

EXISTENCE AND STABILITY OF A PERIODIC SOLUTION OF A GENERAL DIFFERENCE EQUATION WITH APPLICATIONS TO NEURAL NETWORKS WITH A DELAY IN THE LEAKAGE TERMS

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ABSTRACT. In this paper, a new global exponential stability criterion is obtained for a general multidimensional delay difference equation using induction arguments. In the cases that the difference equation is periodic, we prove the existence of a periodic solution by constructing a type of Poincaré map. The results are used to obtain stability criteria for a general discrete-time neural network model with a delay in the leakage terms. As particular cases, we obtain new stability criteria for neural network models recently studied in the literature, in particular for low-order and high-order Hopfield and Bidirectional Associative Memory (BAM).

Keywords: Nonlinear delay difference equation; Exponential stability; Periodic solution; Leakage delay; Neural networks.

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1. INTRODUCTION

Neural network models play an important role in several scientific areas, such as geology [29], medicine [22], and physics [6], due to their usefulness in applications, such as sign and image processing, pattern recognition, optimization problems, and so on [1, 8, 29, 38]. Therefore, since the pioneer works of Cohen and Grossberg [9], Hopfield [17], and Kosko [18], many researchers have devoted themselves to studying the dynamic behavior of neural network models.

The artificial neural network models studied in [9, 17, 18] are described by ordinary differential equations. However, to take into account the transmission speed of signals between different neurons, it is essential to introduce delays in the models. Marcus and Westervelt [23] included a discrete delay in the model studied by Hopfield [17] and observed that delays induce instability in its dynamical behavior (see also [2]).

In 2007, Gopalsamy [15] introduced discrete delays in the negative feedback terms of a continuous-time BAM model because, in several situations, the feedback given by the system is not instantaneous (see also [19]). These terms are also referred to as “forgetting” or leakage terms [18]. Since then, an increasing number of researchers have studied the stability of continuous-time neural network models with delays in the leakage terms (see [4, 21, 26, 28, 37] and references therein).

Discrete-time models play a fundamental role in the computational implementation of continuous models. However, it is important to acknowledge that the process of discretization may not always accurately capture the dynamics of the

original continuous-time model. Therefore, it is crucial to study the dynamic behavior of discrete-time models. In the specific context of neural networks, this issue is addressed in the work by Mohamad and Gopalsamy [25]. They delve into the explanation of how discretizing neural network models with continuous time can impact the dynamics and behavior of the system (see also Dong, Wang, and Zhang [11]).

In the literature there are several works concerning discrete-time neural networks but, as far as we know, just a few works have been dedicated to the study of discrete-time neural networks with delays in the leakage terms [7, 27, 30, 32, 33, 36]. In [7, 27, 30] the global stability was studied for Hopfield type models with delays in the leakage terms and, in [32], for a stochastic impulsive BAM model also with delays in the leakage terms. However, all models studied in [7, 27, 30, 32] have constant parameters such as neuron charging time, interconnection weights, and external inputs. In [33] an asymptotic stability criterion for an uncertain BAM model with delays in the leakage terms was established, while in [36] an exponential stability criterion for an uncertain Hopfield model with delays in the leakage terms was obtained.

Models become more realistic if changes in parameters over time are considered, thus it is important to study nonautonomous neural network models. As far as we know, the global stability of discrete-time neural network models with delays in the leakage terms and changes of parameters over time has not yet been studied. The stability of periodic [3, 10, 42], or almost periodic [35, 44] models have been studied with no delays in the leakage terms.

In the present work, we establish a sufficient condition for the global exponential stability of the following general N -dimensional delay difference equation

$$x_i(m+1) = c_i(m)x_i(m-\tau) + h_i(m, \bar{x}_m), \quad m \in \mathbb{N}_0, i \in \{1, \dots, N\}, \quad (1)$$

and, in the case of (1) being a periodic equation, we prove the existence of a periodic solution as a consequence of its global exponential stability. In the last decade, several studies have focused on the stability of neural network models with time delays in the leakage terms on time scales. In 2014, J. Gao et.al. [14] established a criterion for the existence and global exponential stability of an almost-periodic solution of a Hopfield neural network model with delays in the leakage terms on time scales. Later, this study was extended for a neutral type model in [39] and for a BAM model in [40]. In [41] a criterion for the existence and global exponential stability of an almost automorphic solution for a shunting inhibitory cellular neural network model with time-varying delays in the leakage terms on times scales was obtained. Recently, in 2021, X. Meng and Y. Li [24] published a criterion for the existence and global exponential stability of a pseudo almost periodic solution of a high-order Hopfield neural network model with time-varying delays in the leakage terms on time scales. We emphasize that all the stability criteria presented in [14, 24, 39, 40, 41] are dependent on the delays in the leakage terms. However, the results in this work do not depend on such delays.

In this paper, we only apply our abstract results to neural network type models. But, equation (1) is general enough to include other type of discrete-time models such as the hematopoiesis type models both with monotone production rate [5, 20] and unimodal production rate [34, 43]. As far as we know, for the first time induction arguments are used in this work to obtain a stability criterion for a general delayed difference system, such as (1). In contrast to the Lyapunov's method which

requires a tailored functional for each specific model, our approach enables the derivation of a stability criterion applicable to a wide range of models, including various types of neural network models and eventually other biological models as particular cases. We stress that the proofs usually presented in the literature involve the construction of a suitable Lyapunov function [7, 15, 21, 25, 28, 31, 30, 32, 33, 37, 42], or some Halanay inequalities [36], or other type of inequality analysis techniques [10, 11, 12, 16, 27, 44].

The main novelties in this work are as follows:

1. We established a global exponential stability criterion, Theorem 1, for the nonautonomous difference equation (1). We emphasize the new method used in the proof and the fact that we are dealing with a difference equation with a delay in the linear part.
2. We established a global exponential stability criterion, Theorem 3, for a discrete-time generalized neural network model with a delay in the leakage terms (11), which is general enough to include several neural network models as particular cases such as the low-order Hopfield type model (14), the BAM model (21), and the high-order Hopfield type model (22). We stress that the obtained criterion does not depend on the delay in the leakage terms, nor on the system delay.
3. We provide a criterion for the existence of periodic solutions of (1), Theorem 2. By utilizing this result together with Theorem 3, we establish the existence and global exponential stability of periodic solutions for periodic Hopfield type and BAM models with a delay in the leakage terms. These findings are presented in Corollaries 2, 5, and 6. Previously, the existence and global exponential stability of a periodic solution were established for discrete-time periodic neural network models without delays in the leakage terms [3, 10, 42].

Following the Introduction, Section 2 presents the proof of the main global stability criterion for (1). In Section 3, we consider (1) as a periodic difference equation and employ a Poincaré map to establish the existence of a periodic solution as a consequence of its global exponential stability. Results in Sections 2 and 3 are applied, in Section 4, to obtain criteria for the stability and existence of periodic solutions of nonautonomous discrete-time neural network models with a delay in the leakage terms. Additionally, we compare our results with those in the existing literature. In Section 5, we provide two numerical examples to illustrate the effectiveness of some of our results. Finally, in the last section we present the main conclusions of our work.

2. STABILITY OF MODELS WITH DELAY IN THE LEAKAGE TERM

Consider the difference equation

$$x_i(m+1) = c_i(m)x_i(m-\tau) + h_i(m, \bar{x}_m), \quad m \in \mathbb{N}_0, i \in \{1, \dots, N\}, \quad (2)$$

with $N \in \mathbb{N}$, $\tau \in \mathbb{N}_0$, and, for each $i \in \{1, \dots, N\}$, $c_i : \mathbb{N}_0 \rightarrow]-1, 1[$ and $h_i : \mathbb{N}_0 \times X^N \rightarrow \mathbb{R}$ are functions. The space X^N and the notation $\bar{x}_m$ are explained below.

Given a set $I \subseteq \mathbb{R}$, define $I_{\mathbb{Z}} = I \cap \mathbb{Z}$. Consider $r \in]-\infty, -\tau]_{\mathbb{Z}}$ and denote by X the space of the functions

$$\alpha : [r, 0]_{\mathbb{Z}} \rightarrow \mathbb{R}$$

equipped with the norm

$$\|\alpha\| = \max_{j=r, \dots, 0} |\alpha(j)|.$$

We denote by X^N and $\mathbb{R}^N$ the cartesian products equipped with the supremum norm, i.e., for $\bar{\alpha} = (\alpha_1, \dots, \alpha_N) \in X^N$ and $\bar{y} = (y_1, \dots, y_N) \in \mathbb{R}^N$, we have

$$\|\bar{\alpha}\| = \max_{i=1, \dots, N} \|\alpha_i\| = \max_{i=1, \dots, N} \left(\max_{j=r, \dots, 0} |\alpha_i(j)| \right)$$

and

$$|\bar{y}| = \max_{i=1, \dots, N} |y_i|.$$

We write $\bar{y} = (y_1, \dots, y_N) > 0$ in case of $y_i > 0$ for all $i \in \{1, \dots, N\}$.

Given a function $\bar{x}: [r, +\infty[_{\mathbb{Z}} \rightarrow \mathbb{R}^N$ we denote the i th component by x_i , i.e., $\bar{x} = (x_1, \dots, x_N)$. For each $m \in \mathbb{N}_0$, we define $\bar{x}_m \in X^N$ by

$$\bar{x}_m(j) = \bar{x}(m+j), \quad j = r, r+1, \dots, 0.$$

For each $\bar{\alpha} \in X^N$, we denote by $\bar{x}(\cdot, \bar{\alpha})$ the unique solution

$$\bar{x}: [r, +\infty[_{\mathbb{Z}} \rightarrow \mathbb{R}^N$$

of (2) with initial conditions $\bar{x}_0 = \bar{\alpha}$.

The main purpose in this section is to establish sufficient conditions for the global stability of difference equation (2).

Definition 1. *We say that difference equation (2) is:*

1. *Uniformly stable if*

$$\forall \varepsilon > 0, \exists \delta > 0, \forall \bar{\alpha}, \bar{\beta} \in X^N, \forall m \in \mathbb{N}_0 : \|\bar{\alpha} - \bar{\beta}\| < \delta \Rightarrow \|\bar{x}_m(\cdot, \bar{\alpha}) - \bar{x}_m(\cdot, \bar{\beta})\| < \varepsilon.$$

2. *Globally exponentially stable if there are $C > 0$, and $\zeta \in]0, 1[$ such that, for all $\bar{\alpha}, \bar{\beta} \in X^N$,*

$$\|\bar{x}_m(\cdot, \bar{\alpha}) - \bar{x}_m(\cdot, \bar{\beta})\| \leq C\zeta^m \|\bar{\alpha} - \bar{\beta}\|, \quad \forall m \in \mathbb{N}_0.$$

3. *Globally attractive if*

$$\lim_m \|\bar{x}_m(\cdot, \bar{\alpha}) - \bar{x}_m(\cdot, \bar{\beta})\| = 0, \quad \forall \bar{\alpha}, \bar{\beta} \in X^N.$$

4. *Globally asymptotically stable if it is uniformly stable and globally attractive.*

For each $s \in \{0, \dots, \tau\}$, we denote by $[s]_\tau$ the set

$$[s]_\tau := \{n(\tau+1) - s : n \in \mathbb{N}\}. \quad (3)$$

Consequently, the equality $\mathbb{N} = \bigcup_{s=0}^{\tau} [s]_\tau$ holds and $[s]_\tau \cap [s^*]_\tau = \emptyset$ if $s \neq s^*$ with $s, s^* \in \{0, \dots, \tau\}$.

For the functions c_i and h_i , we assume the following hypotheses

(H1) for each $i \in \{1, \dots, N\}$, there is a function $H_i : \mathbb{N}_0 \rightarrow]0, +\infty[$ such that

$$|h_i(m, \bar{\alpha}) - h_i(m, \bar{\beta})| \leq H_i(m) \|\bar{\alpha} - \bar{\beta}\|, \quad \forall \bar{\alpha}, \bar{\beta} \in X^N, m \in \mathbb{N}_0;$$

(H2) we assume

$$c := \sup \left\{ |c_i(m)| : i \in \{1, \dots, N\}, m \in \mathbb{N}_0 \right\} > 0,$$

and

$$\lambda < 1,$$

where

$$\lambda := \max_{\substack{i=1,\dots,N \\ s=0,\dots,\tau}} \left[\sup_{n \in \mathbb{N}} \sum_{l=0}^{n-1} \left(\prod_{k=l+1}^{n-1} |c_i(k(\tau+1) + \tau - s)| \right) \cdot \frac{H_i(l(\tau+1) + \tau - s)}{c^{n-1-l+\frac{1-\tau+s}{\tau+1}}} \right].$$

In hypothesis (H2), we use the standard convention that a product is equal to one if the number of factors is zero.

By (H2) and the definition of functions c_i , we have $c \in]0, 1]$.

Roughly speaking, condition (H2) says that the perturbations h_i (associated with the Lipschitz constants $H_i(m)$) are controlled by the linear part (associated with the coefficients $c_i(m)$). In the context of neural network models, condition (H2) follows from typical assumptions in the literature.

Before stating our main stability result, we need to prove the following lemma.

Lemma 1. *Let $\bar{\alpha} = (\alpha_1, \dots, \alpha_N)$, $\bar{\beta} = (\beta_1, \dots, \beta_N) \in X^N$.*

If (H1) holds, then the solutions $\bar{x}(\cdot, \bar{\alpha}) = \bar{x}(\cdot) = (x_1(\cdot), \dots, x_N(\cdot))$ and $\bar{x}(\cdot, \bar{\beta}) = \bar{y}(\cdot) = (y_1(\cdot), \dots, y_N(\cdot))$ of (2) verify

$$\begin{aligned} & |x_i(n(\tau+1) - s) - y_i(n(\tau+1) - s)| \\ & \leq \left(\prod_{k=0}^{n-1} |c_i(k(\tau+1) + \tau - s)| \right) \|\bar{\alpha} - \bar{\beta}\| + \sum_{l=0}^{n-1} \left(\prod_{k=l+1}^{n-1} |c_i(k(\tau+1) + \tau - s)| \right) \\ & \quad \cdot H_i(l(\tau+1) + \tau - s) \|\bar{x}_{l(\tau+1)+\tau-s} - \bar{y}_{l(\tau+1)+\tau-s}\|, \end{aligned} \quad (4)$$

for all $i \in \{1, \dots, N\}$, $s \in \{0, \dots, \tau\}$, and $n \in \mathbb{N}$.

Proof. Let $s \in \{0, \dots, \tau\}$ and $i \in \{1, \dots, N\}$.

The proof is done using induction on $n \in \mathbb{N}$.

For $n = 1$, we have, from (2) and (H1),

$$\begin{aligned} & |x_i(\tau+1-s) - y_i(\tau+1-s)| \\ & = |c_i(\tau-s)x_i(\tau-s-\tau) + h_i(\tau-s, \bar{x}_{\tau-s}) \\ & \quad - c_i(\tau-s)y_i(\tau-s-\tau) - h_i(\tau-s, \bar{y}_{\tau-s})| \\ & \leq |c_i(\tau-s)||x_i(-s) - y_i(-s)| + |h_i(\tau-s, \bar{x}_{\tau-s}) - h_i(\tau-s, \bar{y}_{\tau-s})| \\ & \leq |c_i(\tau-s)|\|\bar{\alpha} - \bar{\beta}\| + H_i(\tau-s)\|\bar{x}_{\tau-s} - \bar{y}_{\tau-s}\| \\ & = \left(\prod_{k=0}^{1-1} |c_i(k(\tau+1) + \tau - s)| \right) \|\bar{\alpha} - \bar{\beta}\| + \sum_{l=0}^{1-1} \left(\prod_{k=l+1}^{1-1} |c_i(k(\tau+1) + \tau - s)| \right) \\ & \quad \cdot H_i(l(\tau+1) + \tau - s) \|\bar{x}_{l(\tau+1)+\tau-s} - \bar{y}_{l(\tau+1)+\tau-s}\|. \end{aligned}$$

Now we assume that (4) holds for some $n \in \mathbb{N}$. Consequently, from (2), (H1) and induction hypothesis, we have

$$\begin{aligned}
& |x_i((n+1)(\tau+1)-s) - y_i((n+1)(\tau+1)-s)| \\
&= |x_i(n(\tau+1)+\tau-s+1) - y_i(n(\tau+1)+\tau-s+1)| \\
&= |c_i(n(\tau+1)+\tau-s)x_i(n(\tau+1)-s) + h_i(n(\tau+1)+\tau-s, \bar{x}_{n(\tau+1)+\tau-s}) \\
&\quad - c_i(n(\tau+1)+\tau-s)y_i(n(\tau+1)-s) - h_i(n(\tau+1)+\tau-s, \bar{y}_{n(\tau+1)+\tau-s})| \\
&\leq |c_i(n(\tau+1)+\tau-s)||x_i(n(\tau+1)-s) - y_i(n(\tau+1)-s)| \\
&\quad + H_i(n(\tau+1)+\tau-s)\|\bar{x}_{n(\tau+1)+\tau-s} - \bar{y}_{n(\tau+1)+\tau-s}\| \\
&\leq |c_i(n(\tau+1)+\tau-s)| \left(\prod_{k=0}^{n-1} |c_i(k(\tau+1)+\tau-s)| \right) \|\bar{\alpha} - \bar{\beta}\| \\
&\quad + |c_i(n(\tau+1)+\tau-s)| \sum_{l=0}^{n-1} \left(\prod_{k=l+1}^{n-1} |c_i(k(\tau+1)+\tau-s)| \right) \\
&\quad \cdot H_i(l(\tau+1)+\tau-s)\|\bar{x}_{l(\tau+1)+\tau-s} - \bar{y}_{l(\tau+1)+\tau-s}\| \\
&\quad + H_i(n(\tau+1)+\tau-s)\|\bar{x}_{n(\tau+1)+\tau-s} - \bar{y}_{n(\tau+1)+\tau-s}\| \\
&= \left(\prod_{k=0}^n |c_i(k(\tau+1)+\tau-s)| \right) \|\bar{\alpha} - \bar{\beta}\| + \sum_{l=0}^n \left(\prod_{k=l+1}^n |c_i(k(\tau+1)+\tau-s)| \right) \\
&\quad \cdot H_i(l(\tau+1)+\tau-s)\|\bar{x}_{l(\tau+1)+\tau-s} - \bar{y}_{l(\tau+1)+\tau-s}\|
\end{aligned}$$

and the proof of (4) is concluded. $\square$

Now we are in position to prove the main stability result about the difference equation (2).

Theorem 1. *Assume the hypotheses (H1) and (H2).*

Then, for any $\bar{\alpha}, \bar{\beta} \in X^N$, the solutions $\bar{x}(\cdot, \bar{\alpha})$ and $\bar{x}(\cdot, \bar{\beta})$ of (2) verify

$$\|\bar{x}_m(\cdot, \bar{\alpha}) - \bar{x}_m(\cdot, \bar{\beta})\| \leq \frac{c^{\frac{\tau}{\tau+1}-1}}{1-\lambda} \left(c^{\frac{1}{\tau+1}} \right)^m \|\bar{\alpha} - \bar{\beta}\|, \quad (5)$$

for all $m \in \mathbb{N}_0$.

Proof. Let $\bar{\alpha}, \bar{\beta} \in X^N$ and consider the solutions $\bar{x}(\cdot, \bar{\alpha}) = \bar{x}(\cdot) = (x_1(\cdot), \dots, x_N(\cdot))$ and $\bar{x}(\cdot, \bar{\beta}) = \bar{y}(\cdot) = (y_1(\cdot), \dots, y_N(\cdot))$ of (2).

First we prove that

$$|\bar{x}(m, \bar{\alpha}) - \bar{x}(m, \bar{\beta})| \leq \frac{1}{1-\lambda} c^{\frac{m-(\tau+1)}{\tau+1}} \|\bar{\alpha} - \bar{\beta}\|, \quad \forall m \in \mathbb{N}_0. \quad (6)$$

The proof of inequality (6) is done by induction on $m \in \mathbb{N}_0$.

As $0 < c \leq 1$ and $0 < 1 - \lambda < 1$, then condition (6) holds trivially for $m = 0$.

Assume that, for some $m \in \mathbb{N}$, we have

$$|\bar{x}(t) - \bar{y}(t)| \leq \frac{1}{1-\lambda} c^{\frac{t-(\tau+1)}{\tau+1}} \|\bar{\alpha} - \bar{\beta}\|, \quad \forall t \in [0, m-1]_{\mathbb{Z}}. \quad (7)$$

As $\mathbb{N} = \bigcup_{s=0}^{\tau} [s]_{\tau}$ and $[s]_{\tau} \cap [s^*]_{\tau} = \emptyset$ for $s \neq s^*$, by (3) we conclude there are unique $s \in \{0, \dots, \tau\}$ and $n \in \mathbb{N}$ such that $m = n(\tau+1) - s$. For $l \in [0, n-1]_{\mathbb{Z}}$, we have

$$l(\tau+1) + \tau - s + j \leq (n-1)(\tau+1) + \tau - s = n(\tau+1) - s - 1 < m, \quad \forall j \in \{r, \dots, 0\},$$

thus, by (7),

$$\begin{aligned}
\|\bar{x}_{l(\tau+1)+\tau-s} - \bar{y}_{l(\tau+1)+\tau-s}\| &= \max_{j=r, \dots, 0} |\bar{x}(l(\tau+1) + \tau - s + j) - \bar{y}(l(\tau+1) + \tau - s + j)| \\
&\leq \frac{1}{1-\lambda} \cdot \max_{j=r, \dots, 0} \left\{ c^{\frac{l(\tau+1)+\tau-s+j-(\tau+1)}{\tau+1}} \right\} \|\bar{\alpha} - \bar{\beta}\| \\
&= \frac{1}{1-\lambda} c^{\frac{l(\tau+1)+\tau-s+r-\tau-1}{\tau+1}} \|\bar{\alpha} - \bar{\beta}\| \\
&= \frac{1}{1-\lambda} c^{l+\frac{r-s-1}{\tau+1}} \|\bar{\alpha} - \bar{\beta}\|. \tag{8}
\end{aligned}$$

Let $i \in \{1, \dots, N\}$. From Lemma 1 and (8), we have

$$\begin{aligned}
|x_i(m) - y_i(m)| &= |x_i(n(\tau+1) - s) - y_i(n(\tau+1) - s)| \\
&\leq \left(\prod_{k=0}^{n-1} |c_i(k(\tau+1) + \tau - s)| \right) \|\bar{\alpha} - \bar{\beta}\| \\
&\quad + \sum_{l=0}^{n-1} \left(\prod_{k=l+1}^{n-1} |c_i(k(\tau+1) + \tau - s)| \right) \\
&\quad \cdot H_i(l(\tau+1) + \tau - s) \|\bar{x}_{l(\tau+1)+\tau-s} - \bar{y}_{l(\tau+1)+\tau-s}\| \\
&\leq \left(\prod_{k=0}^{n-1} |c_i(k(\tau+1) + \tau - s)| \right) \|\bar{\alpha} - \bar{\beta}\| \\
&\quad + \sum_{l=0}^{n-1} \left(\prod_{k=l+1}^{n-1} |c_i(k(\tau+1) + \tau - s)| \right) \\
&\quad \cdot H_i(l(\tau+1) + \tau - s) \frac{1}{1-\lambda} c^{l+\frac{r-s-1}{\tau+1}} \|\bar{\alpha} - \bar{\beta}\| \\
&= \left[\left(\prod_{k=0}^{n-1} \frac{|c_i(k(\tau+1) + \tau - s)|}{c} \right) c \right. \\
&\quad \left. + \frac{1}{1-\lambda} \sum_{l=0}^{n-1} \left(\prod_{k=l+1}^{n-1} |c_i(k(\tau+1) + \tau - s)| \right) H_i(l(\tau+1) + \tau - s) \right. \\
&\quad \left. \cdot c^{l+\frac{r-s-1}{\tau+1}-n+1} \right] c^{n-1} \|\bar{\alpha} - \bar{\beta}\|.
\end{aligned}$$

Denoting

$$\mathcal{A} = \left(\prod_{k=0}^{n-1} \frac{|c_i(k(\tau+1) + \tau - s)|}{c} \right) c$$

and

$$\mathcal{B} = \sum_{l=0}^{n-1} \left[\left(\prod_{k=l+1}^{n-1} |c_i(k(\tau+1) + \tau - s)| \right) \frac{H_i(l(\tau+1) + \tau - s)}{c^{n-1-l+\frac{1+s-r}{\tau+1}}} \right],$$

we have

$$|x_i(m) - y_i(m)| \leq \left(\mathcal{A} + \frac{1}{1-\lambda} \mathcal{B} \right) c^{n-1} \|\bar{\alpha} - \bar{\beta}\|.$$

From **(H2)**, we obtain $\mathcal{B} \leq \lambda < 1$ and $\mathcal{A} \leq 1$, thus, recalling that $n = \frac{m+s}{\tau+1}$, $c \in]0, 1]$ and $s \in \{0, \dots, \tau\}$, we conclude

$$|x_i(m) - y_i(m)| \leq \frac{1}{1-\lambda} c^{n-1} \|\bar{\alpha} - \bar{\beta}\| \leq \frac{1}{1-\lambda} c^{\frac{m-(\tau+1)}{\tau+1}} \|\bar{\alpha} - \bar{\beta}\|.$$

As i is arbitrary, then condition (7) holds for $t = m$ and consequently (7) holds for all $m \in \mathbb{N}_0$. Finally, by (7), we have

$$\begin{aligned} \|\bar{x}_m - \bar{y}_m\| &= \max_{j=r, \dots, 0} \{|\bar{x}(m+j) - \bar{y}(m+j)|\} \\ &\leq \max_{j=r, \dots, 0} \left\{ \frac{1}{1-\lambda} c^{\frac{m+j-(\tau+1)}{\tau+1}} \|\bar{\alpha} - \bar{\beta}\| \right\} \\ &= \frac{1}{1-\lambda} c^{\frac{m+r-(\tau+1)}{\tau+1}} \|\bar{\alpha} - \bar{\beta}\| = \frac{c^{\frac{r}{\tau+1}-1}}{1-\lambda} \left(c^{\frac{1}{\tau+1}}\right)^m \|\bar{\alpha} - \bar{\beta}\| \end{aligned}$$

□

We remark that, Theorem 1 assures that, under hypotheses (H1) and (H2), difference equation (2) is uniformly stable and, if $c < 1$, then (2) is globally exponentially stable.

3. PERIODIC MODEL

In this section we assume that difference equation (2) is periodic, i.e., for some $\omega \in \mathbb{N}$, we consider the difference equation

$$x_i(m+1) = c_i(m)x_i(m-\tau) + h_i(m, \bar{x}_m), \quad m \in \mathbb{N}_0, i \in \{1, \dots, N\}, \quad (9)$$

with $c_i : \mathbb{N}_0 \rightarrow]-1, 1[$ and $h_i : \mathbb{N}_0 \times X^N \rightarrow \mathbb{R}$ functions satisfying the hypotheses:

- (P1) For all $i \in \{1, \dots, N\}$ and $m \in \mathbb{N}_0$, we have $c_i(m) = c_i(m+\omega)$;
- (P2) for all $i \in \{1, \dots, N\}$, $m \in \mathbb{N}_0$, and $\bar{\alpha} \in X^N$, we have $h_i(m+\omega, \bar{\alpha}) = h_i(m, \bar{\alpha})$.

Theorem 2. Assume (P1), (P2), (H1), and (H2) with $c \in]0, 1[$.

Then there is a ω -periodic solution $\bar{x}^*$ of (9) verifying

$$\|\bar{x}_m(\cdot, \bar{\alpha}) - \bar{x}_m^*\| \leq \frac{c^{\frac{r}{\tau+1}-1}}{1-\lambda} \left(c^{\frac{1}{\tau+1}}\right)^m \|\bar{\alpha} - \bar{x}_0^*\|, \quad (10)$$

for all $m \in \mathbb{N}$.

Proof. From Theorem 1, inequality (5), we have

$$\|\bar{x}_m(\cdot, \bar{\alpha}) - \bar{x}_m(\cdot, \bar{\beta})\| \leq Q(m) \|\bar{\alpha} - \bar{\beta}\|, \quad \forall \bar{\alpha}, \bar{\beta} \in X^N, \forall m \in \mathbb{N}_0,$$

where $Q(m) = \frac{c^{\frac{r}{\tau+1}-1}}{1-\lambda} \left(c^{\frac{1}{\tau+1}}\right)^m$. As $c < 1$, then there is $p \in \mathbb{N}$ such that $Q(m) < 1$ for all $m \in [p, +\infty[_{\mathbb{Z}}$.

Define the map $P : X^N \rightarrow X^N$ by $P(\bar{\alpha}) = \bar{x}_\omega(\cdot, \bar{\alpha})$. For $\bar{\alpha}, \bar{\beta} \in X^N$, we have

$$\begin{aligned} \|P^p(\bar{\alpha}) - P^p(\bar{\beta})\| &= \|P(P^{p-1}(\bar{\alpha})) - P(P^{p-1}(\bar{\beta}))\| \\ &= \|\bar{x}_\omega(\cdot, P^{p-1}(\bar{\alpha})) - \bar{x}_\omega(\cdot, P^{p-1}(\bar{\beta}))\| \\ &= \|\bar{x}_\omega(\cdot, \bar{x}_\omega(\cdot, P^{p-2}(\bar{\alpha}))) - \bar{x}_\omega(\cdot, \bar{x}_\omega(\cdot, P^{p-2}(\bar{\beta})))\|, \end{aligned}$$

and from (P1) and (P2) the difference equation (9) is ω -periodic and consequently

$$\begin{aligned} \|P^p(\bar{\alpha}) - P^p(\bar{\beta})\| &= \|\bar{x}_{2\omega}(\cdot, P^{p-2}(\bar{\alpha})) - \bar{x}_{2\omega}(\cdot, P^{p-2}(\bar{\beta}))\| \\ &= \|\bar{x}_{p\omega}(\cdot, \bar{\alpha}) - \bar{x}_{p\omega}(\cdot, \bar{\beta})\| \leq Q(p\omega) \|\bar{\alpha} - \bar{\beta}\|. \end{aligned}$$

As $Q(\omega p) < 1$, then P^p is a contraction map on Banach space X^N , thus there is a unique $\bar{\alpha}^* \in X^N$ such that $P^p(\bar{\alpha}^*) = \bar{\alpha}^*$. Consequently

$$P^p(P(\bar{\alpha}^*)) = P(P^p(\bar{\alpha}^*)) = P(\bar{\alpha}^*),$$

and we obtain $P(\bar{\alpha}^*) = \bar{\alpha}^*$, that is $\bar{x}_\omega(\cdot, \bar{\alpha}^*) = \bar{\alpha}^*$.

As $\bar{x}(m, \bar{\alpha}^*)$ is a solution of the ω -periodic difference equation (9), then $\bar{x}(m + \omega, \bar{\alpha}^*)$ is also a solution of (9) verifying

$$\bar{x}(m, \bar{\alpha}^*) = \bar{x}(m, \bar{x}_\omega(\cdot, \bar{\alpha}^*)) = \bar{x}(m + \omega, \bar{\alpha}^*), \quad \forall m \in \mathbb{N},$$

which means that $\bar{x}^*(m) = \bar{x}(m, \bar{\alpha}^*)$ is a ω -periodic solution of (9).

Finally, the inequality (10) follows from Theorem 1. $\square$

4. APPLICATIONS TO NEURAL NETWORK MODELS

In this section, we apply our main results to obtain criteria for the global exponential stability of several discrete-time neural network type models with delay in the leakage terms. Some criteria for the existence and global exponential stability of a periodic solution of periodic models are also established.

First of all, we consider the following discrete-time generalized neural network model with delay in the leakage terms

$$x_i(m+1) = c_i(m)x_i(m-\tau) + \sum_{j=1}^N h_{ij}(m, \bar{x}_m), \quad m \in \mathbb{N}_0, i = 1, \dots, N, \quad (11)$$

where $N \in \mathbb{N}$, $\tau \in \mathbb{N}_0$, and $c_i : \mathbb{N}_0 \rightarrow]-1, 1[$ and $h_{ij} : \mathbb{N}_0 \times X^N \rightarrow \mathbb{R}$ are functions such that the following hypothesis holds:

(A1) The functions h_{ij} are Lipschitz, i.e. for each $i, j \in \{1, \dots, N\}$ there exists a positive constant H_{ij} such that

$$|h_{ij}(m, \bar{\alpha}) - h_{ij}(m, \bar{\beta})| \leq H_{ij} \|\bar{\alpha} - \bar{\beta}\|, \quad \forall m \in \mathbb{N}_0, \forall \bar{\alpha}, \bar{\beta} \in X^N.$$

Now, we apply Theorem 1 to obtain the following stability result.

Theorem 3. *Assume (A1).*

If

$$1 - c_i^+ > \sum_{j=1}^N H_{ij}, \quad \forall i \in \{1, \dots, N\}, \quad (12)$$

where $c_i^+ = \sup_m |c_i(m)|$, then the model (11) is globally exponentially stable.

Proof. The model (11) is a particular case of (2) with

$$h_i(m, \bar{\alpha}) = \sum_{j=1}^N h_{ij}(m, \bar{\alpha}),$$

for all $\bar{\alpha} \in X^N$, $m \in \mathbb{N}_0$, and $i, j \in \{1, \dots, N\}$.

From (A1), the hypothesis (H1) holds with $H_i(m) = \sum_{j=1}^N H_{ij}$ for all $i \in \{1, \dots, N\}$.

Now we show that (H2) also holds.

For each $i \in \{1, \dots, N\}$ such that $c_i^+ \neq 0$, by (12), we have $c_i^+ \in]0, 1[$ and define $\nu_i = -\ln(c_i^+) \in]0, +\infty[$. Thus, we have $c_i^+ = e^{-\nu_i}$.

For each $i \in \{1, \dots, N\}$ such that $c_i^+ = 0$, by (12) it is possible to choose $\nu_i \in]0, +\infty[$ such that

$$1 - e^{-\nu_i} > \sum_{j=1}^N H_{ij}.$$

From (12), we have

$$\frac{e^{\nu_i} - 1}{e^{\nu_i}} > \sum_{j=1}^N H_{ij}, \quad \forall i \in \{1, \dots, N\}.$$

Consequently, there is a positive number $\mu < \min_i \nu_i$ such that

$$\frac{e^{\nu_i - \mu} - 1}{e^{\nu_i}} e^{\mu \frac{\tau}{\tau+1}} > \sum_{j=1}^N H_{ij}, \quad \forall i \in \{1, \dots, N\}. \quad (13)$$

Defining $c = e^{-\mu}$, we have $|c_i(m)| \leq c$ for all $m \in \mathbb{N}_0$. Consequently, for $i \in \{1, \dots, N\}$ and $s \in \{0, \dots, \tau\}$, we have

$$\begin{aligned} & \sup_{n \in \mathbb{N}} \left[\sum_{l=0}^{n-1} \left(\prod_{k=l+1}^{n-1} |c_i(k(\tau+1) + \tau - s)| \right) H_i(l(\tau+1) + \tau - s) c^{l + \frac{\tau-s-1}{\tau+1} - n + 1} \right] \\ & \leq \sup_{n \in \mathbb{N}} \left[\sum_{l=0}^{n-1} e^{-\nu_i(n-1-l)} e^{-\mu(l + \frac{\tau-s-1}{\tau+1} - n + 1)} \sum_{j=1}^N H_{ij} \right] \\ & = \sup_{n \in \mathbb{N}} \left[\sum_{l=0}^{n-1} e^{(\nu_i - \mu)(l-n)} e^{\nu_i} e^{-\mu(\frac{\tau-s-1}{\tau+1} + 1)} \sum_{j=1}^N H_{ij} \right] \\ & = \sup_{n \in \mathbb{N}} \left[\left(\sum_{l=0}^{n-1} e^{(\nu_i - \mu)(l-n)} \right) e^{\nu_i} e^{-\mu(\frac{\tau-s-1}{\tau+1} + 1)} \sum_{j=1}^N H_{ij} \right] \\ & = \sup_{n \in \mathbb{N}} \left[\left(\sum_{l=0}^{n-1} e^{(\nu_i - \mu)l} \right) e^{-(\nu_i - \mu)n} e^{\nu_i} e^{-\mu(\frac{\tau-s-1}{\tau+1} + 1)} \sum_{j=1}^N H_{ij} \right] \\ & = \sup_{n \in \mathbb{N}} \left[\frac{1 - e^{(\nu_i - \mu)n}}{1 - e^{\nu_i - \mu}} e^{-(\nu_i - \mu)n} e^{\nu_i} e^{-\mu(\frac{\tau-s-1}{\tau+1} + 1)} \sum_{j=1}^N H_{ij} \right] \\ & = \sup_{n \in \mathbb{N}} \left[\frac{1 - e^{-(\nu_i - \mu)n}}{e^{\nu_i - \mu} - 1} e^{\nu_i} e^{-\mu(\frac{\tau-s-1}{\tau+1} + 1)} \sum_{j=1}^N H_{ij} \right] \\ & = \sup_{n \in \mathbb{N}} \left(1 - e^{-(\nu_i - \mu)n} \right) e^{-\mu(\frac{\tau-s-1}{\tau+1} + 1)} \frac{e^{\nu_i}}{e^{\nu_i - \mu} - 1} \sum_{j=1}^N H_{ij}. \end{aligned}$$

As $\sup_{n \in \mathbb{N}} (1 - e^{-(\nu_i - \mu)n}) = 1$ and $s \in \{1, \dots, \tau\}$, we have

$$\lambda \leq \max_{i=1, \dots, N} \left(e^{-\mu \frac{\tau}{\tau+1}} \frac{e^{\nu_i}}{e^{\nu_i - \mu} - 1} \sum_{j=1}^N H_{ij} \right)$$

and, by (13), we obtain

$$\lambda < \max_{i=1, \dots, N} \left(e^{-\mu \frac{\tau}{\tau+1}} \frac{e^{\nu_i}}{e^{\nu_i - \mu} - 1} \frac{e^{\nu_i - \mu} - 1}{e^{\nu_i}} e^{\mu \frac{\tau}{\tau+1}} \right) = 1,$$

thus hypotheses (H2) holds. From Theorem 1, we obtain that model (11) is globally exponentially stable. $\square$

We emphasize that condition (12) does not depend on delays τ and r . In the literature, stability conditions for neural network models with delays in the leakage terms typically depend on the delays in those terms. For example, see [4, 21, 26] for continuous-time models, [7, 36] for discrete-time models, and [14, 24, 39, 40, 41] for models on time scales.

If (11) is a periodic model then, from Theorem 2, we establish sufficient conditions for the existence and global exponential stability of a periodic solution.

Theorem 4. *Assume (A1) and there is $\omega \in \mathbb{N}$ such that*

$$c_i(m) = c_i(m + \omega) \quad \text{and} \quad h_{ij}(m + \omega, \bar{\alpha}) = h_{ij}(m, \bar{\alpha})$$

for all $m \in \mathbb{N}_0$, $\bar{\alpha} \in X^N$, and $i, j \in \{1, \dots, N\}$.

If condition (12) holds, then there is a ω -periodic solution of (11) which is globally exponentially stable.

Proof. By the hypotheses, conditions (P1) and (P2) hold. From the proof of Theorem 3, the conditions (H1) and (H2) also hold with $c \in]0, 1[$. Thus, the result follows from Theorem 2. $\square$

As a particular case of (11) we consider the following discrete-time low-order Hopfield neural network model with delay in leakage term

$$x_i(m+1) = c_i(m)x_i(m-\tau) + \sum_{j=1}^N \sum_{k=1}^K b_{ijk}(m)f_{ijk}(x_j(m-\tau_{ijk}(m))) + I_i(m), \quad (14)$$

for $m \in \mathbb{N}_0$, $i \in \{1, \dots, N\}$, where $N, K \in \mathbb{N}$, $\tau \in \mathbb{N}_0$, and $c_i : \mathbb{N}_0 \rightarrow]-1, 1[$, $\tau_{ijk} : \mathbb{N}_0 \rightarrow \mathbb{N}_0$, $b_{ijk}, I_i : \mathbb{N}_0 \rightarrow \mathbb{R}$, and $f_{ijk} : \mathbb{R} \rightarrow \mathbb{R}$ are functions such that the following hypotheses hold:

- (B1) The functions b_{ijk} and τ_{ijk} are bounded;
- (B2) the functions f_{ijk} are Lipschitz, i.e. for each $i, j \in \{1, \dots, N\}$ and $k \in \{1, \dots, K\}$, there exists a constant F_{ijk} such that

$$|f_{ijk}(x) - f_{ijk}(y)| \leq F_{ijk}|x - y|, \quad \forall x, y \in \mathbb{R}.$$

The discrete-time autonomous Hopfield neural network model, studied in [16],

$$x_i(m+1) = c_i x_i(m) + \sum_{j=1}^N b_{ij} f_j(x_j(m - \tau_{ij})), \quad m \in \mathbb{N}_0, i = 1, \dots, N, \quad (15)$$

and the discrete-time Hopfield neural network model with delay in leakage term and constant coefficients, studied in [30],

$$x_i(m+1) = c_i x_i(m - \tau) + \sum_{j=1}^N a_{ij} f_j(x_j(m)) + \sum_{j=1}^N b_{ij} f_j(x_j(m - \sigma(m))) + I_i, \quad (16)$$

for $m \in \mathbb{N}_0, i = 1, \dots, N$, are particular case of model (14).

For $i, j \in \{1, \dots, N\}$ and $k \in \{1, \dots, K\}$, in what follows we use the notation

$$r = -\max_{i,j,k,m} \{\tau_{ijk}(m), \tau\}, \quad c_i^+ = \sup_m |c_i(m)|, \quad \text{and} \quad b_{ijk}^+ = \sup_m |b_{ijk}(m)|.$$

From Theorem 3, we obtain the global exponential stability of (14).

Theorem 5. *Assume (B1) and (B2).*

If

$$1 - c_i^+ > \sum_{j=1}^N \sum_{k=1}^K b_{ijk}^+ F_{ijk}, \quad \forall i \in \{1, \dots, N\},$$

then the model (14) is globally exponentially stable.

Proof. The model (14) is a particular case of model (11) with

$$h_{ij}(m, \bar{\alpha}) = \left(\sum_{k=1}^K b_{ijk}(m) f_{ijk}(\alpha_j(-\tau_{ijk}(m))) \right) + \frac{I_i(m)}{N},$$

for all $\bar{\alpha} = (\alpha_1, \dots, \alpha_N) \in X^N, m \in \mathbb{N}_0$, and $i, j \in \{1, \dots, N\}$.

By (B1) and (B2), the hypothesis (A1) holds with $H_{ij} = \sum_{k=1}^K b_{ijk}^+ F_{ijk}$, for all $i, j \in \{1, \dots, N\}$, thus the conclusion follows from Theorem 3. $\square$

The previous result is improved in the following result.

Consider the $N \times N$ -matrix $\mathcal{M}$ defined by

$$\mathcal{M} = \mathcal{I} - \text{diag}(c_1^+, \dots, c_N^+) - \left[\left(\sum_{k=1}^K b_{ijk}^+ F_{ijk} \right)_{ij} \right], \quad (17)$$

where $\mathcal{I}$ is the $N \times N$ identity matrix.

Corollary 1. *Assume (B1) and (B2).*

If $\mathcal{M}$ is a non-singular M-matrix, then the model (14) is globally exponentially stable.

Proof. If $\mathcal{M}$ is a non-singular M-matrix, then (see [13]) there is $\bar{d} = (d_1, \dots, d_N) > 0$ such that $\mathcal{M}\bar{d} > 0$, i.e.,

$$d_i(1 - c_i^+) > \sum_{j=1}^N d_j \left(\sum_{k=1}^K b_{ijk}^+ F_{ijk} \right), \quad \forall i \in \{1, \dots, N\}. \quad (18)$$

The change $y_i(m) = d_i^{-1}x_i(m)$, $m \in \mathbb{N}$ and $i \in \{1, \dots, N\}$, transforms (14) into

$$y_i(m+1) = c_i(m)y_i(m-\tau) + \sum_{j=1}^N \sum_{k=1}^K \tilde{b}_{ijk}(m) \tilde{f}_{ijk}(y_j(m-\tau_{ijk}(m))),$$

where

$$\tilde{b}_{ijk}(m) = d_i^{-1}b_{ijk}(m) \quad \text{and} \quad \tilde{f}_{ijk}(u) = f_{ijk}(d_j u),$$

for $m \in \mathbb{N}_0$ and $u \in \mathbb{R}$. As f_{ijk} are Lipschitz functions with constant F_{ijk} , then $\tilde{f}_{ijk}$ verify (B2) with constant $\tilde{F}_{ijk} = d_j F_{ijk}$. From (18) we have

$$1 - c_i^+ > \sum_{j=1}^N \sum_{k=1}^K \left(d_i^{-1} b_{ijk}^+ \right) (d_j F_{ijk}), \quad \forall i \in \{1, \dots, N\}.$$

which is equivalent to

$$1 - c_i^+ > \sum_{j=1}^N \sum_{k=1}^K \tilde{b}_{ijk}^+ \tilde{F}_{ijk}, \quad \forall i \in \{1, \dots, N\}.$$

and the result follows from the Theorem 5. $\square$

Now we consider the model (14) with periodic coefficients, i.e., we assume that there is $\omega \in \mathbb{N}$ such that $c_i(m)$, $b_{ijk}(m)$, $\tau_{ijk}(m)$ and $I_i(m)$ are ω -periodic functions. Naturally we have

$$c_i^+ = \max \{|c_i(1)|, \dots, |c_i(\omega)|\} \quad \text{and} \quad b_{ijk}^+ = \max \{|b_{ijk}(1)|, \dots, |b_{ijk}(\omega)|\},$$

for $i, j \in \{1, \dots, N\}$, $k \in \{1, \dots, K\}$.

From Theorem 4 and Corollary 1 we have the following result, which extends, to models with delay in the leakage terms, the result [3, Theorem 4].

Corollary 2. *Assume c_i , b_{ijk} , τ_{ijk} , and I_i are ω -periodic function, and (B2).*

If the matrix $\mathcal{M}$, defined by (17), is a non-singular M-matrix, then model (14) has a unique ω -periodic solution which is globally exponentially stable.

Considering model (14) with autonomous coefficients that is, for each $i, j \in \{1, \dots, N\}$ and $k \in \{1, \dots, K\}$ we have

$$c_i(m) = c_i, \quad I_i(m) = I_i, \quad \text{and} \quad b_{ijk}(m) = b_{ijk}, \quad \forall m \in \mathbb{N}_0,$$

with $c_i \in]-1, 1[$ and $b_{ijk}, I_i \in \mathbb{R}$, the previous Corollary 2 allows us to obtain the following result.

Corollary 3. *Assume (B2).*

If the matrix

$$\mathcal{N} = \mathcal{I} - \text{diag}(|c_1|, \dots, |c_N|) - \left[\left(\sum_{k=1}^K |b_{ijk}| F_{ijk} \right)_{ij} \right] \quad (19)$$

is a non-singular M-matrix, then the model

$$x_i(m+1) = c_i x_i(m-\tau) + \sum_{j=1}^N \sum_{k=1}^K b_{ijk} f_{ijk}(x_j(m-\tau_{ijk}(m))) + I_i \quad (20)$$

for $m \in \mathbb{N}_0$, $i \in \{1, \dots, N\}$, has a unique equilibrium which is globally exponentially stable.

In [16, Theorem 3.1], the global attractivity of (15), a particular case of (20), is obtained assuming that activation functions $f_j : \mathbb{R} \rightarrow \mathbb{R}$ are differentiable such that

$$f_j(0) = 0, \quad \lim_{x \rightarrow +\infty} f_j(x) = 1, \quad \lim_{x \rightarrow -\infty} f_j(x) = -1,$$

and

$$f'_j(0) = \sup_x f'_j(x) = 1, \quad f'_j(u) > 0, \forall u \in \mathbb{R},$$

joint with the condition of

$$\mathcal{I} - \text{diag}(|c_1|, \dots, |c_n|) - [|b_{ij}|]$$

being an M-matrix. By hypotheses in [16], it is clear that f_j are Lipschitz functions with Lipschitz constant equal to 1. We remark that Corollary 3 states the global exponential stability of (15), instead of the global attractivity as in [16, Theorem 3.1], but we assume that $\mathcal{N}$, defined by (19), is a non-singular M-matrix which is more restrictive than $\mathcal{N}$ just be an M-matrix as is assumed in [16, Theorem 3.1].

Model (16), studied in [30], is also a particular case of (20), but the asymptotic stability of (16) is established in [30] under a different hypotheses set.

It is relevant to observe that model (11) is general enough to include, as particular cases, some BAM neural network models with delay in leakage term.

Considering, in the general model (11), $N = N_1 + N_2$, for $N_1, N_2 \in \mathbb{N}$,

$$c_i(m) = \begin{cases} \hat{c}_i(m), & i = 1, \dots, N_1 \\ \tilde{c}_{i-N_1}(m), & i = N_1 + 1, \dots, N_1 + N_2 \end{cases}, \quad \forall m \in \mathbb{N}_0,$$

and

$$h_{ij}(m, \bar{\alpha}) = \begin{cases} 0, & i = 1, \dots, N_1, j = 1, \dots, N_1 \\ \hat{a}_{i(j-N_1)}(m)f_{j-N_1}(\alpha_j(0)) \\ + \hat{b}_{i(j-N_1)}(m)f_{j-N_1}(\alpha_j(-\hat{\tau}_{i(j-N_1)}(m))) \\ + \frac{\hat{I}_i(m)}{N_2}, & i = 1, \dots, N_1, j = N_1 + 1, \dots, N_1 + N_2 \\ \tilde{a}_{(i-N_1)j}(m)g_j(\alpha_j(0)) \\ + \tilde{b}_{(i-N_1)j}(m)g_j(\alpha_j(-\tilde{\tau}_{(i-N_1)j}(m))) \\ + \frac{\tilde{I}_i(m)}{N_1}, & i = N_1 + 1, \dots, N_1 + N_2, j = 1, \dots, N_1 \\ 0, & i = N_1 + 1, \dots, N_1 + N_2, \\ & j = N_1 + 1, \dots, N_1 + N_2 \end{cases},$$

for all $m \in \mathbb{N}_0$ and $\bar{\alpha} \in X^{N_1+N_2}$, we have the following BAM neural network model

$$\left\{ \begin{aligned} x_i(m+1) &= \hat{c}_i(m)x_i(m-\tau) + \sum_{j=1}^{N_2} \hat{a}_{ij}(m)f_j(y_j(m)) \\ &\quad + \sum_{j=1}^{N_2} \hat{b}_{ij}(m)f_j(y_j(m-\hat{\tau}_{ij}(m))) + \hat{I}_i(m), \quad i = 1, \dots, N_1, \\ y_j(m+1) &= \tilde{c}_j(m)y_j(m-\tau) + \sum_{i=1}^{N_1} \tilde{a}_{ji}(m)g_i(x_i(m)) \\ &\quad + \sum_{i=1}^{N_1} \tilde{b}_{ji}(m)g_i(x_i(m-\tilde{\tau}_{ji}(m))) + \tilde{I}_j(m), \quad j = 1, \dots, N_2, \end{aligned} \right. \quad (21)$$

where $\hat{c}_i, \tilde{c}_j : \mathbb{N}_0 \rightarrow]-1, 1[$, $\hat{a}_{ij}, \tilde{a}_{ji} : \mathbb{N}_0 \rightarrow \mathbb{R}$, $\hat{\tau}_{ij}, \tilde{\tau}_{ji} : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ are bounded functions, $\hat{I}_i, \tilde{I}_j : \mathbb{N}_0 \rightarrow \mathbb{R}$ are functions, and $f_j, g_i : \mathbb{R} \rightarrow \mathbb{R}$ are Lipschitz functions with Lipschitz constants F_j and G_i respectively.

For the functions in the model (21), we use the notation

$$r = -\max_{i,j,m} \{\hat{\tau}_{ij}(m), \tilde{\tau}_{ji}(m), \tau\}, \quad \hat{c}_i^+ = \sup_m |\hat{c}_i(m)|, \quad \tilde{c}_j^+ = \sup_m |\tilde{c}_j(m)|,$$

$$\hat{a}_{ij}^+ = \sup_m |\hat{a}_{ij}(m)|, \quad \hat{b}_{ij}^+ = \sup_m |\hat{b}_{ij}(m)|, \quad \tilde{a}_{ji}^+ = \sup_m |\tilde{a}_{ji}(m)|, \quad \text{and} \quad \tilde{b}_{ji}^+ = \sup_m |\tilde{b}_{ji}(m)|.$$

We also define the matrix $\mathcal{P}$ by

$$\mathcal{P} := \begin{bmatrix} \mathcal{I}_{N_1} - \hat{C} & -U \\ -S & \mathcal{I}_{N_2} - \tilde{C} \end{bmatrix},$$

where, for $k = 1, 2$, $\mathcal{I}_{N_k}$ is the $N_k \times N_k$ identity matrix, $\hat{C} = \text{diag}(\hat{c}_1^+, \dots, \hat{c}_{N_1}^+)$, $\tilde{C} = \text{diag}(\tilde{c}_1^+, \dots, \tilde{c}_{N_2}^+)$, $U = [(\hat{a}_{ij}^+ + \hat{b}_{ij}^+)F_j]$, and $S = [(\tilde{a}_{ji}^+ + \tilde{b}_{ji}^+)G_i]$. Following the same arguments presented in the proofs of Theorem 5 and Corollary 1, we obtain the next exponential stability criterion.

Theorem 6. *If $\mathcal{P}$ is a non-singular M-matrix, then the model (21) is globally exponentially stable.*

From Theorem 4 and Theorem 6, we obtain the following stability results for model (21) with periodic and constant coefficients, respectively.

Corollary 4. *Assume $\hat{c}_i$, $\tilde{c}_j$, $\hat{a}_{ij}$, $\tilde{a}_{ji}$, $\hat{b}_{ij}$, $\tilde{b}_{ji}$, $\hat{\tau}_{ij}$, $\tilde{\tau}_{ji}$, $\hat{I}_i$, and $\tilde{I}_j$ are ω -periodic functions.*

If $\mathcal{P}$ is a non-singular M-matrix, then the model (21) has a unique ω -periodic solution which is globally exponentially stable.

Corollary 5. *Assume $\hat{c}_i(m) = \hat{c}_i$, $\tilde{c}_j(m) = \tilde{c}_j$, $\hat{a}_{ij}(m) = \hat{a}_{ij}$, $\tilde{a}_{ji}(m) = \tilde{a}_{ji}$, $\hat{b}_{ij}(m) = \hat{b}_{ij}$, $\tilde{b}_{ji}(m) = \tilde{b}_{ji}$, $\hat{I}_i(m) = \hat{I}_i$, $\tilde{I}_j(m) = \tilde{I}_j$ for all $m \in \mathbb{N}_0$.*

If

$$\mathcal{I} - \text{diag}(|\hat{c}_1|, \dots, |\hat{c}_{N_1}|, |\tilde{c}_1|, \dots, |\tilde{c}_{N_2}|) - \begin{bmatrix} 0 & ((|\hat{a}_{ij}| + |\hat{b}_{ij}|)F_j)_{ij} \\ ((|\tilde{a}_{ji}| + |\tilde{b}_{ji}|)G_i)_{ji} & 0 \end{bmatrix}$$

is a non-singular M-matrix, then the model (21) has a unique equilibrium which is globally exponentially stable.

The global exponential stability of (21), with constant coefficients but without delay in the leakage terms, $\tau = 0$, also was established in [31] but with a different hypotheses set. Also with a different hypotheses set, the global asymptotic stability of (21) with constant coefficients and different delays in the leakage terms, was established in [33].

Now we consider the following discrete-time high-order Hopfield neural network model with delay in leakage terms

$$\begin{aligned}
x_i(m+1) &= c_i(m)x_i(m-\tau) + \sum_{j=1}^N a_{ij}(m)f_j(x_j(m)) \\
&\quad + \sum_{j=1}^N \sum_{l=1}^N b_{ijl}(m)g_j(x_j(m-\tau_{ijl}(m)))g_l(x_l(m-\xi_{ijl}(m))), \quad (22)
\end{aligned}$$

for $m \in \mathbb{N}_0$, $i \in \{1, \dots, N\}$, where $N \in \mathbb{N}$, $\tau \in \mathbb{N}_0$, and $c_i : \mathbb{N}_0 \rightarrow]-1, 1[$, $\tau_{ijl}, \xi_{ijl} : \mathbb{N}_0 \rightarrow \mathbb{N}_0$, $a_{ij}, b_{ijl} : \mathbb{N}_0 \rightarrow \mathbb{R}$, and $f_j, g_j : \mathbb{R} \rightarrow \mathbb{R}$ are functions such that the following hypotheses hold:

(HO1) The functions a_{ij} , b_{ijl} , and τ_{ijl}, ξ_{ijl} are bounded, and consider

$$\begin{aligned}
r &= -\max_{i,j,l,m} \{\tau_{ijl}(m), \xi_{ijl}(m), \tau\}, \quad c_i^+ = \sup_m |c_i(m)|, \\
a_{ij}^+ &= \sup_m |a_{ij}(m)|, \quad b_{ijl}^+ = \sup_m |b_{ijl}(m)|;
\end{aligned}$$

(HO2) for each $j \in \{1, \dots, N\}$, the functions f_j and g_j are Lipschitz i.e., there exist constants F_j and G_j such that

$$|f_j(x) - f_j(y)| \leq F_j|x - y| \text{ and } |g_j(x) - g_j(y)| \leq G_j|x - y|, \quad \forall x, y \in \mathbb{R};$$

(HO3) for each $j \in \{1, \dots, N\}$, the function g_j is bounded i.e., there exists $m_j > 0$ such that

$$|g_j(u)| \leq m_j, \quad \forall u \in \mathbb{R}.$$

From Theorem 3 we obtain the following global exponential stability result.

Theorem 7. Assume (HO1), (HO2), and (HO3).

If there is $\bar{q} = (q_1, \dots, q_N) > 0$ such that

$$d_i(1 - c_i^+) > \sum_{j=1}^N \left(d_j F_j a_{ij}^+ + \sum_{l=1}^N b_{ijl}^+ (m_j d_l G_l + m_l d_j G_j) \right), \quad \forall i \in \{1, \dots, N\}, \quad (23)$$

then model (22) is globally exponentially stable.

Proof. The change $y_i(m) = d_i^{-1}x_i(m)$, $m \in \mathbb{N}$ and $i \in \{1, \dots, N\}$, transforms (22) into

$$\begin{aligned}
y_i(m+1) &= c_i(m)y_i(m-\tau) + \sum_{j=1}^N \tilde{a}_{ij}(m)\tilde{f}_j(y_j(m)) \\
&\quad + \sum_{j=1}^N \sum_{l=1}^N \tilde{b}_{ijl}(m)\tilde{g}_j(y_j(m-\tau_{ijl}(m)))\tilde{g}_l(y_l(m-\xi_{ijl}(m))), \quad (24)
\end{aligned}$$

where

$\tilde{a}_{ij}(m) = d_i^{-1}a_{ij}(m)$, $\tilde{b}_{ijl}(m) = d_i^{-1}b_{ijl}(m)$, $\tilde{f}_j(u) = f_j(d_j u)$, and $\tilde{g}_j(u) = g_j(d_j u)$, for all $m \in \mathbb{N}_0$, $i, j, l \in \{1, \dots, N\}$, and $u \in \mathbb{R}$. Model (24) is a particular case of model (11) with

$$h_{ij}(m, \bar{\alpha}) = \tilde{a}_{ij}(m)\tilde{f}_j(\alpha_j(0)) + \sum_{l=1}^N \tilde{b}_{ijl}(m)\tilde{g}_j(\alpha_j(-\tau_{ijl}(m)))\tilde{g}_l(\alpha_l(-\xi_{ijl}(m))), \quad (25)$$

for all $\bar{\alpha} = (\alpha_1, \dots, \alpha_N) \in X^N$, $m \in \mathbb{N}_0$, and $i, j \in \{1, \dots, N\}$.

For each $i, j \in \{1, \dots, N\}$, from (HO2) and (HO3) the function h_{ij} , defined by (25), verifies

$$\begin{aligned}
& |h_{ij}(m, \bar{\alpha}) - h_{ij}(m, \bar{\beta})| \leq d_i^{-1} a_{ij}^+ |\tilde{f}_j(\alpha_j(0)) - \tilde{f}_j(\beta_j(0))| \\
& + \sum_{l=1}^N d_i^{-1} b_{ijl}^+ |\tilde{g}_j(\alpha_j(-\tau_{ijl}(m))) \tilde{g}_l(\alpha_l(-\xi_{ijl}(m))) - \tilde{g}_j(\beta_j(-\tau_{ijl}(m))) \tilde{g}_l(\beta_l(-\xi_{ijl}(m)))| \\
& \leq d_i^{-1} a_{ij}^+ F_j d_j |\alpha_j(0) - \beta_j(0)| \\
& + \sum_{l=1}^N d_i^{-1} b_{ijl}^+ |\tilde{g}_j(\alpha_j(-\tau_{ijl}(m))) \tilde{g}_l(\alpha_l(-\xi_{ijl}(m))) - \tilde{g}_j(\alpha_j(-\tau_{ijl}(m))) \tilde{g}_l(\beta_l(-\xi_{ijl}(m)))| \\
& + \sum_{l=1}^N d_i^{-1} b_{ijl}^+ |\tilde{g}_j(\alpha_j(-\tau_{ijl}(m))) \tilde{g}_l(\beta_l(-\xi_{ijl}(m))) - \tilde{g}_j(\beta_j(-\tau_{ijl}(m))) \tilde{g}_l(\beta_l(-\xi_{ijl}(m)))| \\
& \leq d_i^{-1} a_{ij}^+ F_j d_j \|\bar{\alpha} - \bar{\beta}\| + \sum_{l=1}^N d_i^{-1} b_{ijl}^+ m_j G_l d_l |\alpha_l(-\xi_{ijl}(m)) - \beta_l(-\xi_{ijl}(m))| \\
& + \sum_{l=1}^N d_i^{-1} b_{ijl}^+ m_l G_j d_j |\alpha_j(-\tau_{ijl}(m)) - \beta_j(-\tau_{ijl}(m))| \\
& \leq \left(d_i^{-1} a_{ij}^+ d_j F_j + \sum_{l=1}^N d_i^{-1} b_{ijl}^+ (m_j d_l G_l + m_l d_j G_j) \right) \|\bar{\alpha} - \bar{\beta}\|,
\end{aligned}$$

for all $m \in \mathbb{N}_0$, and $\bar{\alpha}, \bar{\beta} \in X^N$. Consequently hypothesis (A1) holds with

$$H_{ij} = \left(d_i^{-1} a_{ij}^+ d_j F_j + \sum_{l=1}^N d_i^{-1} b_{ijl}^+ (m_j d_l G_l + m_l d_j G_j) \right), \quad \forall i, j \in \{1, \dots, N\}.$$

By hypothesis (23), condition (12) also holds and the result follows from Theorem 3. $\square$

Now we consider model (22) with periodic delays and coefficients functions. From Theorem 4 and the proof of Theorem 7, we obtain the next result.

Corollary 6. *Assume c_i , a_{ij} , b_{ijl} , τ_{ijl} , and ξ_{ijl} are ω -periodic functions.*

If there is $\bar{q} = (q_1, \dots, q_N) > 0$ such that condition (23) holds, then (22) has a unique ω -periodic solution which is globally exponentially stable.

The exponential stability of (22) with constants coefficients, $\tau_{ijl}(m) = \xi_{ijl}(m)$ for all $i, j, l \in \{1, \dots, N\}$ and $m \in \mathbb{N}_0$, and without delay in the leakage terms ($\tau = 0$) was recently studied in [11]. The authors also assume that

$$f_i(0) = g_i(0) = 0, \quad \forall i \in \{1, \dots, N\},$$

which implies that $x = 0$ is an equilibrium point of

$$\begin{aligned}
x_i(m+1) &= c_i x_i(m) + \sum_{j=1}^N a_{ij} f_j(x_j(m)) \\
&+ \sum_{j=1}^N \sum_{l=1}^N b_{ijl} g_j(x_j(m - \tau_{ijl}(m))) g_l(x_l(m - \tau_{ijl}(m))), \quad (26)
\end{aligned}$$

where $c_i \in]-1, 1[$ and $a_{ij}, b_{ijl} \in \mathbb{R}$. Under all these restrictions, in [11] the global exponential stability of the zero solution of (26) is obtained with the hypothesis:

there is $\bar{q} = (q_1, \dots, N) > 0$ such that

$$d_i(1 - |c_i|) > \sum_{j=1}^N \left(d_j F_j |a_{ij}| + \sum_{l=1}^N |b_{ijl}| m_j d_l G_l \right), \quad \forall i \in \{1, \dots, N\},$$

which is a slight weaker condition than (23).

5. NUMERICAL SIMULATIONS

In this section, we give two numerical examples to illustrate the effectiveness of some the new results presented in this paper.

Example 1. In model (14) with $N = 2$, let $K = 2$ and

$$\begin{aligned} c_1(m) &= \frac{1}{4} \cos m, & c_2(m) &= \frac{1}{12} \sin m, \\ b_{111}(m) &= \frac{\cos m}{8}, & b_{121}(m) &= 0, & b_{112}(m) &= \frac{\sin m}{8}, & b_{122}(m) &= \frac{\sin m}{6}, \\ b_{211}(m) &= \frac{\cos m}{4}, & b_{221}(m) &= -\frac{\sin m}{6}, & b_{212}(m) &= \frac{\sin m}{4}, & b_{222}(m) &= -\frac{5 \sin m}{12}, \\ f_{111}(u) &= f_{121}(u) = f_{211}(u) = f_{221}(u) = \arctan u, \\ f_{112}(u) &= f_{122}(u) = f_{212}(u) = f_{222}(u) = \tanh u \\ \tau_{111}(m) &= \tau_{121}(u) = \tau_{211}(m) = \tau_{221}(m) = 0, \\ \tau_{112}(m) &= \tau_{122}(u) = \tau_{212}(m) = \tau_{222}(m) = 2 + (-1)^m, \\ I_1(m) &= 0, & I_2(m) &= \frac{\cos m}{2}, & \tau &= 2, & r &= 3, \end{aligned}$$

we have $F_{ijk} = 1$ for all $i, j, k \in \{1, 2\}$ and the M -matrix, defined by (17), has the form

$$\mathcal{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{12} \end{bmatrix} - \begin{bmatrix} \frac{1}{4} & \frac{1}{6} \\ \frac{1}{2} & \frac{7}{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{6} \\ -\frac{1}{2} & \frac{1}{3} \end{bmatrix}.$$

As $\mathcal{M}$ is a non-singular M -matrix (the principal minors are positive [13]), by Corollary 1 this example is globally exponentially stable. Figures 1 and 2 give the plot of first and second, respectively, component of three solutions of this numerical example.

Example 2. In model (14) with $N = 2$, let $K = 2$ and

$$\begin{aligned} c_1(m) &= \frac{1}{4} \cos \frac{2m\pi}{\omega}, & c_2(m) &= \frac{1}{12} \sin \frac{2m\pi}{\omega}, & b_{111}(m) &= \frac{1}{8} \cos \frac{2m\pi}{\omega} \\ b_{112}(m) &= \frac{1}{8} \sin \frac{2m\pi}{\omega}, & b_{121}(m) &= 0, & b_{221}(m) &= -\frac{1}{6} \sin \frac{2m\pi}{\omega}, & b_{122}(m) &= \frac{1}{6} \sin \frac{2m\pi}{\omega} \\ b_{211}(m) &= \frac{1}{4} \cos \frac{2m\pi}{\omega}, & b_{212}(m) &= \frac{1}{4} \sin \frac{2m\pi}{\omega}, & b_{222}(m) &= -\frac{5}{12} \sin \frac{2m\pi}{\omega}, \\ \tau_{111}(m) &= \tau_{121}(u) = \tau_{211}(m) = \tau_{221}(m) = 0, \\ \tau_{112}(m) &= \tau_{122}(u) = \tau_{212}(m) = \tau_{222}(m) = 2 + (-1)^m, \\ f_{111}(u) &= f_{121}(u) = f_{211}(u) = f_{221}(u) = \arctan u, & I_1(m) &= 0, & I_2(m) &= \frac{1}{2} \cos \frac{2m\pi}{\omega}, \\ f_{112}(u) &= f_{122}(u) = f_{212}(u) = f_{222}(u) = \tanh u, & \omega &= 10, & \tau &= 2, & r &= 3, \end{aligned}$$

thus all coefficients and delay functions are 10-periodic.

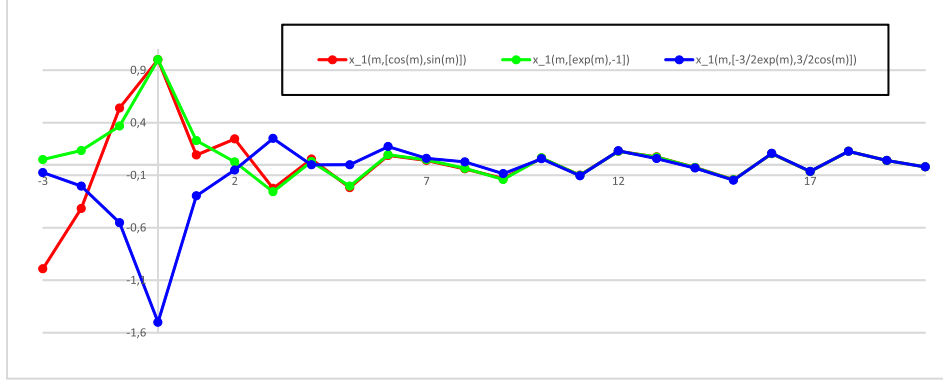


FIGURE 1. The first component, $x_1(m)$, of three solutions of Example 1 with initial condition $\bar{x}_0(j) = (\cos(j), \sin(j))$, $j \in [-3, 0]_{\mathbb{Z}}$, $\bar{x}_0(j) = (e^j, -1)$, $j \in [-3, 0]_{\mathbb{Z}}$, and $\bar{x}_0(j) = (-\frac{3}{2}e^j, \frac{3}{2}\cos(j))$, $j \in [-3, 0]_{\mathbb{Z}}$ respectively.

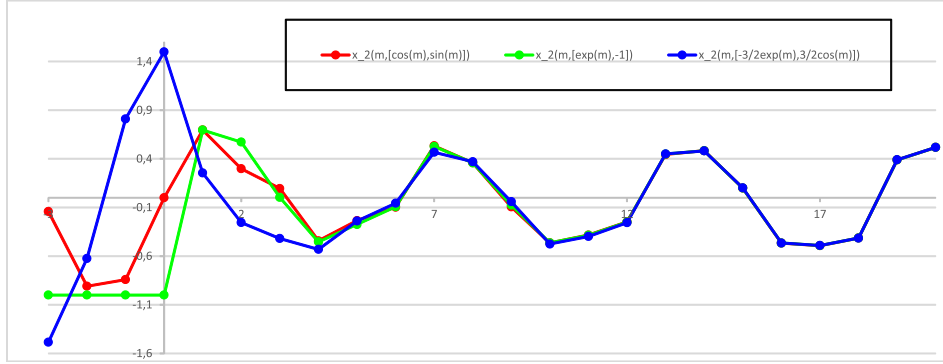


FIGURE 2. The second component, $x_2(m)$, of three solutions Example 1 with initial condition $\bar{x}_0(j) = (\cos(j), \sin(j))$, $j \in [-3, 0]_{\mathbb{Z}}$, $\bar{x}_0(j) = (e^j, -1)$, $j \in [-3, 0]_{\mathbb{Z}}$, and $\bar{x}_0(j) = (-\frac{3}{2}e^j, \frac{3}{2}\cos(j))$, $j \in [-3, 0]_{\mathbb{Z}}$ respectively.

As in Example 1, we have $F_{ijk} = 1$ for all $i, j, k \in \{1, 2\}$ and the M -matrix, defined by (17), has the form

$$\mathcal{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{12} \end{bmatrix} - \begin{bmatrix} \frac{1}{4} & \frac{1}{6} \\ \frac{1}{2} & \frac{7}{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{6} \\ -\frac{1}{2} & \frac{1}{3} \end{bmatrix}$$

and is non-singular. By Corollary 2 this example has a unique 10-periodic solution which is globally exponentially stable. Figures 3 gives the plot of the periodic solution of this illustrative numerical example. Figures 4 and 5 give the plot of first and second, respectively, component of three solutions of this example, together with the periodic solution.

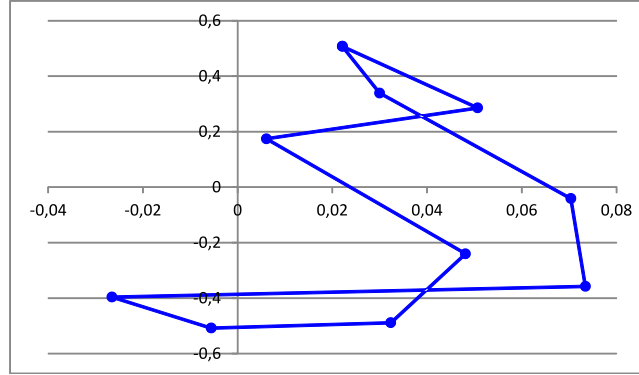


FIGURE 3. The 10-periodic solution of Example 2.

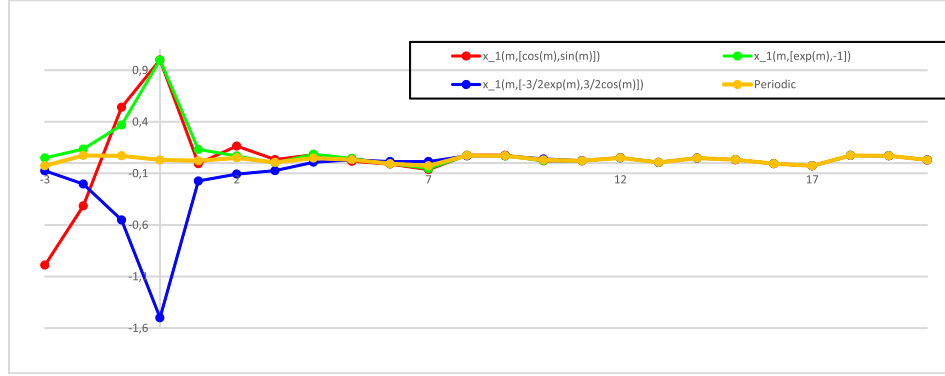


FIGURE 4. The first component, $x_1(m)$, of three solutions of Example 2 with initial condition $\bar{x}_0(j) = (\cos(j), \sin(j))$, $j \in [-3, 0]_{\mathbb{Z}}$, $\bar{x}_0(j) = (e^j, -1)$, $j \in [-3, 0]_{\mathbb{Z}}$, and $\bar{x}_0(j) = (-\frac{3}{2}e^j, \frac{3}{2}\cos(j))$, $j \in [-3, 0]_{\mathbb{Z}}$ respectively.

6. CONCLUSIONS

In optimization problems [8] and in computational image processing [1], the global stability of neural network models holds significant importance.

Through the use of induction arguments, we were able to obtain a global exponential stability criterion for a broad class of delayed difference equations (Theorem 1). In the particular case of periodic coefficients, we built a Poincaré type map to guarantee the existence of a periodic solution without requiring additional conditions (Theorem 2).

Our main results allowed us to obtain new stability criteria for several Hopfield and BAM neural network models with a delay in the leakage terms (Theorems 5, 6 and 7).

An important distinction of our work lies in the fact that our hypotheses do not depend on the delay in the leakage term, contrary to the conditions typically assumed in existing literature on neural network models with delayed leakage terms.

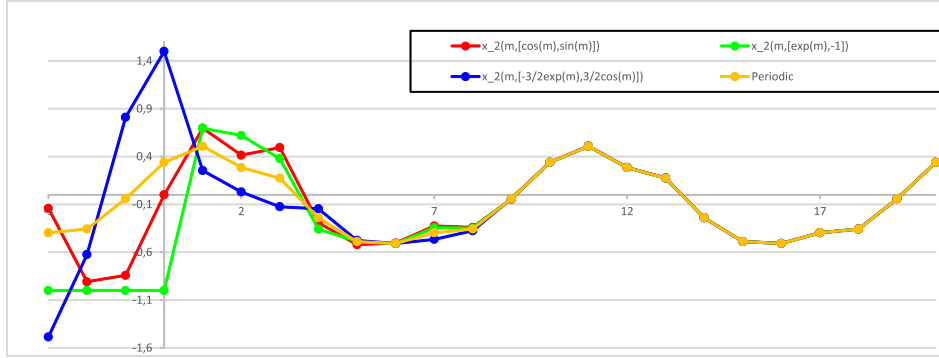


FIGURE 5. The second component, $x_2(m)$, of three solutions of Example 2 with initial condition $\bar{x}_0(j) = (\cos(j), \sin(j))$, $j \in [-3, 0]_{\mathbb{Z}}$, $\bar{x}_0(j) = (e^j, -1)$, $j \in [-3, 0]_{\mathbb{Z}}$, and $\bar{x}_0(j) = (-\frac{3}{2}e^j, \frac{3}{2}\cos(j))$, $j \in [-3, 0]_{\mathbb{Z}}$ respectively.

Furthermore, our conditions are easily verifiable, as demonstrated through the illustrative examples herein.

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