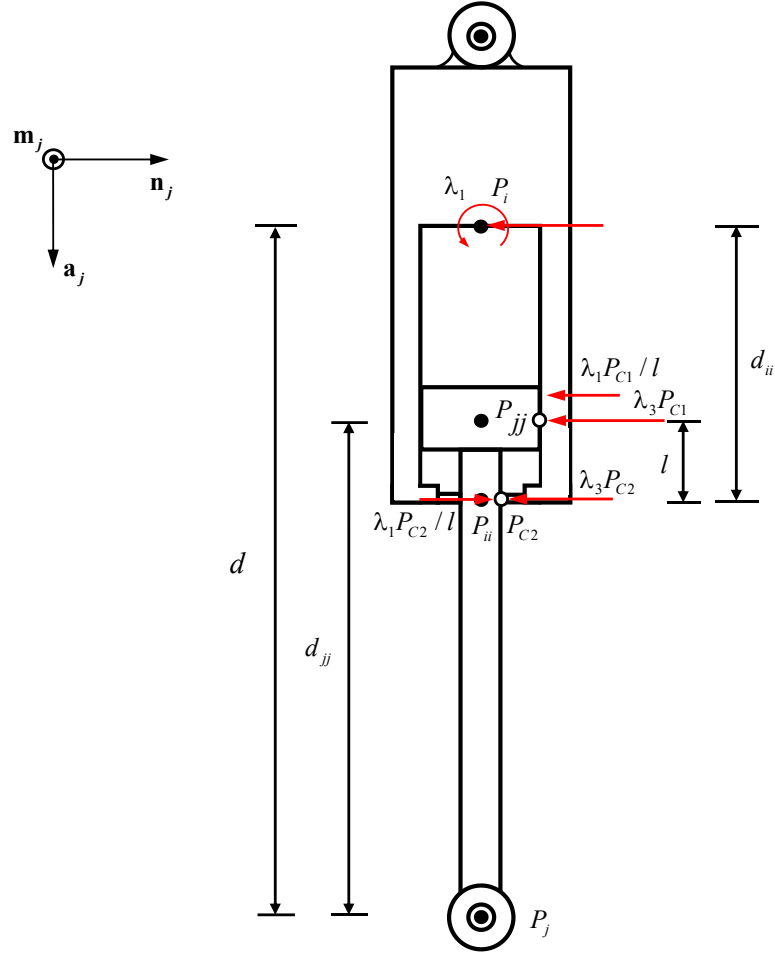


Figure 9.20 – Schematic of the SA projected in the $\mathbf{a}_j \mathbf{n}_j$ plane and the resultant reaction forces and moments.

It must be indicated that for each contact zone (C1 and C2), only a contact point is considered between the bodies, being the contact point located in the position of the auxiliary points P_{jj} and P_{ii} . In the $\mathbf{a}_j \mathbf{n}_j$ plane acts the reaction force λ_3 , which is divided into the normal contact force $\lambda_3 P_{C1}$ in contact point 1 (C1) and into the normal contact force $\lambda_3 P_{C2}$ in contact point 2 (C2). In this plane, it also acts the reaction moment λ_1 , which forms a torque produced by a couple of equal and opposite forces $\lambda_1 P_{C1} / l$ and $\lambda_1 P_{C2} / l$, acting in contact point 1 (C1) and contact point 2 (C2), respectively. The static equilibrium equations for each time-step can be described by the following linear system of equations

$$\begin{cases} -\lambda_3 P_{C1} - \lambda_3 P_{C2} = -\lambda_3 \\ -\lambda_3 P_{C1} (d - d_{jj}) - \lambda_3 P_{C2} (d_{ii}) = 0 \end{cases} \quad (9.23)$$

which allows to obtain the value of the normal contact forces $\lambda_3 P_{C1}$ and $\lambda_3 P_{C2}$ due to the kinematic reaction force λ_3 . In Equation (9.23) d is the norm of the vector that is defined between the points P_i and P_j , d_{ii} is the distance between the point P_i and the P_{ii} auxiliary point, which is immutable, since the

bodies of the model are considered rigid, being equal to 0.0860 m, and d_{jj} is also an immutable dimension, defined between the point P_j and the auxiliar point P_{jj} , that is equal to 0.0898 m. The moment arm, l , varies during the compression and rebounding phases of motion and is calculated using the following equation

$$l = d_{jj} - (d - d_{ii}) \quad (9.24)$$

In order to determine the total normal force in the two considered contact points, it is necessary to sum the contribution of the normal contact force due to the reaction moment λ_1 in each contact point, thus the total normal contact force in each contact point in the $\mathbf{a}_j\mathbf{n}_j$ plane can be written as

$$\begin{cases} f_{NC1\mathbf{a}_j\mathbf{n}_j} = -\lambda_3 P_{C1} - \lambda_1 P_{C1} / l \\ f_{NC2\mathbf{a}_j\mathbf{n}_j} = -\lambda_3 P_{C2} + \lambda_1 P_{C2} / l \end{cases} \quad (9.25)$$

The same procedure was applied to the $\mathbf{m}_j\mathbf{a}_j$ plane, where the reaction moment λ_2 and the reaction force λ_4 acts. Figure 9.21 shows the decomposition of λ_2 and λ_4 kinematic reactions in the respective normal forces.

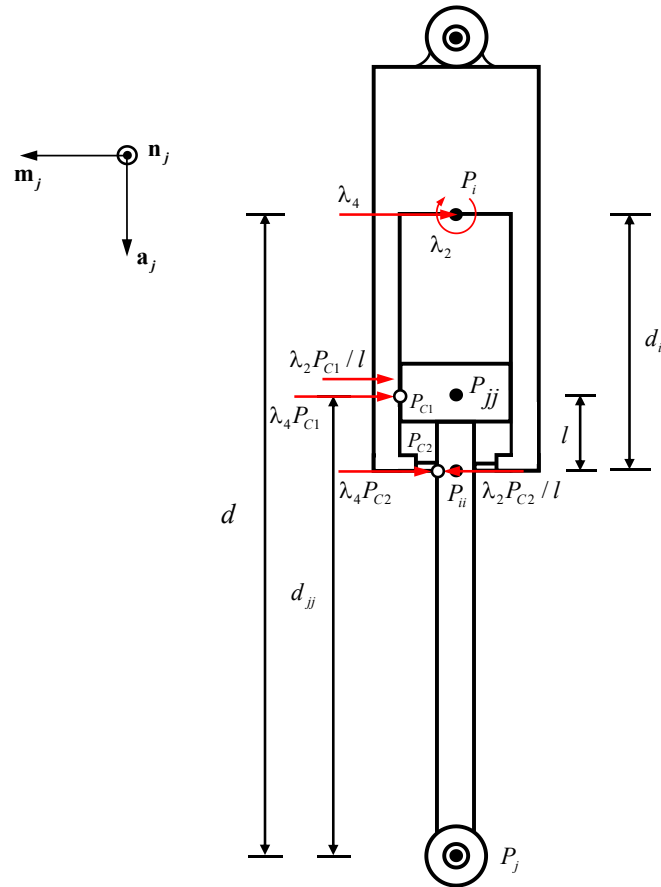


Figure 9.21 – Schematic of the SA projected in the $\mathbf{m}_j\mathbf{a}_j$ plane and the corresponding reaction forces and moments.

The equations that describe the static equilibrium for each time-step in the $\mathbf{m}_j\mathbf{a}_j$ plane are defined as

$$\begin{cases} -\lambda_4 P_{C1} - \lambda_4 P_{C2} = -\lambda_4 \\ -\lambda_4 P_{C1} (d - d_{jj}) - \lambda_4 P_{C2} (d_{ii}) = 0 \end{cases} \quad (9.26)$$

and by summing the contributions of the reaction moment λ_2 to the total normal force yields the total normal contact force in each contact point in the $\mathbf{m}_j\mathbf{a}_j$ plane as

$$\begin{cases} f_{NC1\mathbf{m}_j\mathbf{a}_j} = -\lambda_4 P_{C1} - \lambda_2 / l \\ f_{NC2\mathbf{m}_j\mathbf{a}_j} = -\lambda_4 P_{C2} + \lambda_2 / l \end{cases} \quad (9.27)$$

All the information needed to determine the total normal contact force in both contact point 1 and contact point 2 has been calculated. The total normal force in contact point 1 can be determined by the following equation

$$\mathbf{f}_{NC1} = f_{NC1\mathbf{a}_j\mathbf{n}_j} \cdot \mathbf{n}_j + f_{NC1\mathbf{m}_j\mathbf{a}_j} \cdot \mathbf{m}_j \quad (9.28)$$

and for the contact point 2 is given by

$$\mathbf{f}_{NC2} = f_{NC2\mathbf{a}_j\mathbf{n}_j} \cdot \mathbf{n}_j + f_{NC2\mathbf{m}_j\mathbf{a}_j} \cdot \mathbf{m}_j \quad (9.29)$$

where the magnitude of the normal forces can simply be determined as

$$f_{NC1} = \sqrt{\mathbf{f}_{NC1} \mathbf{f}_{NC1}^T} \quad (9.30)$$

$$f_{NC2} = \sqrt{\mathbf{f}_{NC2} \mathbf{f}_{NC2}^T} \quad (9.31)$$

The magnitude of the normal forces can then be used to calculate the friction forces acting between the contacting bodies using one of the many friction force models presented in literature. For contact point 1, a tribological pair of PTFE and Aluminum was considered between the piston's outer seal and inner walls of the upper shock absorber's body. Between the piston rod and the scraper seal, which together form contact point 2, a tribological pair of NBR82 rubber and Aluminum was considered. The static and kinetic friction coefficients utilized for both contact points are presented in Table 9.3.

Table 9.3 – Static and kinetic friction coefficients for contact points 1 and 2 of the shock absorber (Axsom, 2023; Giesbers, 2012).

Contact Point	Tribological Pair	Parameter	Value
1	PTFE – Aluminum	μ_k	0.18
		μ_s	0.19
2	NBR82 – Aluminum	μ_k	0.60
		μ_s	0.80

The friction forces acting on both contact points of the shock absorber are functions of the normal contact forces and act in the opposite direction of the tangential relative velocity between the two contacting bodies. Figure 9.17 shows the two degrees of freedom of a kinematic cylindrical joint. Since the vast majority of the friction force models take the relative tangential velocity between the contacting bodies as an input, it is necessary to determine the relative linear velocity along the joint axis, $\mathbf{v}_t$, associated with the translation degree of freedom.

The translational relative velocity is equal to the time-rate of the $\mathbf{d}$ vector, $\dot{\mathbf{d}}$, which will be explained further ahead. For each contact point, there will be a friction force and a resultant friction torque that arises from the fact that the contact points have an offset relatively to the joint points P_i and P_j that define vector $\mathbf{d}$, being the offset equal to the radius of the piston or the piston's shaft. The friction force, f_f , and the friction torque, $\mathbf{n}_f$, for each contact point are formulated as

$$f_{fcp} = f_f(f_{ncp}, v) \quad (9.32)$$

$$\mathbf{n}_{fcp} = f_f(f_{ncp}, v) \cdot \mathbf{r} \quad (9.33)$$

in which $f_f(f_{ncp}, v)$ represents the friction force value obtained from a friction force model that takes the normal contact force for the specific contact point in analysis, f_{ncp} , and the relative velocity, $\mathbf{v}_t$, as inputs. The friction torque, $\mathbf{n}_{fcp}$, is simply obtained by multiplying the friction force, f_{fcp} , by the correspondent contact radius, which in this case is equal to the piston's radius.

After calculating the magnitude of the friction forces, it is necessary to know in which direction these forces should be acting between the bodies that compose the cylindrical joint. The strategy applied consists in defining the direction of the friction forces by calculating the time-rate of the vector $\mathbf{d}$ magnitude. The magnitude of the vector $\mathbf{d}$, defined between the P_i and P_j points, is determined by

$$d = \sqrt{\mathbf{d}^T \mathbf{d}} \quad (9.34)$$

and the time-rate magnitude is given by

$$\dot{d} = \frac{\mathbf{d}^T \dot{\mathbf{d}}}{d} \quad (9.35)$$

In this work, the upper part of the shock absorber was considered the i -index body and the lower part the j -index body, in which the latter body moves relatively to the i -index. The $\mathbf{a}_j$ vector has been defined as being collinear with the $\mathbf{a}_i$ vector along the longitudinal axis of body j . Considering the time-rate magnitude of vector $\mathbf{d}$, $\dot{d}$, and the vector $\mathbf{a}_j$ it is possible to define the following direction function

$$\text{dir}(\dot{d}, \mathbf{a}_j) = \begin{cases} -\mathbf{a}_j & \text{if } \dot{d} > 0 \\ \mathbf{a}_j & \text{if } \dot{d} < 0 \\ [0 \ 0 \ 0]^T & \text{if } \dot{d} = 0 \end{cases} \quad (9.36)$$

that relates the direction of the friction forces with the approaching and pulling away motions between bodies i and j , which in the shock absorbers context is related to the compression and rebounding motions, respectively. Considering that when the time-rate magnitude of vector $\mathbf{d}$, $\dot{d}$, assumes positive values, the body j moves away from body i , so body j moves in the direction of the $\mathbf{a}_j$ vector. The dissipative friction forces should then act in the opposite direction of the relative tangential velocity of the bodies, so the direction of these forces should be opposite to the $\mathbf{a}_j$ vector for the j -index body. When bodies i and j are approaching ($\dot{d} < 0$), the friction forces should act in the direction of the $\mathbf{a}_j$ vector for the j -index body. When the relative tangential velocity between the contacting bodies is null ($\dot{d} = 0$), the direction of the friction forces should yield a null value of friction forces.

The total friction force in vector form acting on the cylindrical joint, composed by the friction force vectors for contact point 1, $\mathbf{f}_{\text{fcp1}}$ and for contact point 2, $\mathbf{f}_{\text{fcp2}}$, is determined as

$$\mathbf{f}_{\text{fcp1}} = f_{\text{fcp1}} \cdot \text{dir}(\dot{d}, \mathbf{a}_j) \quad (9.37)$$

$$\mathbf{f}_{\text{fcp2}} = f_{\text{fcp2}} \cdot \text{dir}(\dot{d}, \mathbf{a}_j) \quad (9.38)$$

where f_{fcp1} and f_{fcp2} are simply the magnitudes of the friction forces calculated for contact points 1 and 2. These forces should then be added as external generalized forces to body i and body j as follows

$$\mathbf{f}_i = \mathbf{f}_{\text{fcp1}} + \mathbf{f}_{\text{fcp2}} \quad (9.39)$$

$$\mathbf{f}_j = -\mathbf{f}_i \quad (9.40)$$

Since the cylindrical joint reaction forces and consequently the friction forces act on points P_i and P_j , which are different from the center of gravity of bodies i and j , the respective transport moments are given by

$$\mathbf{n}'_i = \mathbf{A}_i^T (\tilde{\mathbf{s}}_i^P \mathbf{f}_i) \quad (9.41)$$

$$\mathbf{n}'_j = \mathbf{A}_j^T (\tilde{\mathbf{s}}_j^P \mathbf{f}_j) \quad (9.42)$$

where $\mathbf{A}_i^T$ is the transpose of the transformation matrix of body i and $\tilde{\mathbf{s}}_i^P$ is a skew-symmetric matrix made with the components of vector $\mathbf{s}_i^P$.

Similarly, the friction torques at contact points 1 and 2 are defined as

$$\mathbf{n}_{\text{fcp1}} = n_{\text{fcp1}} \cdot \text{dir}(\text{sign}(f_{\text{fcp1}}), \mathbf{m}_j + \mathbf{n}_j) \quad (9.43)$$

$$\mathbf{n}_{\text{fcp2}} = n_{\text{fcp2}} \cdot \text{dir}(\text{sign}(f_{\text{fcp2}}), \mathbf{m}_j + \mathbf{n}_j) \quad (9.44)$$

in which $\text{dir}(\text{sgn}(f_{\text{fcp}}), \mathbf{m}_j + \mathbf{n}_j)$ is a direction function similar to Equation (9.36) that establish a relation between the sign of the friction force on the contact point in analysis and the vectorial sum $\mathbf{m}_j + \mathbf{n}_j$, to determine the direction of the friction torque in each body of the joint, that corresponds to the direction of the total contact force in both contact points. The friction torques are then added to the equations of motions as

$$\mathbf{n}_i = \mathbf{n}_{\text{fcp1}} + \mathbf{n}_{\text{fcp2}} \quad (9.45)$$

$$\mathbf{n}_j = -\mathbf{n}_i \quad (9.46)$$

being the transport moments determined by Equation (9.41) and Equation (9.42) where $\mathbf{f}_i$ and $\mathbf{f}_j$ are substituted by $\mathbf{n}_i$ and $\mathbf{n}_j$, respectively.

The only parameters left to fully characterize the multibody model in order to study the influence of friction forces in the dynamic behavior of the suspension subsystem is to determine the front and rear spring stiffness and damping coefficients.

9.2.3 Spring Stiffness, Damping Coefficient and Bumpstop Modeling

The kinematic cylindrical joint has four constraint equations that restrain four degrees of freedom: two translations and two rotations. This type of joint allows relative rotations between the two connected bodies and also relative translations in the longitudinal/axial direction of the bodies. One of the main disadvantages of this perfect kinematic joint type relatively to a dynamic constraint based on force reaction models and clearances is the fact that the bodies can exhibit translational motions with unlimited displacements.

The problem with the numeric displacements associated with a kinematic cylindrical joint is that they are not fully defined in terms of their maximum and minimum limits. Thus, this characteristic is incompatible with the real-world behavior of a shock absorber that presents a limited travel. In order to enhance the realistic behavior of a shock absorber, two bump stops were modeled to limit the SA's travel both in compression and rebounding. The set of equations that describe the spring-damper multibody dynamics force element considering bump stops is given by

$$f_{sbs} = \begin{cases} k_s \cdot (-0.076) + f_{pl} + k_{bs}(\delta + 0.076) & \text{if } \delta < -0.076 \\ k_s \delta + f_{pl} & \text{if } -0.076 \leq \delta \leq 0 \\ k_{bs} \delta + f_{pl} & \text{if } \delta > 0 \end{cases} \quad (9.47)$$

where k_s is the spring stiffness, which assumes different values for the front and rear SA, f_{pl} is the preload that can be applied to the shock absorber, k_{bs} is the bump stop stiffness that has been defined by a simple finite element analysis simulation, δ is the spring's deformation. From Equation (9.47) it is possible to understand that the first equation is related to the compression bump stop that is activated when the shock absorber is fully compressed, the second equation describes the SA normal behavior between its maximum (-0.076 m) and minimum deformation (0 m), which corresponds to its constructive travel, and the last equation describes the rebounding bump stop. Table 9.4 shows the different values considered for the spring-damper element and bump stop parameters for both front and rear wheels.

Table 9.4 – Front and rear shock absorbers and bump stops parameters.

Wheels	Parameter	Value
Front	k_s	43780 N/m
	c_d	3748 N·s/m
	l_o	0.2273 m
Rear	k_s	96320 N/m
	c_d	6938 N·s/m
	l_o	0.2273 m
Front and Rear	k_{bs}	$6 \cdot 10^8$ N/m

Figure 9.22 exhibits the spring curve for both rear and front wheels without preload, considering the compression and rebounding bump stop modeling.

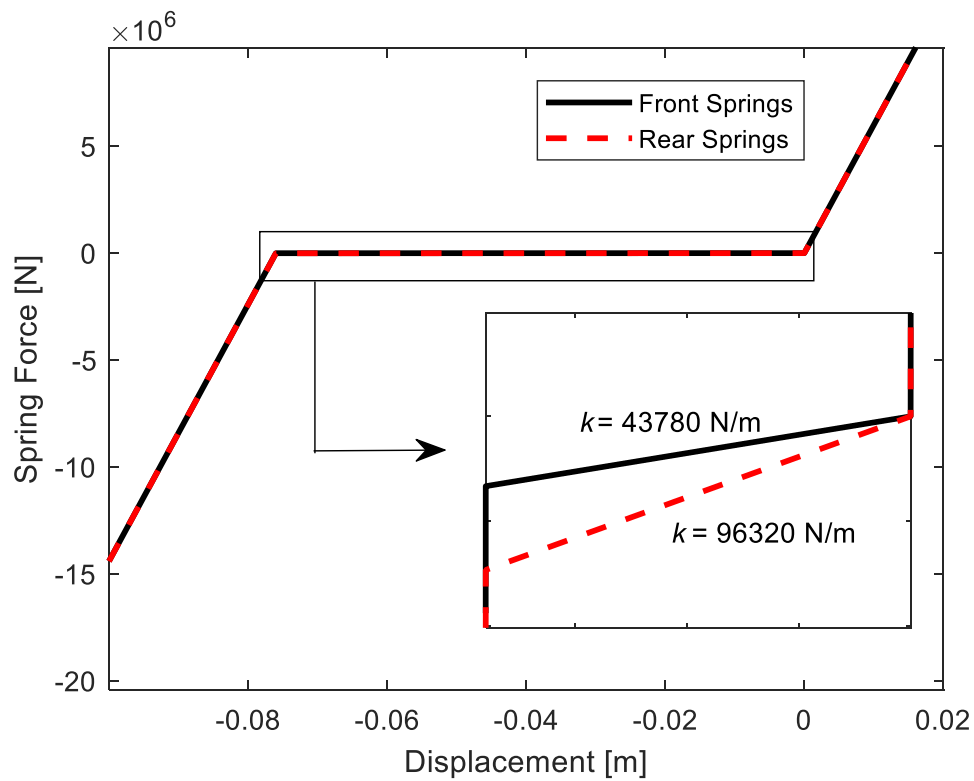


Figure 9.22 – Front and rear springs curves considering bumpstops.

For the middle region of the spring curve, it is necessary to add the damping term as indicated in the following equation

$$f_{sd} = k_s \delta + f_{pl} + c_d \dot{\delta} \quad (9.48)$$

where c_d is the damping coefficient and $\dot{\delta}$ is the time-rate of the spring's deformation. In these conditions, the shock absorber's stiffness and damping parameters are all defined as well as the physical travel through the bump stop modeling.

9.3 Dynamic Simulations of the Vehicle Multibody Model

The FSUM multibody car model has been built to be simulated in MUBODYNA, which is an in-house MATLAB program that performs forward dynamic analysis of general spatial multibody systems. In terms of code structure, its formulation closely follows the one presented in the textbook of Nikravesh (1988). The MUBODYNA code is capable of performing vehicle dynamics simulations since it was validated by Marques et al. (2017) using the European automobile industry standard 4x4 BOMBARDIER ILTIS benchmark problem.

The solutions of the forward dynamics simulations were achieved by using the ode45 solver, which is a variable time-step continuous explicit solver built-in in MATLAB. Since the position and velocity constraint equations are not explicitly utilized during the integration process, it becomes necessary to either eliminate or control the violations of these constraints.

The stabilization of kinematic constraint violations was achieved through Baumgarte's stabilization method. The α and β parameters were defined considering the parametric study performed by Flores et al. (2011) by choosing values that yield smooth results without oscillations, while being consistent with the physical behavior of the car. The initial conditions of the dynamic model have not suffered any type of correction. Table 9.5 summarizes the solver options and stabilization method parameters chosen for the dynamic tests applied to the FSUM multibody car model.

Table 9.5 – Solver and stabilization method options for the equations of motion.

Solver/Stabilization Method	Parameter	Value
ode 45	AbsTol	10^{-5}
	RelTol	10^{-5}
Baumgarte	α	80
	β	80

9.3.1 Obstacle Avoidance Dynamic Test

The analysis of the dynamic behavior of a vehicle in a controlled environment is very important not only to predict its performance, but also to assess its maneuverability and safety during dangerous conditions that can contribute to avoid accidents in real-life situations.

The ISO 3888-2:2011 standard defines the dimensions of the test track for a closed-loop, severe lane-change maneuver that is utilized to subjectively determine the capability of a road vehicle to avoid obstacles at relatively high speeds. Although this test is commonly applied to passenger cars and light commercial vehicles, it was considered to be a demanding dynamic test that can show the influence of the friction phenomena in the suspension subsystem behavior and overall drivability of the FSUM multibody model. The test involves rapidly driving a vehicle from its initial lane at a constant speed to another lane parallel to the first, and returning to the initial lane, without exceeding lane boundaries and without any braking or acceleration inputs from the driver. The objective is to have the vehicle reach a certain sequence of alternate high, lateral accelerations such that the vehicle's lateral dynamics can be subjectively evaluated (ISO 3888-2:2011). Figure 9.23 depicts the different sections of the test track and its dimensions.

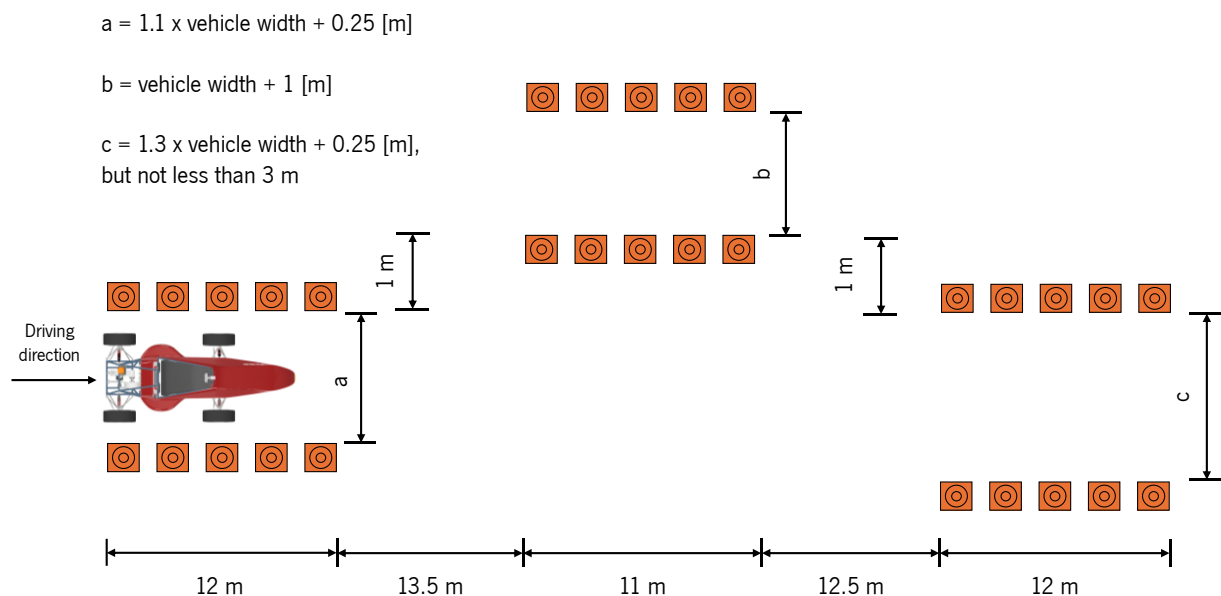


Figure 9.23 – Track layout and dimensions for severe lane-change maneuver according to ISO 3888-2.

The ISO 3888-2:2011 standard procedure does not indicate a minimum or maximum velocity limit that the car should start the test. Breuer (1998) states that the evasive maneuver should be executed at a speed between 60 and 65 km/h. Many automotive publications unofficially consider nowadays that 72 km/h (45 mph) is considered to be the minimum speed for approval in this specific dynamic test.

The FSUM multibody car model was then subjected to the lane-change maneuver for the initial constant speed of 72 km/h. Figure 9.24 a) exhibits the steering wheel input angles given to the multibody model for the dynamic test at 72 km/h, while Figure 9.24 b) exhibits the trajectory of the FSUM car center of gravity for the evasive maneuver executed, confirming that the car is able to perform the intended dynamic test.

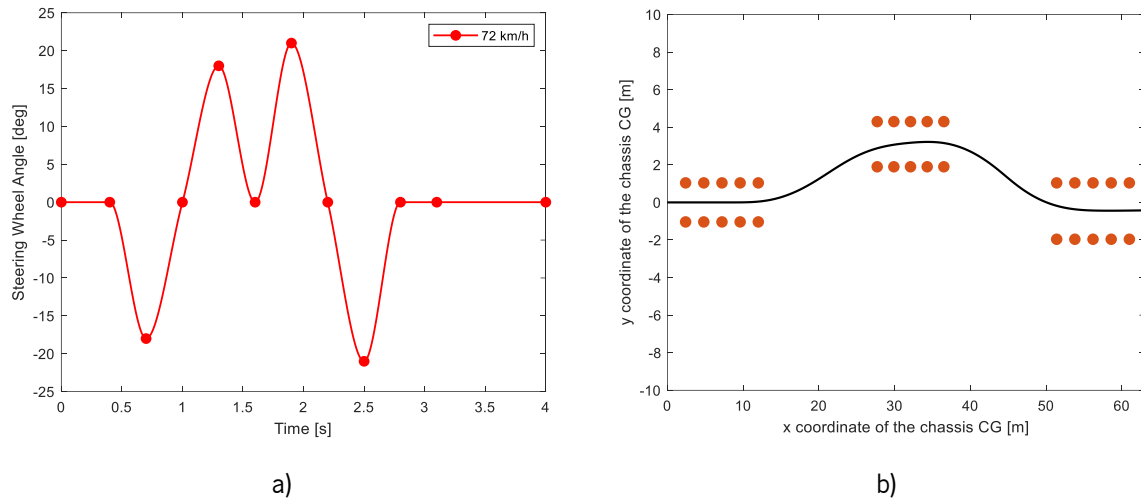


Figure 9.24 – a) Steering wheel input angles defined for the severe lane-change maneuver at the initial speed of 72 km/h b) Trajectory of the center of gravity of the car during the maneuver.

The abruptness of the steering inputs and the relatively high speed at which the dynamic test is taken makes the rear wheels of the car lose contact with the ground at certain time intervals, as it is depicted in Figure 9.25 a) graph where the tire vertical forces assume null values. Figure 9.25 b) also proves the previous statement, since the rear spring's displacement reaches null values, which means that the wheel lifts off the ground and the rebounding bump stop has been reached.

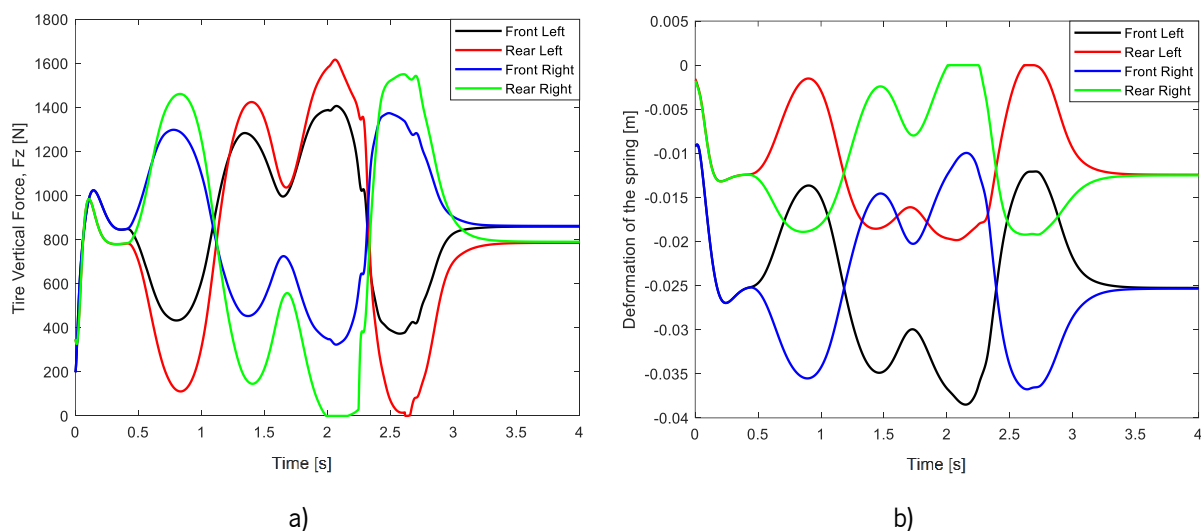


Figure 9.25 – Preliminary analysis of the FSUM car behavior during the obstacle avoidance maneuver a) Tire vertical forces on all four wheels b) Spring deformation on all four shock absorbers of the car.

The study of the influence of friction forces on the dynamic behavior of the suspension system has been made considering four different friction force models. Furthermore, two static friction force models were studied, Rooney and Deravi and Bengisu and Akay models, and also two dynamic friction model, Dahl and LuGre. The choice of static models was made considering that Rooney and Deravi is a very simple regularized approach that does not consider any complex friction phenomena, not even static friction, and its claimed by its authors to be very suitable for automotive applications, and Bengisu and Akay for being a static friction model that can model the Stribeck effect and reaching the stiction level. The same strategy was considered to choose dynamic friction models. The Dahl model is relatively simple and does not consider a static friction force level, while LuGre is a widely utilized friction force model due to its capabilities of modeling several friction phenomena. Table 9.6 shows the numeric values for the different parameters of the simulated friction force models.

Table 9.6 – Parameters for the friction force models utilized in the FSUM car shock absorbers.

Parameter	Symbol	Value	Parameter	Symbol	Value
Velocity threshold (Rooney and Deravi, Bengisu and Akay)	v_0	0.001 m/s	Bristle stiffness (Dahl & LuGre)	σ_0	10^5 N/m
Slope of the sliding state (Bengisu and Akay)	ξ	100	Damping coefficient (LuGre)	σ_1	$\sqrt{10^5}$ N·s/m
Stribeck velocity (LuGre)	v_s	0.001 m/s	Viscous coefficient (LuGre)	σ_2	0.1 N·s/m

Figure 9.26 shows the comparison of the total shock absorber force for the left front wheel of the FSUM car by considering the dissipative effects of friction forces and without considering friction. Analyzing the graph it is possible to verify that in general, the introduction of friction forces in the shock absorbers leads to a reduction of the physically allowed compression and rebound behaviors.

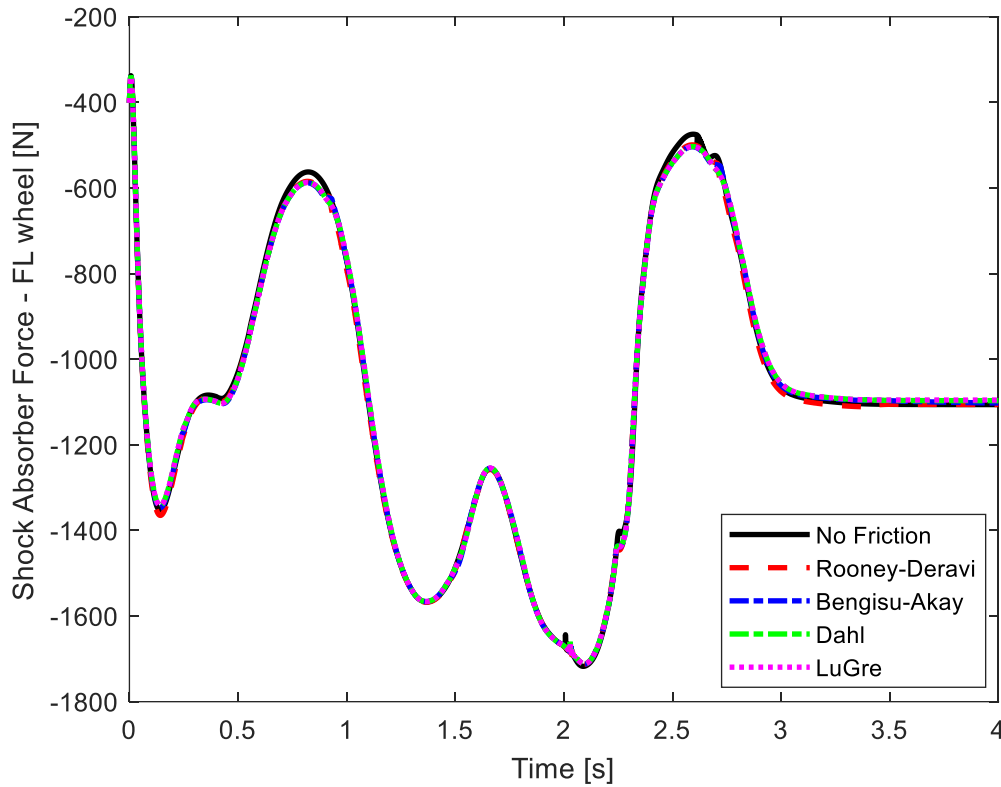


Figure 9.26 – Comparison of the shock absorber force for the front left wheel of the FSUM car considering an ideal no friction scenario and considering friction forces calculated by different models.

Figure 9.27 a) shows a detailed view of a rebounding phase while Figure 9.27 b) exhibits the differences verified between the different friction force models and a non-friction scenario during a compression phase.

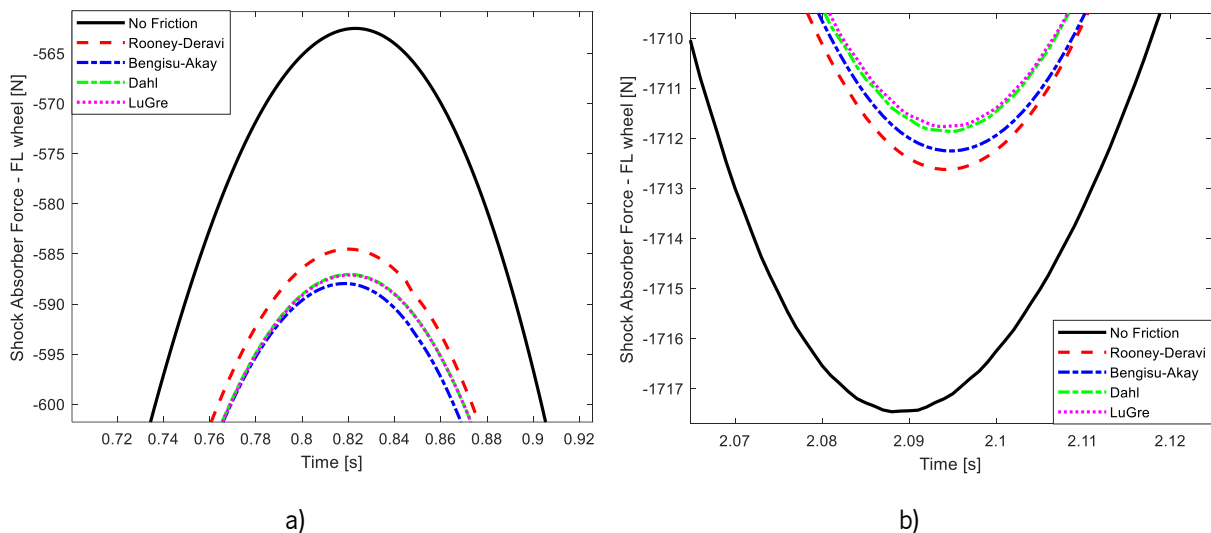


Figure 9.27 – Detailed views of the total force acting on the shock absorber during the evasive maneuver for a a) rebounding phase b) compression phase.

For the rebounding phase, the results are in conformity with what was predicted: (i) the friction forces dissipate energy which prevents the shock absorber from fully rebounding to the force level without

considering the effects of friction; (ii) the friction force models that are capable of modeling stiction, LuGre and Bengisu and Akay, require an higher initial breakaway force, which leads to an additional resistance to the rebounding behavior of the SA, yielding an incapacity to rebound to the rebounding force level of the Rooney and Deravi model. The Dahl model has a very similar behavior to the models that model the stiction behavior, although it was expected that its behavior was more identical to the Rooney and Deravi model. Regardless of the friction force model chosen, by considering the friction modeling within the shock absorber's dynamic behavior, an approximately 5% decrease of the maximum rebounding force is obtained compared to an ideal non-friction rebounding behavior.

The compression phase revealed similar results to the ones observed for the rebounding, where the additional resistance provoked by the friction forces lead to a reduction of the shock absorber's compression capacity. It is worth mentioning that in this situation, the Bengisu and Akay model showed a non-expected behavior, since that the modeling of stiction behavior from this model should yield a behavior more similar to the one represented by LuGre, creating more resistance to the compression motion of the SA. For the compression phase, a reduction in the total shock absorber force of approximately 0.33% was verified considering friction phenomena in comparison with a non-friction scenario.

Figure 9.28 shows the total magnitude of the friction force for the two contact points considered during the simulation time.

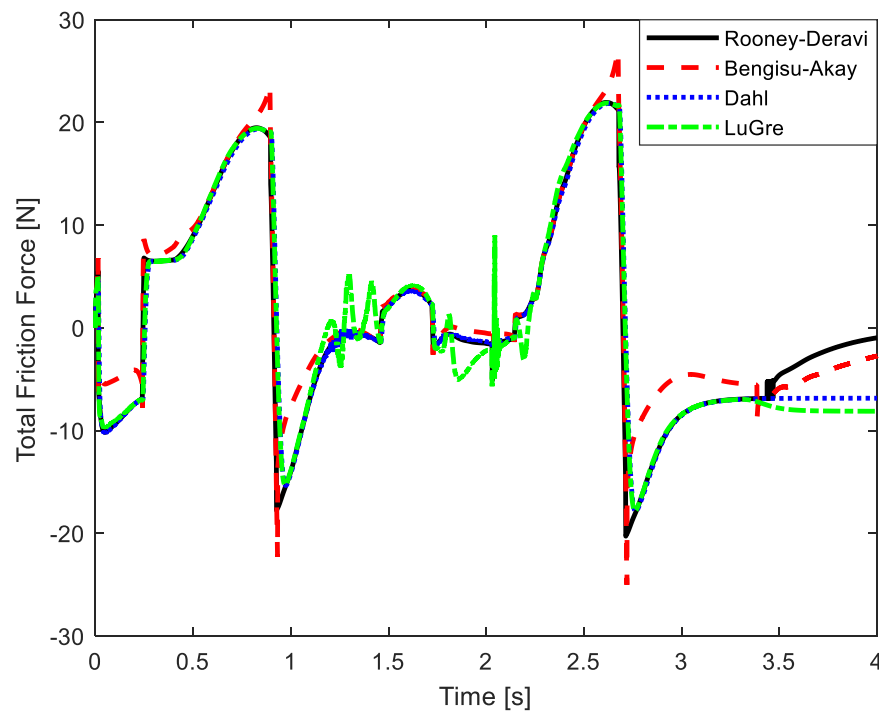


Figure 9.28 – Total friction force acting on the front left shock absorber during the evasive maneuver.

The Bengisu and Akay model exhibits the highest friction force value of all the models, and it was expected that the LuGre model reached a higher friction force level, corresponding to the stiction phase. The LuGre model also exhibits some numerical instabilities that could benefit from stricter absolute and relative tolerances of the ODE solver. The amplitude of the friction force results for both compression and rebounding are within the reported values reported in literature by other authors (Lizarraga et al. 2008; Körner and Mayer 2020).

In terms of computational efficiency, Table 9.7 shows the total computational time and the number of function evaluations of each simulation.

Table 9.7 – Comparison of the simulation time for the evasive maneuver for the ideal non-friction scenario and the different simulations considering static and dynamic friction force models. Values inside brackets correspond to quantities normalized to the lowest value.

Simulation Scenario	CPU time [s]	Number of Function Evaluations
No Friction	780 (1.00)	12313 (1.04)
Rooney and Deravi	931 (1.19)	15415 (1.30)
Bengisu and Akay	786 (1.01)	11833 (1.00)
Dahl	3155 (4.04)	53893 (4.55)
LuGre	3739 (4.79)	63109 (5.33)

It is possible to state that including friction phenomena in the simulations of the dynamic behavior of the car through the Bengisu and Akay friction model has a computational cost very similar to the no friction ideal scenario. The dynamic friction models, Dahl and LuGre, as expected, require a high computational effort of around 4 to 5 times more than the ideal case or the simulations where static friction force models were utilized. This fact is understandable and justifiable due to the additional state variables that need to be integrated by the ODE solver during the solving process of the problem.

9.3.2 Four-Poster Rig

The second dynamic test applied to the multibody model of the FSUM car corresponds to a suspension excitation through a four-poster rig. The use of four-poster rigs is well widespread across the automotive industry. This type of indoor testing allows the reproduction of real-world environment scenarios where data from road profiles such as speed bumps, potholes and other obstacles can be converted in displacements that are emulated by hydraulic actuators (Chindamo et al., 2017).

The usage of this type of testing platform is very important in racing environments, since it allows a more repeatable and less expensive test of suspension setups through track data. Testing rigs with seven or even eight actuators are also reported in literature, allowing a more detailed simulation of track loads, such as aerodynamic forces that play a key role in the dynamic behavior of racing cars (Kelly et al. 2002; Boggs et al. 2009).

The FSUM car suspension test in the four-poster rig consisted of a 40 mm displacement sinewave excitation of all four wheels in which the front wheels were simultaneously acted vertically and the rear wheels are also simultaneously acted vertically with an amplitude of 40 mm, but with a phase delay of π rad relatively to the front wheels. Figure 9.29 shows the excitation profile with a 40 mm amplitude displacement for both front and rear wheels of the FSUM car on the four-poster rig.

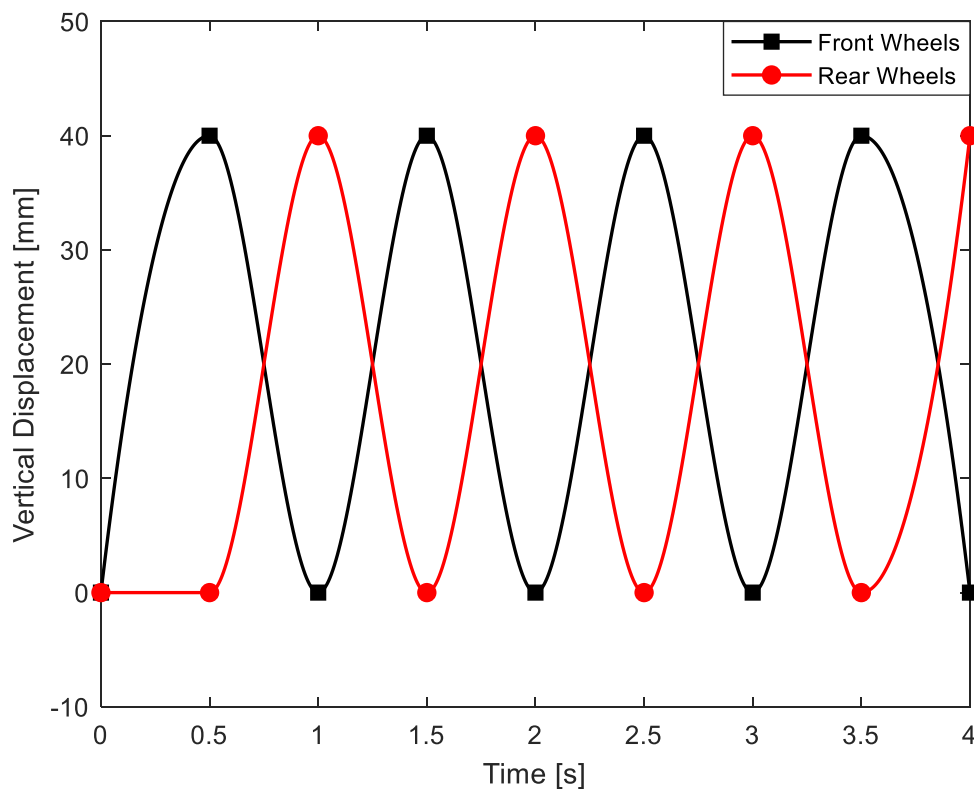


Figure 9.29 – Vertical displacement of 40 mm applied to both front and rear wheels on the four-poster rig with a π rad phase delay.

An important aspect of the modeling of this dynamic test is that the tire-ground contact, which was previously determined using the Magic Formula Tire model, was replaced by a simplification where the tire is modeled as a set of three spring-damper (s-d) elements in the three cartesian directions. Figure 9.30 depicts the details of the tire-ground spring-damper modeling.

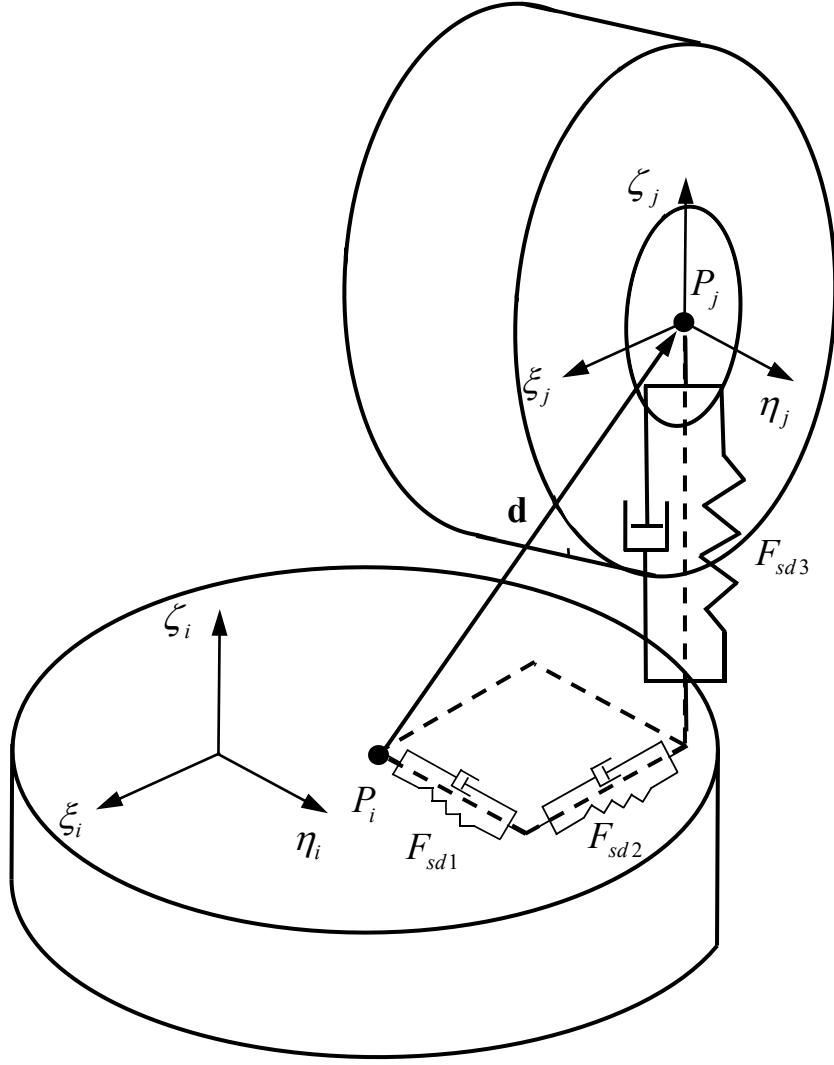


Figure 9.30 – Tire modeling as a set of three spring-damper elements for the four-poster rig simulation.

The F_{sd1} represents the structural stiffness and damping force of the tire in the width direction while F_{sd2} and F_{sd3} represent the radial stiffness and damping forces of the tire. In terms of the dynamic behavior of the system, only the vertical F_{sd3} force is important in this test, since the wheels are static and only the vertical dynamics of the tire matter. By adding the F_{sd2} and F_{sd3} , the system is stabilized, because it is eliminated the tendency that the car may have of exhibiting excessive roll, pitch or yaw motions, that would lead the model to have excessive constraint violations, thus falling off the four-poster platforms.

The total force for the tire-platform contact in this situation is given by the following expression

$$F_t = \left[\mathbf{k}_s \cdot \{\delta_{\xi} \quad \delta_{\eta} \quad \delta_{\zeta}\} + \mathbf{c}_d \cdot \{\dot{\delta}_{\xi} \quad \dot{\delta}_{\eta} \quad \dot{\delta}_{\zeta}\} \right] \cdot \{\xi_i \quad \eta_i \quad \zeta_i\}^T \quad (9.49)$$

where $\mathbf{k}_s$ is a 3x1 vector containing the stiffness of the springs of the s-d elements in the three spatial directions, $\{\delta_\xi \ \delta_\eta \ \delta_\zeta\}$ is a 1x3 vector containing the spring's deformations in all three directions, $\mathbf{c}_d$ is a 3x1 vector composed of the damping constants of the dampers of the s-d elements in the three spatial directions, $\{\dot{\delta}_\xi \ \dot{\delta}_\eta \ \dot{\delta}_\zeta\}$ is a 1x3 vector of the time-rate of the s-d elements deformation and $\{\xi_i \ \eta_i \ \zeta_i\}^T$ is a 3x1 vector utilized to define the direction of actuation of each force in local coordinates.

Figure 9.31 shows the comparison of the total shock absorber force for the front left wheel of the car during the four-poster dynamic test for the no friction ideal scenario and for the scenarios considering the incorporation of friction phenomena.

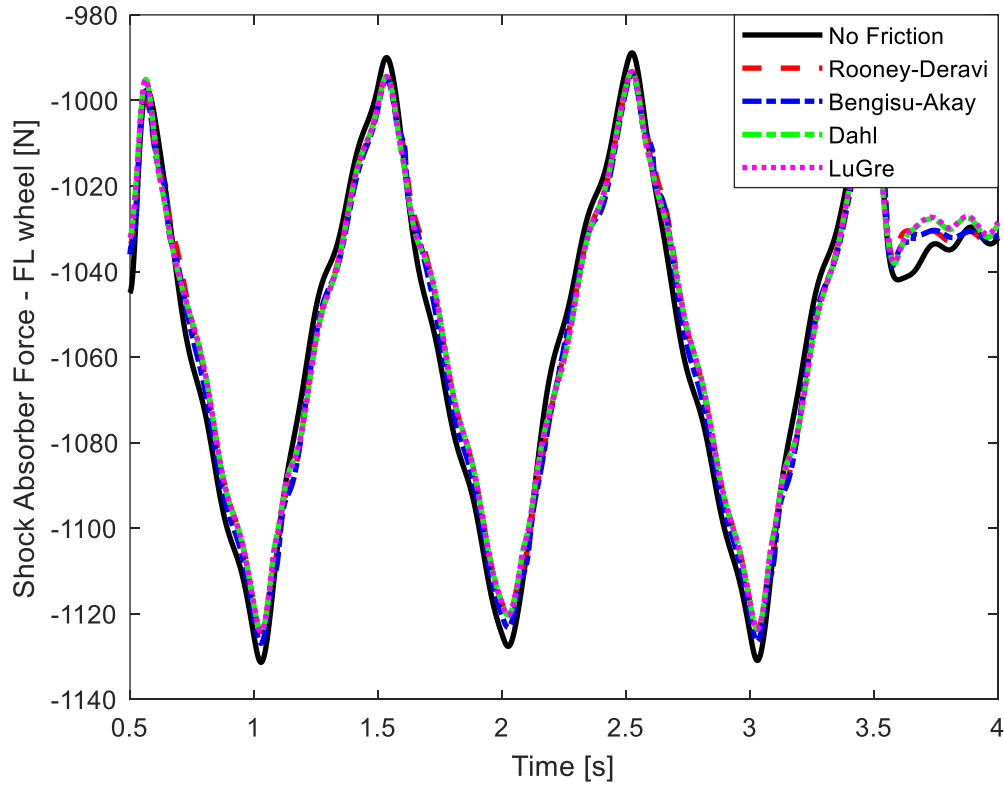


Figure 9.31 – Total shock absorber force for the front left wheel during the four-poster rig test.

In general terms, it is possible to verify that the inclusion of friction forces in the dynamic behavior of the suspension system leads to a reduction of the compressing and rebounding maximum forces, as it was previously verified for the obstacle avoidance maneuver. Another fact worth mentioning is that the imposed displacement of 40 mm to all four wheels of the car does not lead to a major compressing nor rebounding of the shock absorbers, since the minimum and maximum forces materialized by the peak forces in Figure 9.31 have a relatively small amplitude. This behavior can be explained due to the stiffness

of the tires and the front and rear suspensions being of the same order of magnitude, so the tires absorb a significant amount of deformation that should be transmitted to the suspension system.

Figure 9.32 a) shows the rebounding behavior of the front suspension system, while Figure 9.32 b) shows a compression phase of the front left shock absorber's motion during the simulation.

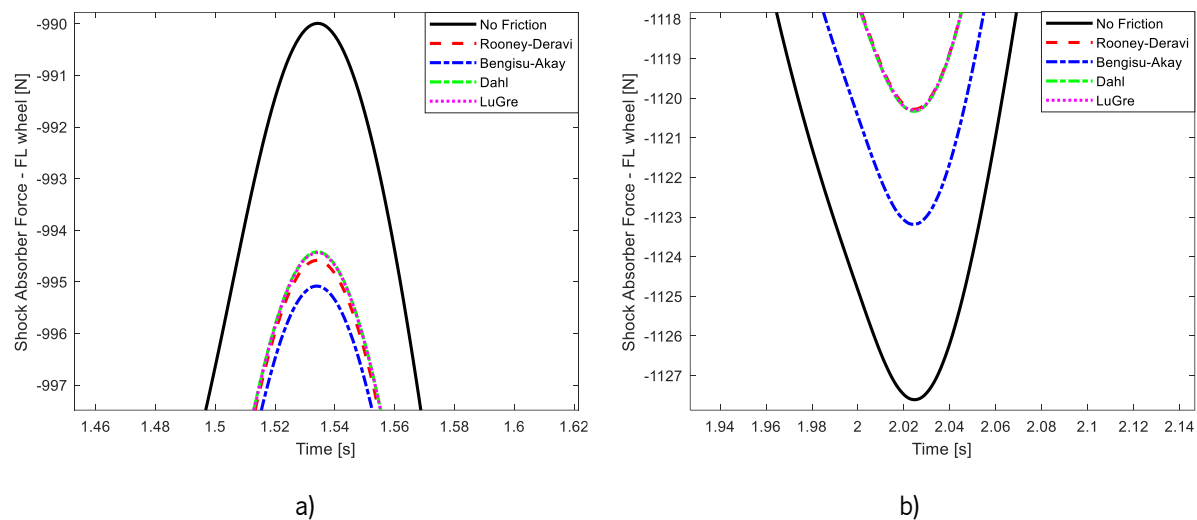


Figure 9.32 – Detailed views of the shock absorbers motion during the four-poster rig simulation a) Rebounding phase b) Compression phase.

In the rebounding phase, the Bengisu and Akay model capability of modeling stiction is shown by the additional dissipative effect that prevents the SA from fully rebounding to a non-friction ideal force level. It was expected that LuGre had a similar behavior to the Bengisu model in this phase, since it is also capable of modeling static friction level. For the compression phase, the results are not very satisfactory in terms of the expected behavior from the models, since both LuGre and Bengisu and Akay should yield lower compression forces due to the additional dissipative effect associated with the static friction force level and consequent breakaway phenomena, thus introducing additional resistance to the relative motion between the two bodies that constitute the shock absorber computational mode. Instead, it was verified that Rooney and Deravi model presents the biggest global resistance to the compression motion of the SA, leading to lower compression force values.

For the total friction force verified in both contact point 1 and contact point 2, the results are depicted in Figure 9.33.

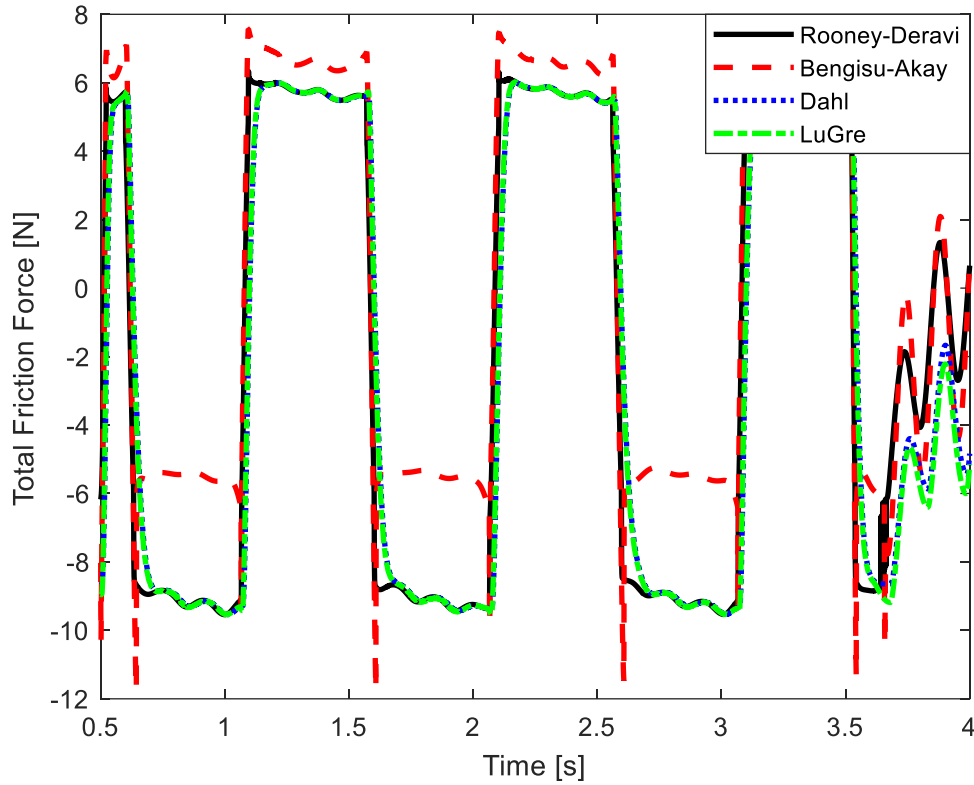


Figure 9.33 – Total friction force verified in the front left shock absorber during the four-poster rig simulations.

The Bengisu and Akay exhibit the highest friction forces due to the modeling of static friction force level, while the rest of the models show a very similar behavior and friction force values. The total friction force values are in general lower by 50% relatively to the ones verified for the obstacle avoidance maneuver. This fact can be explained due to the non-existence of lateral forces acting on the tire during the four-poster dynamic test. This affects the kinematic reactions acting on the suspension contact points, leading to lower normal contact forces, and consequently, lower total tangential friction forces.

A distinct analysis made for the four-poster rig test was the visualization of the total friction force data as a function of the tangential relative velocity between the contacting bodies. The results shown in the graph of Figure 9.34 are in accordance with the graphical representation of the friction force models often presented by authors in literature (Marques et al., 2016; Pennestri et al., 2016).

It is quite visible the stiction behavior of the Bengisu and Akay model, as well as the hyperbolic tangent central region of the Rooney and Deravi model. For the dynamic friction force models, Dahl and LuGre, the existence of extra state variables allows to capture the hysteresis behavior of the contacting materials, materialized by the hysteresis loops on the central region of the graph.

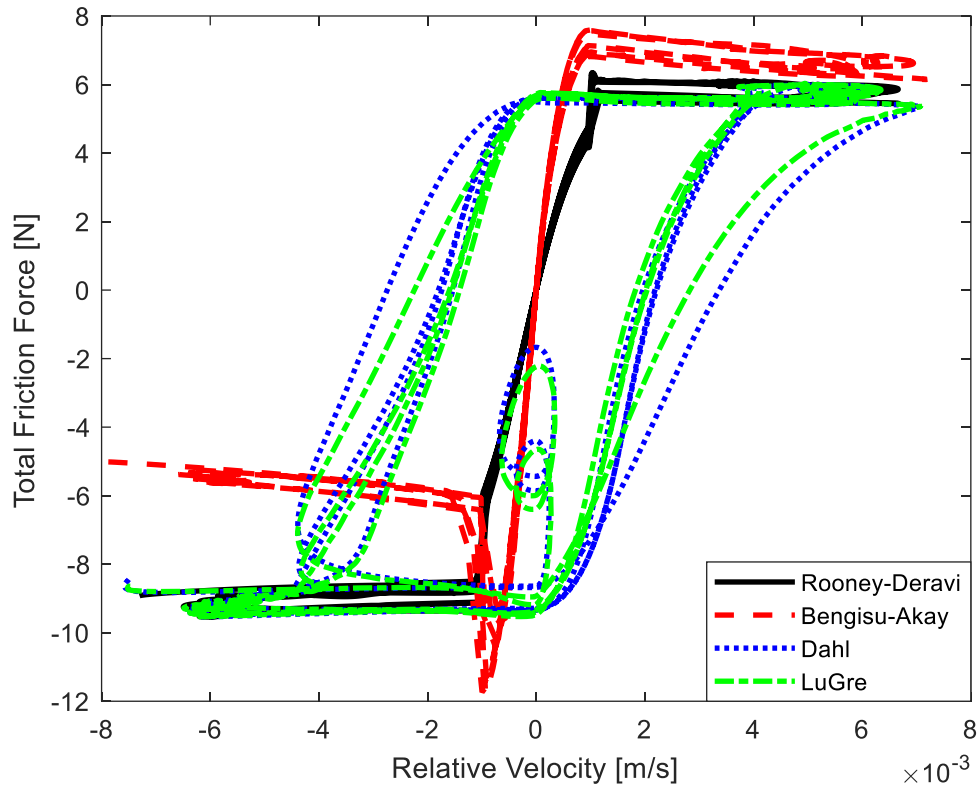


Figure 9.34 – Total friction force as function of the relative tangential velocity between the shock absorber bodies during the four-poster rig test.

The computational efficiency of the different friction force simulations was also studied for this dynamic test. Table 9.8 shows the total computational time and the number of function evaluations of each simulation.

Table 9.8 – Comparison of the simulation time and number of function evaluations for the four-poster rig test. Values inside brackets correspond to quantities normalized to the lowest value.

Simulation Scenario	CPU time [s]	Number of Function Evaluations
No Friction	269 (1.00)	3547 (1.00)
Rooney and Deravi	1015 (3.77)	16315 (4.60)
Bengisu and Akay	702 (2.61)	12175 (3.43)
Dahl	618 (2.30)	10645 (3.00)
LuGre	2668 (9.92)	50377 (14.20)

Without consideration of friction phenomena, the simulation has the lower computational time and the lower number of function evaluations, as expected. The most efficient model in this test was the Dahl model, opposed of what was anticipated, where the static models should have an advantage due to the non-existence of state variables that need to be integrated during the solving process. Once again,

the LuGre model exhibited a very low computational efficiency, yielding a total simulation time that is almost 10 times higher and a total number of function evaluations of around 14 times compared to the no friction ideal scenario simulation.

9.4 Summary and Discussion

In this chapter the effects that friction forces have on the dynamic behavior of a suspension system were analyzed. For this purpose, a multibody model of the FSUM car was created entirely in-house and implemented in the forward dynamics analysis program MUBODYNA. This included the definition of all the inertial properties and positions of the bodies, joints that connect the different bodies and force elements. The modeling of the tire-ground contact was done utilizing the Magic Formula Tire model where the manufacturer's raw tire data was fitted to the model to obtain the optimized characteristic parameters.

Regarding the specific study of friction forces within suspension systems, firstly were identified the contact zones between the different parts that constitute a shock absorber. For the subsequent analysis, two contact points were considered: (i) the piston's outer seal and inner walls of the shock absorber's body; (ii) the piston rod and scraper seal.

The implementation of friction phenomena in the cylindrical joint used to model the shock absorber in the framework of multibody dynamics was made by analyzing the four different kinematic reaction forces and moments. A novel methodology was proposed and explained in detail to determine the normal contact forces for the two contact points where friction can be generated on a SA that are required for the calculation of the friction forces.

For the obstacle avoidance maneuver, defined accordingly to the ISO 3888 standard, it was verified that the inclusion of friction forces in the dynamic behavior of the suspension results in a reduction of both compression and rebounding maximum forces compared to a non-friction ideal scenario. Furthermore, a more detailed analysis revealed that a 5% decrease of the total shock absorber force is verified when modeling friction phenomena for the rebounding phase, while for the compression phase, the friction forces only result in a 0.33% reduction of the total SA force. The amplitude of friction force results for both compression and rebounding phases are within the values reported in literature.

In terms of computational efficiency, the Bengisu and Akay model showed great results, very similar to the non-friction simulation, while the LuGre model exhibited higher computational effort both in terms of computational time and number of function evaluations. The inefficiency of the later model is

intuitively understood due to the additional state variables that need to be integrated during the solving process of the equations of motion.

The four-poster rig dynamic test was modeled considering a simplification of the tire-ground contact where a set of three spring-damper elements were used. These elements represent the tire's structural stiffness and damping in the three spatial directions.

The results obtained in this test are in accordance with the ones obtained from the obstacle avoidance test, where the inclusion of friction lead to a reduction of the compression and rebounding maximum forces, due to the additional resistance to the shock absorbers motion provoked by friction. The total friction force values verified in this test were around 50% lower compared to the ones of the evasive maneuver test, a fact that can be explained by the non-existence of lateral forces acting on the tires and the suspension components, thus reducing the kinematic reaction forces and consequently the normal contact forces and the friction forces generated.

The efficiency analysis of the different models yielded the Dahl model as the most efficient on this test and the LuGre as the least efficient. In one hand, the Dahl model results were unusual for this type of dynamic models, since it was expected that the static friction models (Rooney and Deravi and Bengisu and Akay models) required less computational time. On the other hand, the behavior of the LuGre model was expected, due to the existence of state variables.

In conclusion, it was confirmed that friction force has a non-negligible effect on the dynamic behavior of suspension systems. Although it was found that in the global perspective of vehicle dynamics the friction force levels only result in a reduction of compression and rebounding maximum forces of 0.33% and 5%, respectively, these percentual differences can have a major impact in suspension setups for racing setups, as well as for comfort analysis assessments for passenger cars. It is expected that if the modeling of the shock absorber's perfect kinematic cylindrical joint is substituted by a dynamic reaction force constraint, considering clearances and contact-impact events, that the normal contact forces will have higher values and, consequently, the tangential friction forces verified in the suspension system will also be higher.

10. CONCLUSION

10.1 Concluding Remarks

The study presented in this work provides a critical and comprehensive analysis of friction force models and their application to dynamical systems, emphasizing their theoretical formulations, practical implementations, and impacts on the dynamic behavior of mechanical systems. The work underscores the essential role of friction modeling in modern mechanical engineering, where accurate simulation tools are indispensable for designing efficient, reliable, and high-performance systems and mechanisms.

Friction is a complex and multifaceted phenomenon that significantly influences the dynamic behavior of mechanical systems. The review of historical developments demonstrated the remarkable progress from the rudimentary concepts of friction proposed by Leonardo da Vinci to the sophisticated mathematical and phenomenological models available today. Fundamental laws of friction, such as those proposed by Amontons and Coulomb, form the basis for modern friction models, which attempt to capture the complex dissipative phenomena associated to friction.

The distinction between static and dynamic models emerged as a central theme. The analysis of various regularized approaches to static and dynamic friction models was conducted using a 1 degree of freedom benchmark problem widely recognized by the scientific community, for which an intuitive graphical interface was created in MATLAB. This benchmark problem provided a controlled environment to compare the computational performance, physical realism, and ability of different models to incorporate complex friction phenomena. A broad analysis and review of various dry friction force models commonly implemented in multibody simulations was presented in the context of this work. It was verified that static models without stiction prioritize computational efficiency, making them suitable for applications where simplicity and numerical efficiency are of paramount importance. However, their inability to represent critical phenomena like stiction limits their applicability for scenarios requiring high fidelity, such as robotics and precise control applications. On the other hand, static models with stiction provide enhanced realism by accurately distinguishing between sticking and sliding phases. While computationally more demanding, these models are invaluable for applications requiring precise simulation of transient and steady-state behaviors.

Dynamic friction models represent a significant advancement, capturing complex phenomena such as hysteresis, pre-sliding displacement, and velocity-dependent effects such as non-reversibility. The inclusion of state variables, such as the bristle deflection, allows these models to replicate friction's

complex behaviors across a wide range of conditions. Despite their increased computational cost and the need for parameter calibration, dynamic models are particularly suited for high-precision applications and systems where a detailed understanding of friction dynamics is critical.

The results highlighted the trade-offs between computational efficiency and fidelity, emphasizing the importance of tailoring model selection to the specific requirements of the application in analysis. This benchmark study also underscored the critical role of solver choice in ensuring convergence and stability in numerical simulations.

The experimental apparatus designed and implemented in this study highlights the importance of tailored experimental setups to collect accurate friction force data and validate theoretical findings by refining model parameters. Although challenges related to manufacturing precision and data acquisition limitations were identified, the apparatus successfully validated key friction phenomena such as stiction, hysteresis, and non-reversibility. This validation not only reinforced the theoretical findings but also provided a basis for parameter optimization of advanced friction models like LuGre.

Optimization results demonstrated the limitations of using generalized parameters from literature. System-specific parameter calibration significantly improved the ability of models to capture real-world friction phenomena, underlining the necessity of experimental data for accurate friction modeling. The genetic algorithm employed in this work proved to be an effective tool for achieving robust parameter optimization, ensuring that the simulated dynamic behavior closely aligns with the physical performance of mechanical systems.

The complex multibody case study related to the suspension system of a Formula Student type car, underscored the importance of considering friction phenomena in the dynamic behavior of vehicles and their subsystems. For the FSUM car suspension system, friction modeling revealed that dissipative effects on shock absorber dynamics result in the reduction of the total capacity of compression and rebounding motions of SA. While these reductions appeared small in magnitude, their implications for racing setups and passenger comfort assessments are significant, particularly when considering the stringent performance requirements in these domains.

The research presented in this work emphasizes the critical role of friction modeling in advancing the reliability and realism of simulations for dynamic mechanical systems. By integrating increasingly sophisticated friction models, engineers can optimize performance, durability, and energy efficiency across diverse applications, ranging from automotive and aerospace systems to robotics and control applications.

Future research should aim to address the identified challenges, including the development of models that balance computational efficiency with the ability to capture complex friction phenomena. Advancements in experimental techniques, such as the adoption of higher-frequency data acquisition and non-intrusive and high-precision measurement technologies, will further enhance the fidelity of parameter identification processes. Moreover, extending the analysis to more complex multibody systems and incorporating additional phenomena like wear and thermal effects can provide deeper insights into the role of friction in real-world engineering systems.

10.2 Future Works

This work presented an extensive, broad and critical overview of the influence of friction in the dynamic behavior of mechanical systems from numerical and experimental approaches. Nonetheless, there are still various improvements that can be achieved for both numerical and experimental approaches. In the following topics, some considerations about the experimental apparatus designed and built in the context of this work as well as the complex multibody vehicle model are exposed, which can lead to a variety of future research themes in order to enhance the prediction of the dynamic behavior of mechanical systems considering dissipative phenomena associated with friction.

10.2.1 Experimental Apparatus

- Improvement of the design of the experimental apparatus to increase its versatility for testing different combinations of normal loads and contact materials.
- Improvement of the manufacturing processes for obtaining the different parts that compose the structure of the experimental apparatus, ensuring that geometric and dimensional tolerances are fulfilled.
- Select and implement advanced technologies to enable more precise determination of the slider's displacement, velocity, and acceleration. These technologies should preferably be contact-free to avoid influencing the dynamic behavior of the system. For velocity, for example, a laser Doppler velocimeter can be utilized and for acceleration, accelerometers are recommended. It is not advised to determine acceleration solely by using accelerometers and then perform successive integrations to obtain velocity and position, as indirect measurements may not fully represent the system's dynamics and can introduce errors in the experimental data.

- In case of opting for a non-real time approach to determine the kinematic properties of the slider, it is advised that a higher resolution and frame rate camera is used, as well as a more accurate tracking software such as TEMA lite, for example, which is an industry standard.
- Once a substantial amount of experimental data is collected, additional studies can be conducted using machine learning algorithms to predict the dynamic behavior of systems under different velocity ranges, normal loads, contact materials, and lubrication conditions. These studies are crucial for better understanding the dynamics of such systems considering friction phenomena. Improved parameter estimation from these studies would enhance the numerical friction force models and enable the development of precise control algorithms for mechanical and mechatronic systems capable of compensating for dissipative phenomena.

10.2.2 FSUM Car Multibody Model

- Creation of a flexible multibody model of the FSUM car with clearances, since the compliance of materials and the contact-impact events within the various components of the subsystems of the car can significantly affect the dynamics of the car. In this case, the reaction forces could be calculated through finite element software, which would then feed the car's multibody model in a co-simulation approach. It would also be interesting to study the effects of lubrication fluids in the shock absorber through computational fluid dynamics (CFD) co-simulation.
- Creation of a digital twin of the car and hardware-in-the-loop simulations to feed the numerical model in order to predict a greater variety of dynamic tests, reducing the time and increasing the computational efficiency of the model.
- Creation of an active suspension system for the FSUM multibody model to compensate for the dissipative effects of friction within the suspension subsystem.
- Study of wear and aging effects of how components, such as bushings and shock absorbers, age and degrade over time could help predict their behavior under varying load and environmental conditions. This can be integrated into digital twins to simulate how wear and tear affect vehicle dynamics and safety.
- Multi-scale friction simulation where the different scales at which friction occurs in suspension systems are studied. A macro-scale could be used for joints and a micro-scale for surfaces in contact, which would involve coupling simulations of micro-level surface interactions with macro-level system dynamics, improving the overall dynamic performance of suspension systems.

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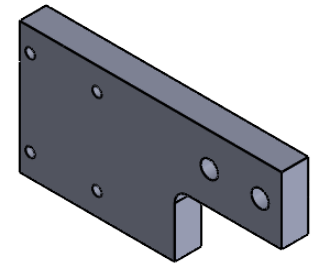
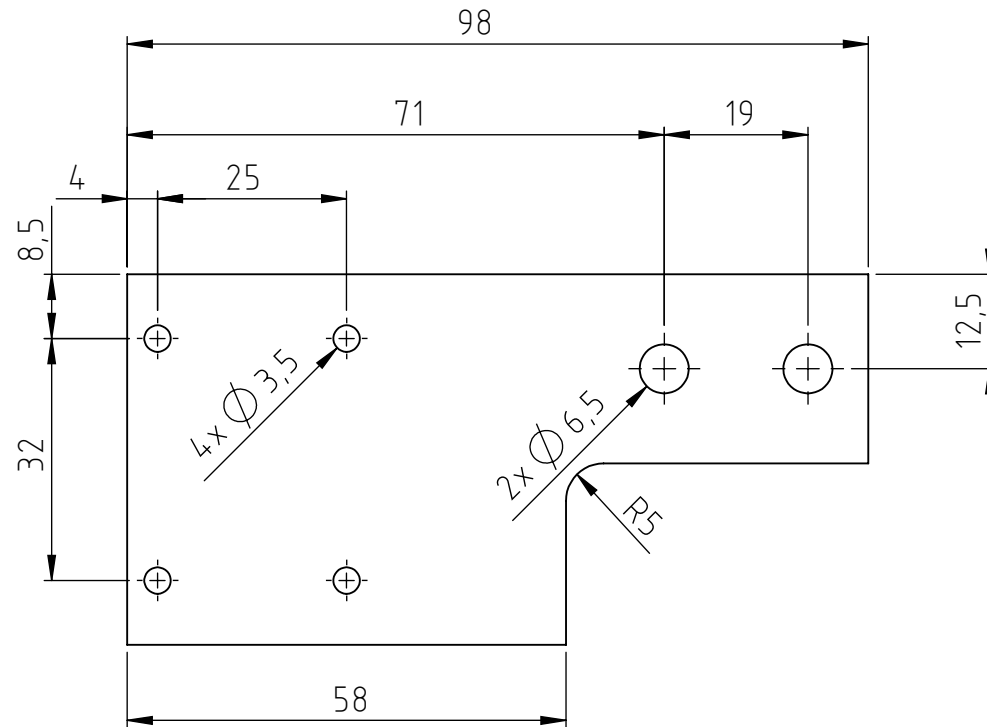
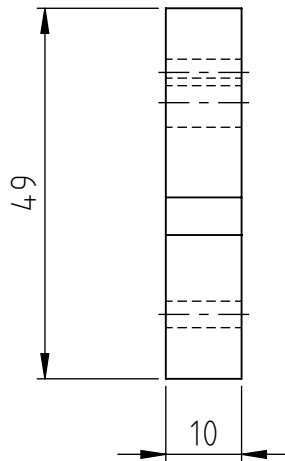
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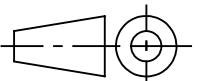
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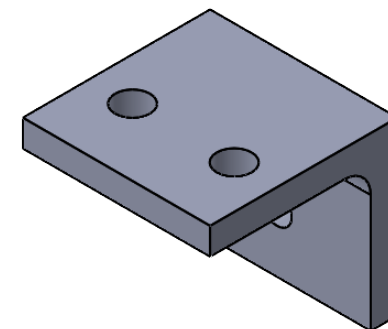
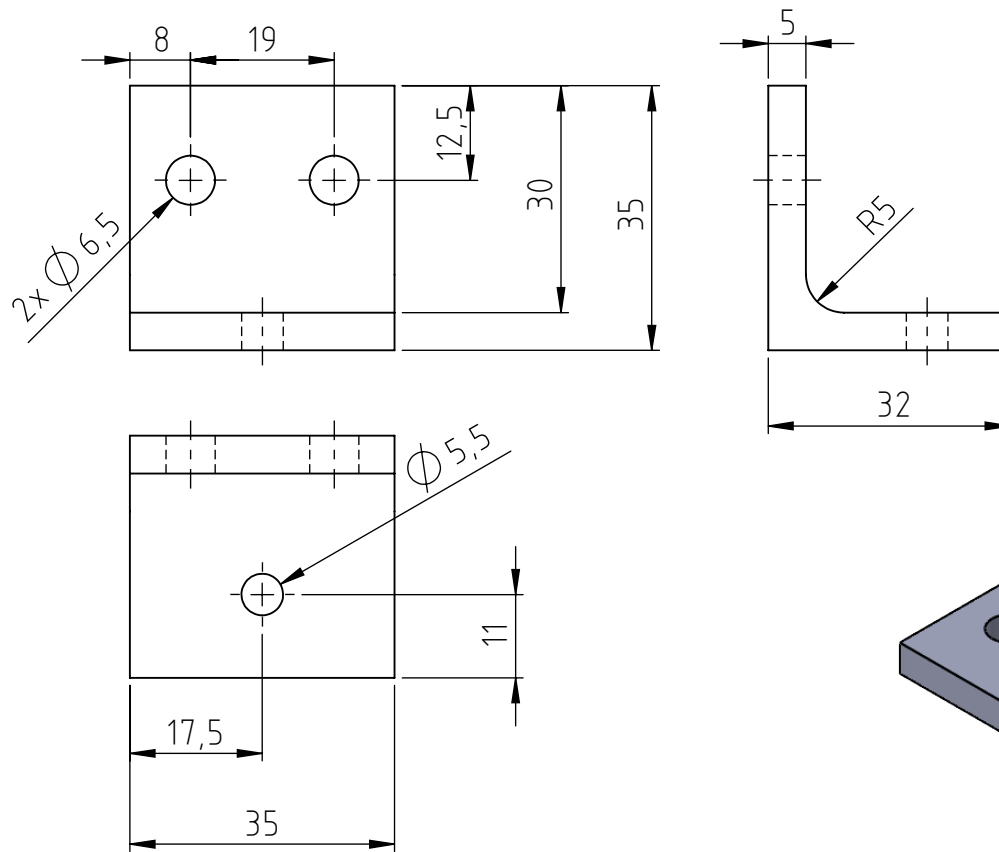
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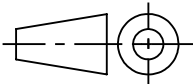
APPENDIX I – EXPERIMENTAL APPARATUS TECHNICAL DRAWINGS

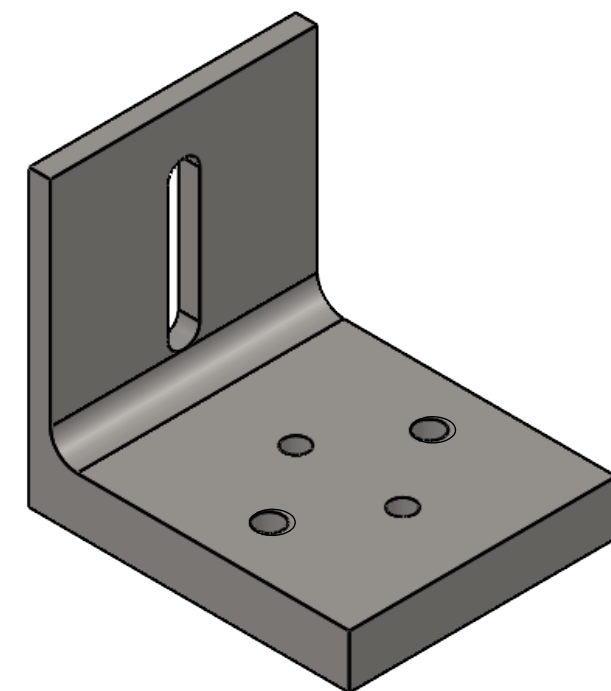
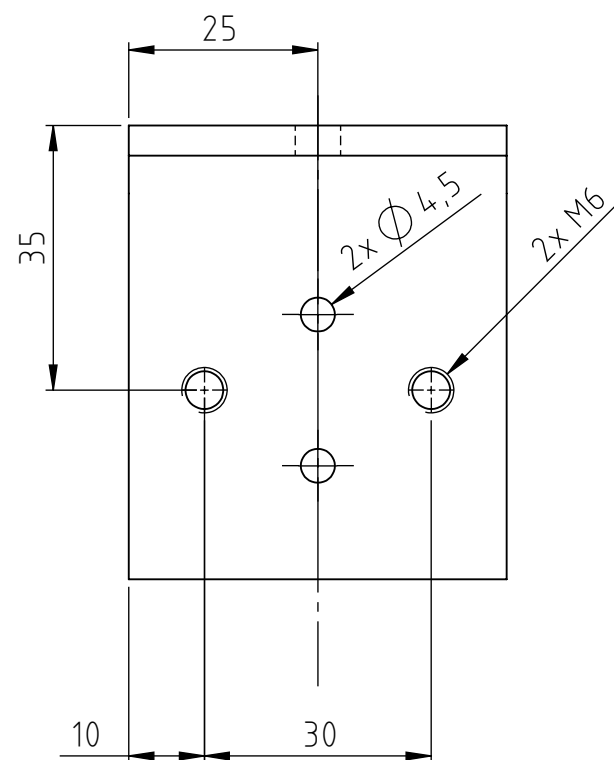
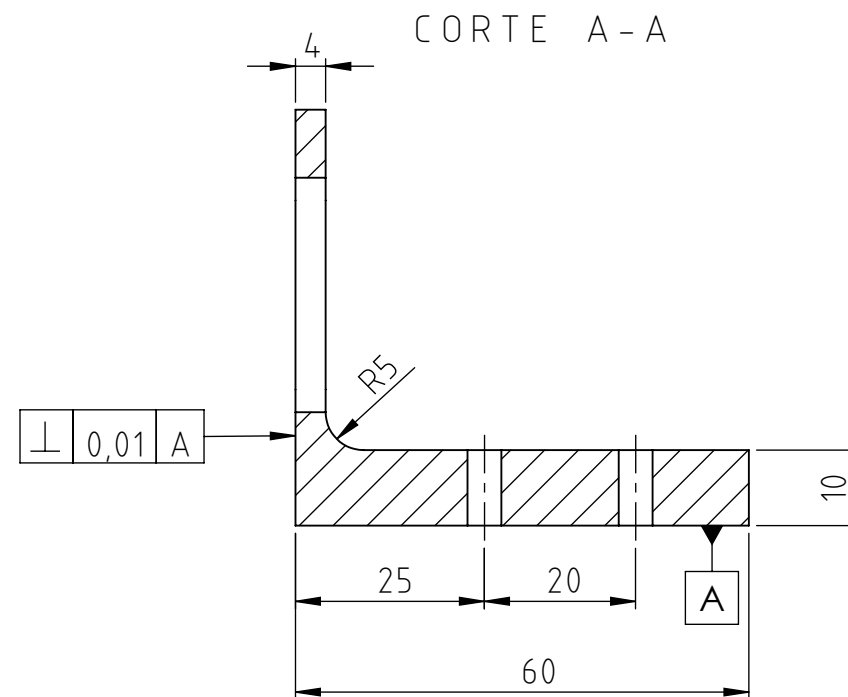
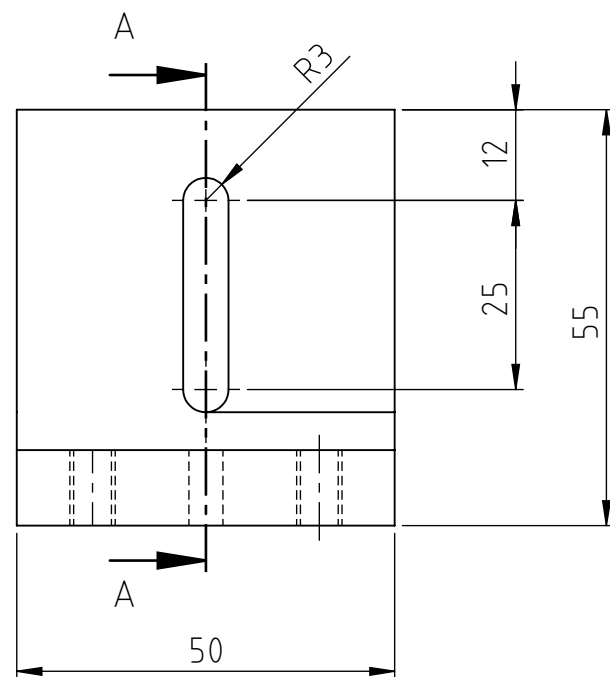
In this Appendix are presented the different technical drawings elaborated for the tailored-made parts of the experimental apparatus that were entirely manufactured in the workshops of the Mechanical Engineering Department of Universidade do Minho.

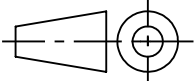


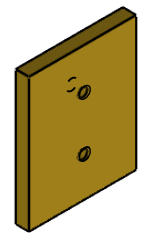
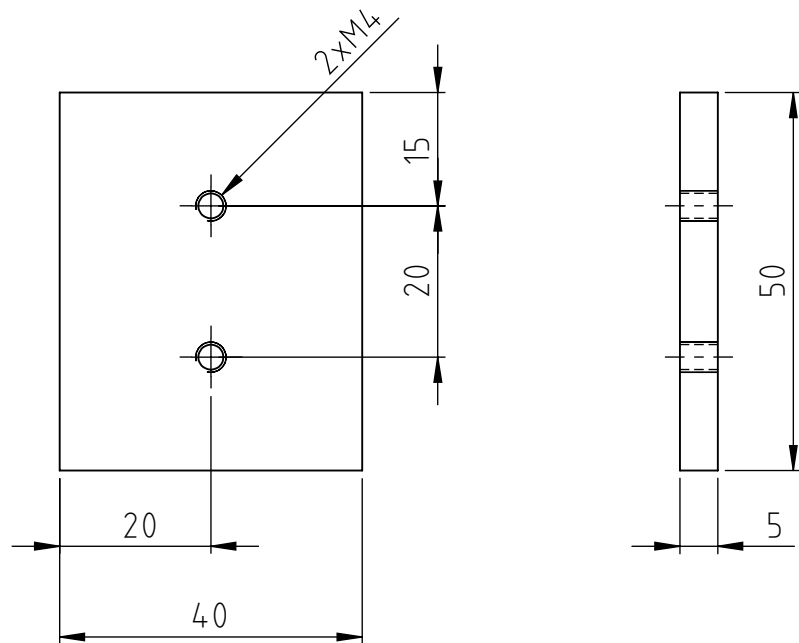
TOLERÂNCIAS GERAIS (mm)								
ATÉ 6	DE 6 A 30	DE 30 A 120	DE 120 A 315	DE 315 A 1000	DE 1000 A 2000	DE 2000 A 4000	ACIMA DE 4000	
+0,1	+0,2	+0,3	+0,5	+0,8	+1,2	+2,0	+3,0	
TOLERÂNCIAS ANGULARES			DATA	ASSIN.	UNIVERSIDADE DO MINHO Departamento de Engenharia Mecânica			
		Desenh.	17/09/2024	Francisco Novais				
		Verific.	14/01/2025	Francisco Novais				
DE 3 A 6	±10'	ESCALA 1:1	skate_plate					
DE 6 A 30	±20'							
DE 30 A 120	±30'	Toler.						
ACIMA DE 120	±120°							
								Substituí:
								Substituído:

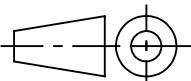


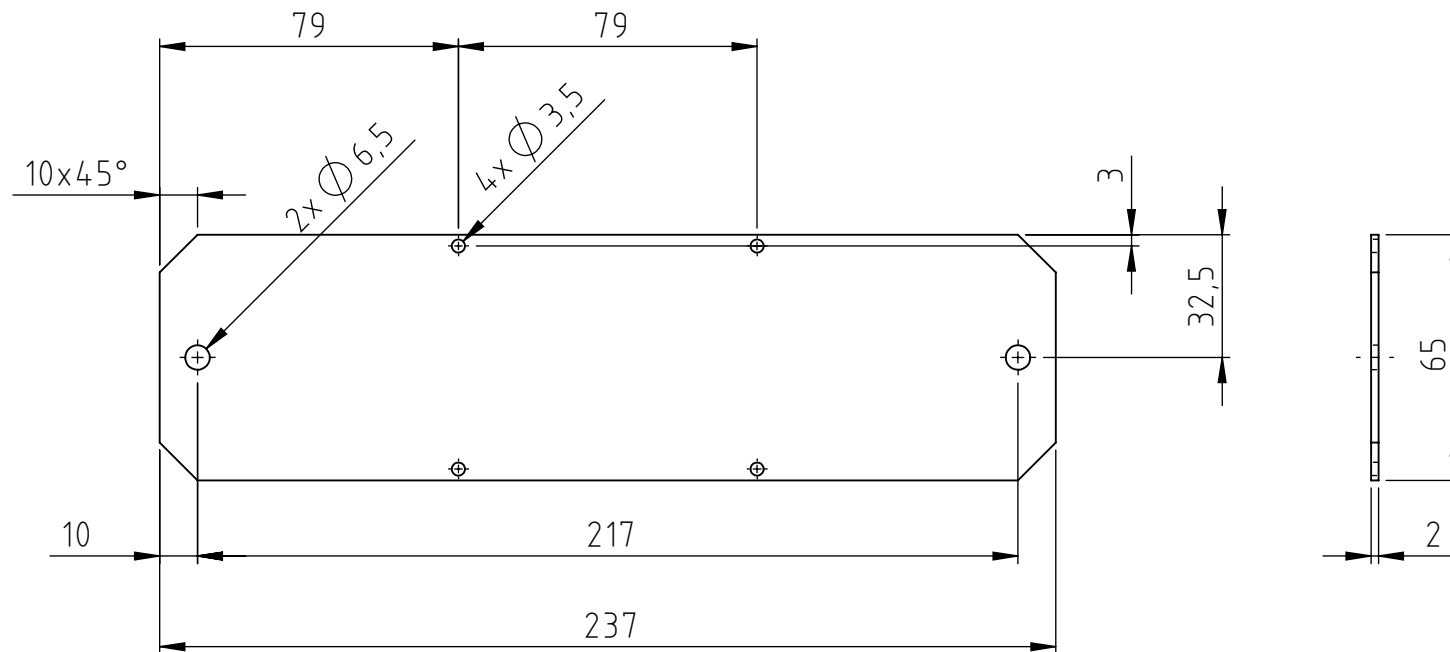
TOLERÂNCIAS GERAIS (mm)								
ATÉ 6	DE 6 A 30	DE 30 A 120	DE 120 A 315	DE 315 A 1000	DE 1000 A 2000	DE 2000 A 4000	ACIMA DE 4000	
+0,1	+0,2	+0,3	+0,5	+0,8	+1,2	+2,0	+3,0	
TOLERÂNCIAS ANGULARES			DATA	ASSIN.	UNIVERSIDADE DO MINHO Departamento de Engenharia Mecânica			
		Desenh.	17/09/2024	Francisco Novais				
		Verific.	29/09/2024	Francisco Novais				
DE 3 A 6	±10'	ESCALA 1:1 Toler.	load_cell_L					
DE 6 A 30	±20'							
DE 30 A 120	±30'							
ACIMA DE 120	±120°							
								Substituído:
								Substituído:

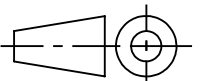


TOLERÂNCIAS GERAIS (mm)								
ATÉ 6	DE 6 A 30	DE 30 A 120	DE 120 A 315	DE 315 A 1000	DE 1000 A 2000	DE 2000 A 4000	ACIMA DE 4000	
+0,1	+0,2	+0,3	+0,5	+0,8	+1,2	+2,0	+3,0	
TOLERÂNCIAS ANGULARES			DATA	ASSIN.	UNIVERSIDADE DO MINHO Departamento de Engenharia Mecânica			
		Desenh.	25/04/2024	Francisco Novais				
		Verific.	14/01/2025	Francisco Novais				
DE 3 A 6	±10'	ESCALA 1:1	slider					
DE 6 A 30	±20'							
DE 30 A 120	±30'	Toler.						
ACIMA DE 120	±120°							
								Substituí:
								Substituído:



TOLERÂNCIAS GERAIS (mm)								
ATÉ 6	DE 6 A 30	DE 30 A 120	DE 120 A 315	DE 315 A 1000	DE 1000 A 2000	DE 2000 A 4000	ACIMA DE 4000	
+0,1	+0,2	+0,3	+0,5	+0,8	+1,2	+2,0	+3,0	
TOLERÂNCIAS ANGULARES			DATA	ASSIN.	UNIVERSIDADE DO MINHO Departamento de Engenharia Mecânica			
		Desenh.	25/04/2024	Francisco Novais				
		Verific.	14/01/2025	Francisco Novais				
DE 3 A 6	±10'	ESCALA 1:1	slider_changeable_material					
DE 6 A 30	±20'							
DE 30 A 120	±30'	Toler.						
ACIMA DE 120	±120°							
								Substituído:
								Substituído:



TOLERÂNCIAS GERAIS (mm)								
ATÉ 6	DE 6 A 30	DE 30 A 120	DE 120 A 315	DE 315 A 1000	DE 1000 A 2000	DE 2000 A 4000	ACIMA DE 4000	
+0,1	+0,2	+0,3	+0,5	+0,8	+1,2	+2,0	+3,0	
TOLERÂNCIAS ANGULARES			DATA	ASSIN.	UNIVERSIDADE DO MINHO Departamento de Engenharia Mecânica			
		Desenh.	25/04/2024	Francisco Novais				
		Verific.	29/09/2024	Francisco Novais				
DE 3 A 6	±10'	ESCALA 1:2 Toler.	workbench_test_sheet					
DE 6 A 30	±20'							
DE 30 A 120	±30'							
ACIMA DE 120	±120°							
								Substituí:
								Substituído:

APPENDIX II – INERTIAL PROPERTIES OF THE VEHICLE MBD MODEL

In this Appendix are presented the inertial properties of the left-side components of the multibody model of the FSUM car. The inertial properties of the right-side components were defined taking advantage of the symmetry of the model.

Assembly	Body number	Components description	Mass [kg]	Inertia Matrix [kg/m ²] $\begin{bmatrix} \xi\xi & \eta\eta & \zeta\zeta \end{bmatrix}$
Chassis	1	Chassis + driver	275.885	$\begin{bmatrix} 15.355358 & 110.034295 & 107.989598 \end{bmatrix}$
Steering	2	Steering box	0.500	$\begin{bmatrix} 0.001857 & 0.000096 & 0.001887 \end{bmatrix}$
	3	Left steering tie-rod	0.204	$\begin{bmatrix} 0.003417 & 0.003416 & 0.000009 \end{bmatrix}$
Front suspension	4	Left lower arm	0.887	$\begin{bmatrix} 0.010950 & 0.007666 & 0.018536 \end{bmatrix}$
	5	Left upright/knuckle	1.157	$\begin{bmatrix} 0.006148 & 0.001660 & 0.006783 \end{bmatrix}$
	6	Left wheel	9.583	$\begin{bmatrix} 0.374506 & 0.224036 & 0.224032 \end{bmatrix}$
	7	Left bottom spring-damper	0.623	$\begin{bmatrix} 0.000209 & 0.001466 & 0.001476 \end{bmatrix}$
	8	Left top damper	0.490	$\begin{bmatrix} 0.000326 & 0.000529 & 0.000796 \end{bmatrix}$
	9	Left top arm	0.420	$\begin{bmatrix} 0.001541 & 0.001651 & 0.003150 \end{bmatrix}$
Rear suspension	10	Left lower arm	0.586	$\begin{bmatrix} 0.010323 & 0.008801 & 0.019070 \end{bmatrix}$
	11	Left upright/knuckle	1.027	$\begin{bmatrix} 0.004115 & 0.001550 & 0.004893 \end{bmatrix}$
	12	Left wheel	10.398	$\begin{bmatrix} 0.374506 & 0.224036 & 0.224032 \end{bmatrix}$
	13	Left top arm	0.727	$\begin{bmatrix} 0.003478 & 0.007231 & 0.010648 \end{bmatrix}$
	14	Left bottom spring-damper	0.623	$\begin{bmatrix} 0.000209 & 0.001466 & 0.001476 \end{bmatrix}$
	15	Left top damper	0.490	$\begin{bmatrix} 0.000326 & 0.000529 & 0.000796 \end{bmatrix}$
	16	Left toe bar	0.124	$\begin{bmatrix} 0.000004 & 0.001239 & 0.001238 \end{bmatrix}$

APPENDIX III – CENTER OF GRAVITY AND EULER PARAMETERS OF THE VEHICLE MBD MODEL

Assembly	Body number	Center of gravity [m] (x, y, z)	Euler parameters (e_0, e_1, e_2, e_3)
Chassis	1	(-0.7509, 0.0000, 0.3181)	(1.000000, 0.000000, 0.000000, 0.000000)
Steering	2	(-0.0512, 0.0000, 0.1656)	(1.000000, 0.000000, 0.000000, 0.000000)
	3	(-0.0676, 0.3137, 0.1605)	(0.358771, 0.393086, -0.572411, 0.623789)
Front left suspension	4	(0.0045, 0.3707, 0.1302)	(0.999952, -0.009774, 0.000000, 0.000000)
	5	(-0.0004, 0.5497, 0.2449)	(0.695789, 0.717363, -0.025674, 0.024658)
	6	(0.0001, 0.6281, 0.2569)	(0.702403, 0.080364, -0.058848, -0.704775)
	7	(0.0087, 0.4133, 0.2100)	(0.673821, 0.214396, 0.673821, 0.214396)
	8	(0.0091, 0.3882, 0.2963)	(0.673821, 0.214396, 0.673821, 0.214396)
	9	(-0.0034, 0.3965, 0.3703)	(0.998688, 0.051203, 0.000000, 0.000000)
Rear left suspension	10	(-1.5463, 0.3796, 0.1472)	(0.000000, 0.000000, 0.001765, 0.999998)
	11	(-1.5517, 0.5497, 0.2685)	(0.700541, 0.712732, -0.025244, 0.024860)
	12	(-1.5497, 0.6316, 0.2563)	(0.660508, 0.252577, -0.241171, -0.664658)
	13	(-1.5544, 0.4223, 0.3567)	(0.995448, 0.095310, 0.000000, 0.000000)
	14	(-1.5583, 0.3793, 0.4244)	(0.628658, 0.323711, 0.628658, 0.323711)
	15	(-1.5580, 0.3211, 0.5021)	(0.628658, 0.323711, 0.628658, 0.323711)
	16	(-1.6964, 0.3853, 0.3040)	(0.625532, -0.570135, -0.418232, 0.329753)

APPENDIX IV – KINEMATIC JOINTS OF THE VEHICLE MBD MODEL

Assembly	N	Joint type	Body i	Body j	ξ_i^P $S_{\xi i}$	η_i^P $S_{\eta i}$	ζ_i^P $S_{\zeta i}$	ξ_j^P $S_{\xi j}$	η_j^P $S_{\eta j}$	ζ_j^P $S_{\zeta j}$
Front Suspension	1	REV	1	2	0.7509	0.2200	-0.1853	-0.0045	-0.1507	-0.0003
					1.0000	0.0000	0.0000	1.0000	0.0000	0.0000
	2	REV	1	4	0.7509	0.3373	0.0460	0.0034	-0.0596	-0.0002
					1.0000	0.0000	0.0000	-1.0000	0.0000	0.0000
	3	SPH	2	3	0.0046	0.1507	-0.0003	0.0011	-0.1175	0.0319
					-	-	-	-	-	-
	4	SPH	3	4	0.0011	0.1365	0.0601	-0.0055	0.0894	-0.0002
					-	-	-	-	-	-
	5	REV	3	16	0.0011	0.0095	-0.0114	0.0674	0.0000	0.0000
					0.0000	0.0000	-1.0000	-1.0000	0.0000	0.0000
	6	UNI	2	23	0.0046	0.0847	0.0210	0.0740	-0.0003	0.0004
					1.0000	0.0000	0.0000	0.0000	1.0000	0.0000
	7	CYL	24	23	-0.0592	-0.0297	0.0000	0.0740	-0.0003	0.0004
					1.0000	0.0000	0.0000	1.0000	0.0000	0.0000
	8	UNI	1	24	0.7600	0.3297	0.0093	-0.0592	-0.0297	0.0000
					1.0000	0.0000	0.0000	0.0000	1.0000	0.0000
Rear Suspension	9	REV	1	5	-0.7991	0.2000	-0.1714	0.0037	0.1796	-0.0011
					1.0000	0.0000	0.0000	1.0000	0.0000	0.0000
	10	REV	1	7	-0.7991	0.3001	0.0143	0.0043	-0.1246	-0.0007
					1.0000	0.0000	0.0000	1.0000	0.0000	0.0000
	11	SPH	5	6	-0.0042	-0.1585	0.0013	0.0010	-0.1207	0.0137
					-	-	-	-	-	-
	12	SPH	6	7	0.0010	0.1033	0.0495	-0.0036	0.0764	-0.0007
					-	-	-	-	-	-
	13	SPH	6	8	-0.0870	0.0613	0.0532	0.1239	0.0000	0.0000
					-	-	-	-	-	-
	14	SPH	1	8	-0.9991	0.2751	-0.0321	-0.1239	0.0000	0.0000
	15	REV	6	17	0.0010	-0.0137	-0.0038	0.0779	0.0000	0.0000
					0.0000	0.0000	1.0000	-1.0000	0.0000	0.0000
	16	UNI	7	25	-0.0036	0.0214	0.0211	0.0740	-0.0003	0.0004
					1.0000	0.0000	0.0000	0.0000	1.0000	0.0000

	17	CYL	26	25	-0.0592	-0.0297	0.0000	0.0740	-0.0003	0.0004
					1.0000	0.0000	0.0000	1.0000	0.0000	0.0000
	18	UNI	1	26	-0.8070	0.2557	0.1942	-0.0592	-0.0297	0.0000
					1.0000	0.0000	0.0000	0.0000	1.0000	0.0000
Steering	19	TRAN	1	22	0.6967	0.0000	-0.1515	-0.0030	0.1000	0.0010
System					0.0000	1.0000	0.0000	-	-	-
	20	SPH	3	20	-0.0868	-0.0825	0.0698	0.0000	0.0000	-0.1696
					-	-	-	-	-	-
	21	SPH	20	22	0.0000	0.0000	0.1696	-0.0028	0.1448	0.0009
					-	-	-	-	-	-

APPENDIX V – MAGIC FORMULA TIRE MODEL EQUATIONS

- Longitudinal Force (Pure Longitudinal Slip, $\alpha = 0$)

$$F_{xo} = D_x \sin \left[C_x \arctan \left\{ B_x \kappa_x - E_x \left(B_x \kappa_x - \arctan(B_x \kappa_x) \right) \right\} \right] + S_{Vx} \quad (A4.1)$$

$$\kappa_x = \kappa + S_{Hx} \quad (A4.2)$$

$$C_x = p_{Cx1} \cdot \lambda_{Cx} \quad (> 0) \quad (A4.3)$$

$$D_x = \mu_x \cdot F_z \cdot \zeta_1 \quad (> 0) \quad (A4.4)$$

$$\mu_x = (p_{Dx1} + p_{Dx2} df_z) (1 + p_{px3} dp_i + p_{px4} dp_i^2) (1 - p_{Dx3} \gamma^2) \cdot \lambda_{\mu x}^* \quad (A4.5)$$

$$E_x = (p_{Ex1} + p_{Ex2} df_z + p_{Ex3} + df_z^2) \{1 - p_{Ex4} \operatorname{sgn}(\kappa_x)\} \cdot \lambda_{Ex} \quad (\leq 1) \quad (A4.6)$$

$$K_{xx} = F_z (p_{Kx1} + p_{Kx2} df_z) \exp(p_{Kx3} df_z) (1 + p_{px1} dp_i + p_{px2} dp_i^2) \quad (A4.7)$$

$$B_x = K_{xx} / (C_x D_x + \varepsilon_x) \quad (A4.8)$$

$$S_{Hx} = (p_{Hx1} + p_{Hx2} df_z) \cdot \lambda_{Hx} \quad (A4.9)$$

$$S_{Vx} = F_z \cdot (p_{Vx1} + p_{Vx2} df_z) \cdot \lambda_{Vx} \lambda_{\mu x}' \zeta_1 \quad (A4.10)$$

- Lateral Force (Pure Side Slip, $\kappa = 0$)

$$F_{yo} = D_y \sin \left[C_y \arctan \left\{ B_y \alpha_y - E_y \left(B_y \alpha_y - \arctan(B_y \alpha_y) \right) \right\} \right] + S_{Vy} \quad (A4.11)$$

$$\alpha_y = \alpha^* + S_{Hy} \quad (A4.12)$$

$$C_y = p_{Cy1} \cdot \lambda_{Cy} \quad (> 0) \quad (A4.13)$$

$$D_y = \mu_y \cdot F_z \cdot \zeta_2 \quad (A4.14)$$

$$\mu_y = (p_{Dy1} + p_{Dy2} df_z) (1 + p_{py3} dp_i + p_{py4} dp_i^2) (1 - p_{Dy3} \gamma^{*2}) \lambda_{\mu y}^* \quad (A4.15)$$

$$E_y = (p_{Ey1} + p_{Ey2} df_z) \{1 + p_{Ey5} \gamma^{*2} - (p_{Ey3} + p_{Ey4} \gamma^*) \operatorname{sgn}(\alpha_y)\} \lambda_{Ey} \quad (\leq 1) \quad (A4.16)$$

$$K_{y\alpha} = p_{Ky1} F_{zo}' (1 + p_{py1} dp_i) (1 - p_{Ky3} |\gamma^*|) \cdot \sin \left[p_{Ky4} \arctan \left\{ \frac{F_z / F_{z0}'}{(p_{Ky2} + p_{Ky5} \gamma^{*2}) (1 + p_{py2} dp_i)} \right\} \right] \cdot \zeta_3 \lambda_{Ky\alpha} \quad (A4.17)$$

$$B_y = K_{y\alpha} / (C_y D_y + \varepsilon_y) \quad (\text{A4.18})$$

$$S_{Hy} = (p_{Hy1} + p_{Hy2} df_z) \lambda_{Hy} + \frac{K_{y\gamma 0} \gamma^* - S_{Vy\gamma}}{K_{y\alpha} + \varepsilon_K} \zeta_0 + \zeta_4 - 1 \quad (\text{A4.19})$$

$$S_{Vy\gamma} = F_z \cdot (p_{Vy3} + p_{Vy4} df_z) \gamma^* \cdot \lambda_{Ky\gamma} \lambda'_{\mu y} \zeta_2 \quad (\text{A4.20})$$

$$S_{Vy} = F_z \cdot (p_{Vy1} + p_{Vy2} df_z) \cdot \lambda_{Vy} \lambda'_{\mu y} \zeta_2 + S_{Vy\gamma} \quad (\text{A4.21})$$

$$K_{y\gamma 0} = F_z \cdot (p_{Ky6} + p_{Ky7} df_z) (1 + p_{py5} dp_i) \cdot \lambda_{Ky\gamma} \quad (\text{A4.22})$$

- Aligning Torque (Pure Side Slip, $\kappa = 0$)

$$M_{zo} = M'_{zo} + M_{zro} \quad (\text{A4.23})$$

$$M'_{zo} = -t_0 \cdot F_{y0} \quad (\text{A4.24})$$

$$t_0 = t(\alpha_t) = D_t \cos \left[C_t \arctan \left\{ B_t \alpha_t - E_t (B_t \alpha - \arctan(B_t \alpha)) \right\} \right] \cdot \cos' \alpha \quad (\text{A4.25})$$

$$\alpha_t = \alpha^* + S_{Ht} \quad (\text{A4.26})$$

$$S_{Ht} = q_{Hz1} + q_{Hz2} df_z + (q_{Hz3} + q_{Hz4} df_z) \gamma^* \quad (\text{A4.27})$$

$$M_{zro} = M_{zr}(\alpha_r) = D_r \cos \left[C_r \arctan(B_r \alpha_r) \right] \cdot \cos' \alpha \quad (\text{A4.28})$$

$$\alpha_r = \alpha^* + S_{Hr} \quad (\text{A4.29})$$

$$S_{Hr} = S_{Hy} + S_{Vy} / K'_{y\alpha} \quad (\text{A4.30})$$

$$K'_{y\alpha} = K_{y\alpha} + \varepsilon_K \quad (\text{A4.31})$$

$$B_t = (q_{Bz1} + q_{Bz2} df_z + q_{Bz3} df_z^2) (1 + q_{Bz5} |\gamma^*| + q_{Bz6} \gamma^{*2}) \lambda_{Ky\alpha} / \lambda'_{\mu y} \quad (> 0) \quad (\text{A4.32})$$

$$C_t = q_{Cz1} \quad (> 0) \quad (\text{A4.33})$$

$$D_{to} = F_z \cdot (R_o / F'_{zo}) \cdot (q_{Dz1} + q_{Dz2} df_z) (1 - p_{pz1} dp_i) \cdot \lambda_t \cdot \text{sgn} V_{cx} \quad (\text{A4.34})$$

$$D_t = D_{to} \cdot (1 + q_{Dz3} |\gamma^*| + q_{Dz4} \gamma^{*2}) \cdot \zeta_5 \quad (\text{A4.35})$$

$$E_t = (q_{Ez1} + q_{Ez2} df_z + q_{Ez3} df_z^2) \cdot \left\{ 1 + (q_{Ez4} + q_{Ez5} \gamma^*) \frac{2}{\pi} \arctan(B_t C_t \alpha_t) \right\} \quad (\leq 1) \quad (\text{A4.36})$$

$$B_r = (q_{Bz9} \cdot \lambda_{Ky\alpha} / \lambda'_{\mu y} + q_{Bz10} B_y C_y) \cdot \zeta_6 \quad (\text{preferred: } q_{Bz9} = 0) \quad (\text{A4.37})$$

$$C_r = \zeta_7 \quad (\text{A4.38})$$

$$D_r = F_z R_o \left[(q_{Dz6} + q_{Dz7} df_z) \lambda_{Mr} \zeta_2 + \left\{ (q_{Dz8} + q_{Dz9} df_z) (1 + p_{pz2} dp_i) + (q_{Dz10} + q_{Dz11} df_z) |\gamma^*| \right\} \gamma^* \lambda_{Kz\gamma} \zeta_0 \right] \lambda_{\mu y}^* \operatorname{sgn} V_{cx} \cos' \alpha + \zeta_8 - 1 \quad (\text{A4.39})$$

$$K_{z\alpha o} = D_{to} K_{y\alpha} \quad (\text{A4.40})$$

$$K_{z\gamma o} = F_z R_o (q_{Dz8} + q_{Dz9} df_z) (1 + p_{pz2} dp_i) \lambda_{Kz\gamma} \lambda_{\mu y}^* - D_{to} K_{y\gamma o} \quad (\text{A4.41})$$

- Longitudinal Force (Combined Slip)

$$F_x = G_{x\alpha} \cdot F_{xo} \quad (\text{A4.42})$$

$$G_{x\alpha} = \cos \left[C_{x\alpha} \arctan \left\{ B_{x\alpha} \alpha_S - E_{x\alpha} (B_{x\alpha} \alpha_S - \arctan(B_{x\alpha} \alpha_S)) \right\} \right] / G_{x\alpha o} \quad (> 0) \quad (\text{A4.43})$$

$$G_{x\alpha o} = \cos \left[C_{x\alpha} \arctan \left\{ B_{x\alpha} S_{Hx\alpha} - E_{x\alpha} (B_{x\alpha} S_{Hx\alpha} - \arctan(B_{x\alpha} S_{Hx\alpha})) \right\} \right] \quad (\text{A4.44})$$

$$\alpha_S = \alpha^* + S_{Hx\alpha} \quad (\text{A4.45})$$

$$B_{x\alpha} = (r_{Bx1} + r_{Bx3} \gamma^{*2}) \cos \left[\arctan(r_{Bx2} \kappa) \right] \cdot \lambda_{x\alpha} \quad (> 0) \quad (\text{A4.46})$$

$$C_{x\alpha} = r_{Cx1} \quad (\text{A4.47})$$

$$E_{x\alpha} = r_{Ex1} + r_{Ex2} df_z \quad (\leq 1) \quad (\text{A4.48})$$

$$S_{Hx\alpha} = r_{Hx1} \quad (\text{A4.49})$$

- Lateral Force (Combined Slip)

$$F_y = G_{y\kappa} \cdot G_{yo} + S_{Vy\kappa} \quad (\text{A4.50})$$

$$G_{y\kappa} = \cos \left[C_{y\kappa} \arctan \left\{ B_{y\kappa} \kappa_S - E_{y\kappa} (B_{y\kappa} \kappa_S - \arctan(B_{y\kappa} \kappa_S)) \right\} \right] / G_{y\kappa o} \quad (> 0) \quad (\text{A4.51})$$

$$G_{y\kappa o} = \cos \left[C_{y\kappa} \arctan \left\{ B_{y\kappa} S_{Hy\kappa} - E_{y\kappa} (B_{y\kappa} S_{Hy\kappa} - \arctan(B_{y\kappa} S_{Hy\kappa})) \right\} \right] \quad (\text{A4.52})$$

$$\kappa_S = \kappa + S_{Hy\kappa} \quad (\text{A4.53})$$

$$B_{y\kappa} = (r_{By1} + r_{By4} \gamma^{*2}) \cos \left[\arctan \left\{ r_{By2} (\alpha^* - r_{By3}) \right\} \right] \cdot \lambda_{y\kappa} \quad (> 0) \quad (\text{A4.54})$$

$$C_{y\kappa} = r_{Cy1} \quad (\text{A4.55})$$

$$E_{y\kappa} = r_{Ey1} + r_{Ey2} df_z \quad (\leq 1) \quad (\text{A4.56})$$

$$S_{Hy\kappa} = r_{Hy1} + r_{Hy2} df_z \quad (\text{A4.57})$$

$$S_{V_{yK}} = D_{V_{yK}} \sin \left[r_{V_{y5}} \arctan(r_{V_{y6}} \kappa) \right] \cdot \lambda_{V_{yK}} \quad (\text{A4.58})$$

$$D_{V_{yK}} = \mu_y F_z \cdot (r_{V_{y1}} + r_{V_{y2}} df_z + r_{V_{y3}} \gamma^*) \cdot \cos \left[\arctan(r_{V_{y4}} \alpha^*) \right] \cdot \zeta_2 \quad (\text{A4.59})$$

- Overturning Couple

$$\begin{aligned} M_x = R_o F_z & \left[q_{sx1} \lambda_{VMx} - q_{sx2} \gamma (1 + p_{pMx1} dp_i) + q_{sx3} \frac{F_y}{F_{zo}} \right. \\ & + q_{sx4} \cos \left\{ q_{sx5} \arctan \left(q_{sx6} \frac{F_z}{F_{zo}} \right)^2 \right\} \sin \{ q_{sx7} \gamma + q_{sx8} \arctan \\ & \left. \left(q_{sx9} \frac{F_y}{F_{zo}} \right) \right\} + q_{sx10} \arctan \left(q_{sx11} \frac{F_z}{F_{zo}} \right) \cdot \gamma \left. \right] \cdot \lambda_{Mx} \end{aligned} \quad (\text{A4.60})$$

- Rolling Resistance Moment

$$\begin{aligned} M_y = F_z R_o & \left\{ q_{sy1} + q_{sy2} \frac{F_x}{F_{zo}} + q_{sy3} \left| \frac{V_x}{V_o} \right| + q_{sy4} \left(\frac{V_x}{V_o} \right)^4 \right. \\ & \left. + \left(q_{sy5} + q_{sy6} \frac{F_z}{F_{zo}} \right) \gamma^2 \right\} \left\{ \left(\frac{F_z}{F_{zo}} \right)^{q_{sy7}} \cdot \left(\frac{p_i}{p_{io}} \right)^{q_{sy8}} \right\} \cdot \lambda_{My} \end{aligned} \quad (\text{A4.61})$$

- Aligning Torque

$$M_z = M'_z + M_{zr} + s \cdot F'_x \quad (\text{A4.62})$$

$$M'_z = -t \cdot F'_y \quad (\text{A4.63})$$

$$t = D_t \cos \left[C_t \arctan \left\{ B_t \alpha_{t,eq} - E_t \left(B_t \alpha_{t,eq} - \arctan(B_t \alpha_{t,eq}) \right) \right\} \right] \cos' \alpha \quad (\text{A4.64})$$

$$F'_y = G_{yK} \cdot F_{y0} \quad (\text{A4.65})$$

$$M_{zr} = D_r \cos \left[C_r \arctan(B_r \alpha_{r,eq}) \right] \quad (\text{A4.66})$$

$$s = R_o \cdot \left\{ s_{sz1} + s_{sz2} (F_y / F'_{zo}) + (s_{sz3} + s_{sz4} df_z) \gamma^* \right\} \cdot \lambda_s \quad (\text{A4.67})$$

$$\alpha_{t,eq} = \sqrt{\alpha_t^2 + \left(\frac{K_{xK}}{K_{y\alpha}} \right)^2} \kappa^2 \cdot \text{sgn}(\alpha_t) \quad (\text{A4.68})$$

$$\alpha_{r,eq} = \sqrt{\alpha_r^2 + \left(\frac{K_{xK}}{K_{y\alpha}} \right)^2} \kappa^2 \cdot \text{sgn}(\alpha_r) \quad (\text{A4.69})$$

APPENDIX VI – MF TIRE MODEL CURVE FITTING OF EXPERIMENTAL DATA IN MATLAB

The graphs presented in this appendix were obtained through curve fitting of experimental tire data using a MATLAB dedicated app called Magic Formula Tyre Tool (Teasdale, 2022). The experimental tire data was obtained from the FSAE TTC (Kasprzak & Gentz, 2006) and the different graph curves represent the different experimental data and respective curve fitting considering various loads for a constant tire pressure of 0.84 bar used during the tests.

- Longitudinal Force, F_x [N] – Slip Ratio [-]

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- Lateral Force, F_y [N] – Slip Angle [deg]

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- Aligning Torque, M_z [N] – Slip Angle [deg]

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- Overturning Moment, M_x [N] – Slip Angle [-]

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