

Contact-impact events with friction in multibody dynamics: Back to basics

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This work is dedicated to Professor Parviz Nikraves, a true and inspiring educator and mentor.

Abstract

Multibody dynamics deals with the study of mechanical systems composed of multiple bodies, whose motion interactions are governed by the presence of kinematic constraints and by the application of external forces. Frictional contact-impact events are among the most complex and important phenomena that can be modeled under the umbrella of multibody systems, since their behavior depend on critical factors, including geometry of the contacting surfaces, material properties of the colliding bodies, and constitutive laws utilized to mimic the contact response. This paper is aimed to present a comprehensive description of the most relevant aspects and the state-of-the-art techniques concerning the modeling collisions in multibody dynamics. For that purpose, the contact geometry, contact detection and contact resolution procedures are subjected to a critical analysis. Particular attention is given to the regularized or continuous models to treat frictional contacts in dynamical systems. Moreover, relevant numerical ingredients associated with contact-impact events in multibody systems are examined with the intent to discuss the computational accuracy and efficiency. Application examples are provided whose results allow to highlight the key features related to the modeling process of frictional contacts and impacts in multibody systems.

Keywords: Contact-impact events, frictional collisions, multibody dynamics, regularized methods, continuous analysis, force-based models, compliant forces, numerical aspects

1. Introduction

This paper aimed at presenting a general introduction and a comprehensive discussion of the state-of-the-art on the computational techniques to address frictional collisions within the discipline contact-impact dynamics in multibody systems. The topic of contact-impact dynamics can be understood as the study of the motion of mechanical systems in the presence of contact-impact forces or impulses. The main motivation for this work comes from the continuous existing interest in developing models for the dynamic simulation of multibody models with frictional collisions. Contact dynamics problems are omnipresent in wide spectrum of practical applications, as in many mechanical systems, the function and performance of the systems strongly depend on the contact-impact phenomena. The spirit of this work lies on the critical analysis of the basics and fundamentals of contact dynamics returning to the simple and most important aspects related to the frictional collisions in multibody systems.

In the broadest sense, a “system” is a set of connected things or parts, and “mechanical” is related to mechanics or mechanisms. In turn, mechanics is the branch of science dealing with motion and tendencies to motion, while mechanism is the structure or adaptation of parts of a machine. Thus, a mechanical system can be seen as a set of components that fulfill a same function. Mechanical systems, as technological parts, are typically man-made objects that can incorporate several components working in them. Mechanical elements can play different basic functions, such as linking and guiding. There are more complex mechanical functions named motion transmission, and motion transformation. Motion transmission is the transference of motion from one element to another without modifying the nature of the motion. Motion transformation is the transference of motion from one element to another while at the same time modifying or changing the nature of the motion. All mechanical systems contain at least a driver and one driven element, and possible, one or more intermediate elements. The windshield wiper mechanism, illustrated in Fig. 1, is based on one of the most common mechanical system, the planar four-bar mechanism. The primary purpose of the windshield is to transform a purely rotational motion into a different type of motion. Depending on the lengths of the links of the four-bar mechanism the produced response will be different. In the particular case of the windshield wiper system, a double crank four-bar linkage is utilized to generate an oscillatory or rotating motion of the driven element. The driver element of the windshield wiper system is an electrical motor that can rotate continuously.

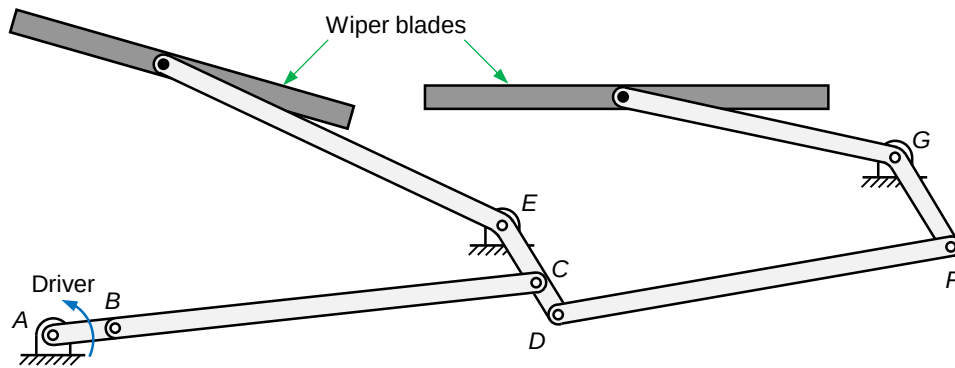


Fig. 1 Schematic representation of the windshield wiper mechanism, which is composed by a double-crank four-bar mechanism.

In mechanical systems, the driver is the element that starts or inputs the motion; the driven element, which is often called the follower because it follows the motion; is the component where the motion gets transmitted to; and the intermediate is the element that connects the driver and the driven elements. The driver receives the power required to set the system in motion; the driven, or follower, receives the motion of the driver and transmits it to another element; and the intermediate, or coupler, is located between the driver and the driven. The intermediate may or may not be present in a mechanical system. A complex mechanical function is a role played by a set of elements in transferring motion inside a technological system. This can either refer to transmitting motion or transforming motion. It must be noted that in motion transmission the driver and the follower have the same motion characteristics, while in motion transformation the driver and the follower do not have the same motion properties. Examples of elementary mechanical systems are linkages, cam-follower mechanisms, friction wheels, chains and sprockets. All the above mentioned features constitute the main ingredients of the development of multibody systems. In short, according to Nikravesh [1] a multibody system can be defined as a collection of bodies interconnected by kinematic joints and force elements, in which some or all of the bodies can move relative to one another. Multibody systems are ubiquitous in science and engineering fields, and a significant number of practical problems can be modeled and simulated in the spirit of the multibody systems formulations. Indeed, a wide variety of situations can be studied in this way, namely mechanisms [2], biomechanical systems [3], vehicle and aircraft systems [4], robots [5], human joints [6], and common toys [7], as depicted in Fig. 2.

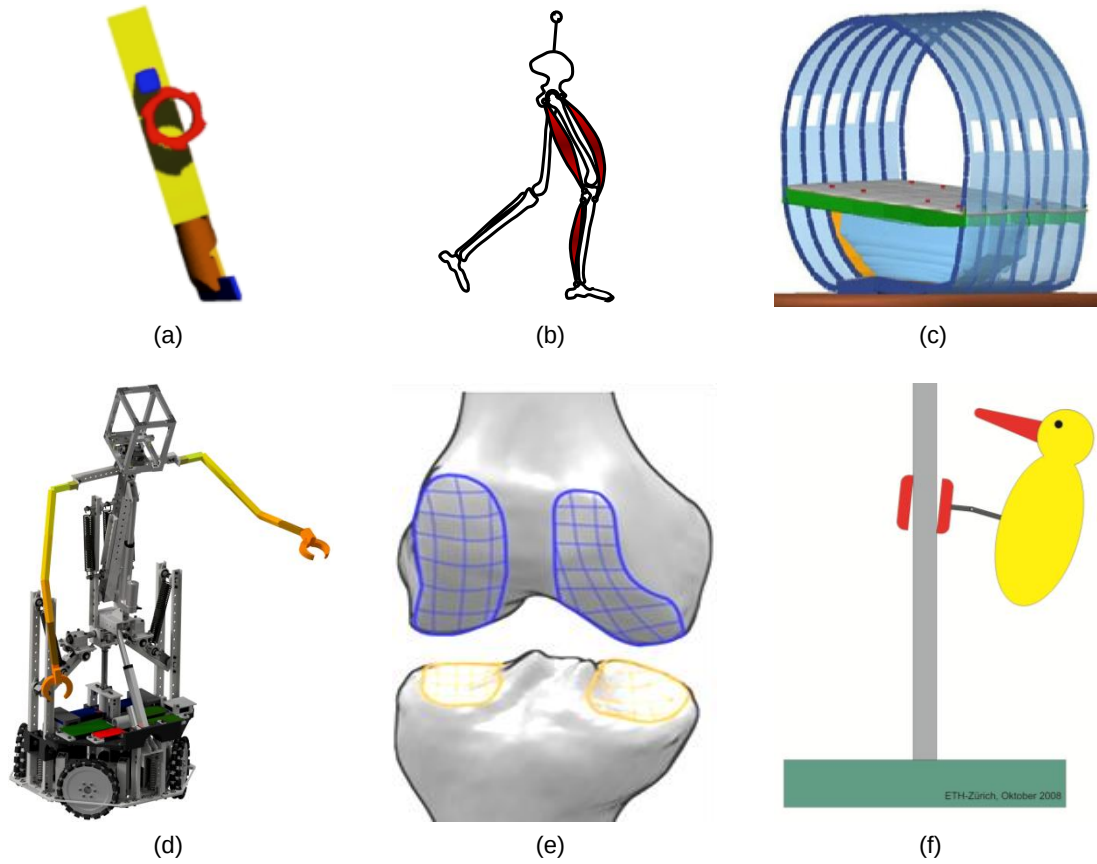


Fig. 2 Examples of application of multibody models for dynamic analysis: **(a)** Cam-follower mechanism [2]; **(b)** Human biomechanical system [3]; **(c)** Aircraft fuselage multibody model [4]; **(d)** Robotic system [5]; **(e)** Human knee articulation [6]; **(f)** Woodpecker toy multibody model [7].

By and large, contact is associated with the study of touching, meeting or communicating. Impact is the action of one body coming forcibly into contact with another. Collision can be seen as a violent impact of a moving body with another or with a fixed object. According to Nikravesh [8], collisions between bodies or sudden contact of one body with the ground can take place in a good number of applications of multibody systems. These situations are often characterized by very short time duration and extremely high force intensity. Frictional collisions deal with the study of contacting bodies in the presence of friction phenomenon. Contact-impact dynamics, in a broad sense, can be understood as the discipline that includes the motion analysis of multibody mechanical systems with frictional collisions. Events involving contact-impact dynamics are likely to occur in different domains, such as robotics and walking machines [9–15], railway vehicles and subsystems [16–22], vehicle and crash models [23–26], biosystems and biomechanical models [27–32], machines and mechanisms [33–38], granular media and powder technologies [39–42], physical toys models [43–47], civil structures [48–51], sounds and musical instruments [52–56], fruit transport and handling [57–60], just to mention some examples under the umbrella of dynamics of mechanical systems.

On the whole and examining the short duration, a collision process can be considered to happen into two different stages, namely the “compression phase” and the “restitution phase” [61]. The compression phase, also named as approaching or loading stage, initiates when the colliding bodies come into contact, and finishes when the maximum local deformation is reached. During this stage, the relative contact velocity is gradually reduced from the maximum impact velocity to zero. In turn, the restitution phase, also called separating or unloading stage, initiates at the end of the compression phase, and finishes when the colliding bodies separate from each other. The coefficient of restitution is a simplified quantification of the type of collision, such that for a fully elastic situation, this parameter is equal to unity, while for a fully inelastic, or plastic case, it is null. The most general type of collision in multibody systems involves a coefficient of restitution that varies from 0 to 1 [61].

The mathematical description of the collision phenomena in multibody systems is strongly dependent on the procedure utilized to represent the contact-impact interactions between the colliding bodies. In general, there are two different strategies that can be followed to model collisions in the framework of multibody systems formulations, namely the “piecewise” method and the “continuous” approach [62]. Either of the two methods is suitable for the computational analysis in multibody dynamics. Both techniques require accurate evaluation of the precise instant of contact between the colliding bodies [63].

In the piecewise method, also known as discrete, rigid or impulse-momentum approach, the impact process is considered to happen instantaneously, and the discontinuous nature of the collision results in an abrupt variation in the velocities at the instant of collision. In fact, in the piecewise approach, the rapid change in the impact velocities is infinitesimal, and at the same time, the position and orientation of the bodies in the multibody system remain unchanged. This assumption is valid if the duration of the collision is extremely small. Thus, in this method, the dynamic analysis of the multibody system is performed based on the numerical resolution of the equations of motion until a discontinuous event occurs at the instant of a collision. At this instant, the resolution of the system’s equations of motion is paused, and a momentum balance is performed to determine the post-impact velocities of the bodies involved in the collision. Subsequently, the resolution of the equations of motion is restarted with the updated velocities until a new collision takes place. In the piecewise method, the response of the system under analysis subjected to a discontinuous event is divided into two distinct stages, namely before and after collision. Typically, the coefficient of restitution is employed to quantify the energy dissipated during the impact process. The coefficient of restitution only relates the relative pre- and post-impact velocities, and ignores what happens in between those two stages. Newton [64] was the pioneer to investigate on impact between two rigid bodies and established his impact theory, usually materialized by the kinematic coefficient of restitution, which is defined as the quotient between the relative normal velocities of the colliding bodies just after and before the impact. When collisions include friction, the Newton’s impact theory might fail in some cases, since it is not able to account for different friction regimens. In these cases, more advanced formulations have to be adopted, such as the Poisson [65] or Stronge [66] approaches. It should be noted that the piecewise method is relatively efficient but the unknown duration of the collision limits its applicability, in the measure that for long contact-impact events, the configuration of the system can change in a significant manner [67]. A difficulty associated with the piecewise method deals with its application to model general contact-impact events, mainly when multiple points of contact occur at the same time. Another issue of this method is the existence of multiple solutions or some inconsistencies when the Coulomb’s friction law is employed [68].

In the continuous approach, also known as compliant, regularized or force-based technique, the bodies are allowed to deform locally and, consequently, the relative contact velocity of the colliding bodies varies in a continuous manner. Thus, the interaction forces developed during the contact-impact process are typically represented by linear or non-linear spring-damper elements that mimic the stiffness and damping terms associated with the resistive contact action. The contact-impact forces are expressed as continuous functions of the local deformations in the contact zone. In multibody dynamics, the local deformation, or indentation, is represented by the geometric overlap or pseudo-penetration that occurs at the colliding zone. Hertz [69] pioneered the investigation on the contact problems and presented a nonlinear and non-dissipative force model, which relates the contact stiffness of the colliding bodies with the local deformation. One drawback associated with the Hertzian force model deals with the difficulty to identify the contact stiffness and the degree of nonlinearity of the deformation. Although, the Hertz contact force model is founded on the elasticity theory, some studies have been performed to extend it to accommodate the energy dissipated during the contact-impact process. Different contact force models have been proposed in the thematic literature, most of them are based on the Hunt and Crossley’s cornerstone work [70].

The continuous analysis technique to treat contact problems in multibody systems is effective, since the contact reaction forces are simply evaluated during the collision period and introduced into the system's equations of motion as external applied forces. In fact, the continuous analysis is quite simple and straightforward to implement in any general multibody code. In this approach, there is no need for the impulse dynamics, and the different contact scenarios, as well as the transition between the several contact situations, are easy to handle, since it is not necessary to distinguish the contact from non-contact cases, neither the different contact regimens. The continuous analysis can be source of high-frequency dynamics into the system's equations of motion, due to the presence of contact related stiff springs to represent the compliant surfaces. This aspect can be prejudicial in terms of computational efficiency. In the continuous approach, the friction forces can also be determined and incorporated in the system dynamics, since the tangential contact forces can be evaluated with basis on any regularized formulation available in the literature [71–76]. It can be stated that this technique overcomes the main difficulties associated with the piecewise method, namely the ability to handle multiple contact points and complex geometries. In short, the continuous analysis has been gaining major importance in the treatment of frictional contact-impact events in multibody dynamics, in virtue of its simplicity and efficiency from computational point of view [77–83].

In the present contribution, the continuous approach is considered to model and simulate multibody systems with frictional collisions. The basic premise of this work is to provide a comprehensive description of the key ingredients that contribute to the modeling process of frictional contact-impact events under the framework of multibody systems formulations. The remaining of this paper is structured as follows.

Section 2 presents a brief overview of the central aspects related to multibody dynamics, where its definition is revisited. The Newton-Euler equations of motion of a simple case are derived from the concept of free-body diagram and kinematic constraints, which results in a set of differential algebraic equations. The resolution of the equations of motion is discussed, which is characterized by an initial value problem, which includes a numerical integration algorithm.

In Section 3, the fundamental aspects related to contact kinematics is revisited, which allows for the comprehensive evaluation of the penetration depth and normal and tangential contact velocities between two colliding bodies within a multibody system. The contact kinematic analysis plays a key in the contact-impact process, since it allows to identify the different contact regimens, as well as the transition between them.

Section 4 deals with the contact-impact dynamics based on both piecewise and continuous methods. For that, the one-dimensional central collision between two solid spheres is considered to address the main issues associated with the classical theory of rigid body contact and impact events. In the sequel of this analysis, the incorporation of the collision solution into the standard equations of motion for constrained multibody systems is discussed.

The main prominent normal and tangential contact forces are presented in Sections 5 and 6, respectively, which are discussed from an historical perspective. In addition, the different normal and tangential contact force models utilized in this analysis are supported with simple application cases, which allows for the assessment of the similarities and differences in terms of the behavior of dynamical systems with contact-impact events.

Section 7 includes several demonstrative application examples of multibody systems involving contact-impact events. The different contact scenarios considered are fundamental to highlight some important aspects when modeling collisions in multibody dynamics, namely in what concerns the contact detection between the colliding surfaces, as well the contact resolution based on the piecewise and continuous approaches. The pros and cons associated with each methodology to handle contact-impact events are also analyzed.

Finally, Section 8 discusses the main conclusions of this work, where future directions for research of the contact-impact problems within of multibody systems are offered.

2. Multibody dynamics

As stated earlier, a multibody system is a collection of two or more interconnected bodies capable of performing large displacements. By and large, a multibody system embraces three major features, namely: (i) mechanical components that can be either rigid or flexible bodies, and that describe large translational and rotational displacements; (ii) kinematic pairs that can impose restrictions on the relative motion of the interconnected bodies; (iii) force elements that represent internal forces and, possibly, the interaction between the bodies and the surrounding environment [1]. A body is formally considered as rigid when the distance between any set of two points located on the body is constant during motion. In other words, the distance between any two points on a rigid body remains constant in time regardless of external forces exerted on it. In essence, rigid body means that the change of dimensions and shape of the body are null or small enough to be neglected during motion. Conversely, a flexible body undergoes large displacements that combine rigid body motion and elastic deformation, being the motion of the flexible body affected by its elastic deformation.

In a simple and comprehensive manner, a multibody system can be defined as a set of bodies, constrained by kinematic pairs, acted by forces that describe large displacements. This last attribute is of paramount importance to identify a multibody system. In order to clarify what a large displacement is, let consider a cantilever beam with a fixed support at the left end, and free at the right end, as Fig. 3a shows. Due to the applied external force, f , the free-end describes a vertical deflection or elastic deformation, δ . Thus, a displacement can be regarded as large when the deformation, δ , is slightly greater than the cantilever thickness, h . It is manifest that the cantilever beam does not look like an ordinary multibody system, since it is constrained, or restricted, in a way that is cannot move. In fact, the cantilever was only considered to revise the concept of large displacement. Let change the cantilever fixed support end to transform it into a hinge joint, resulting in a simple pendulum, as illustrated in Fig. 3b. Hence, the beam or body can rotate around the joint. This rotation, or change in orientation, is referred to as rigid body motion. In short, large displacement, which is the key differentiator in a multibody system, may include rigid body motion and elastic deformation.

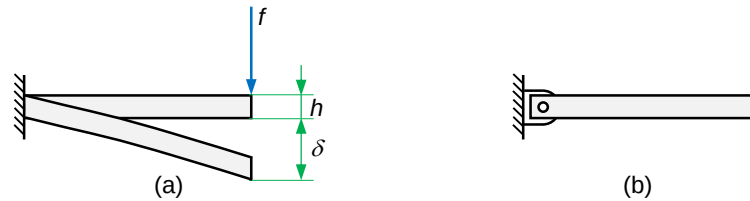


Fig. 3 (a) Cantilever beam; (b) Planar simple pendulum.

One of the characteristics of the simple pendulum is that it only rotates around the point O (see Fig. 4a). In fact, at the point O , the revolute joint connecting the arm and ground, allows one rotation and constrains two translations. In order to be able to write the equations of motion of the simple pendulum, let first establish an inertial, or global, coordinate system xy attached to the ground at point O , which corresponds to the origin to measure the different pose, or configurations, of this multibody model. Moreover, a body, or local, coordinate system $\xi_1 \eta_1$, is placed in the center of the arm, which origin is positioned at point P and ξ -axis is aligned with the longitudinal direction of the arm. Since the simple pendulum system has only one moving body, the maximum number of absolute coordinates that define the system's configuration is equal to three, and can be expressed as

$$\mathbf{q} = \{x_1 \quad y_1 \quad \theta_1\}^T \quad (1)$$

which represents the translation of the origin of the arm coordinate system along the x and y directions of the global reference frame, and the rotation of the arm coordinate system with respect to the global reference frame.

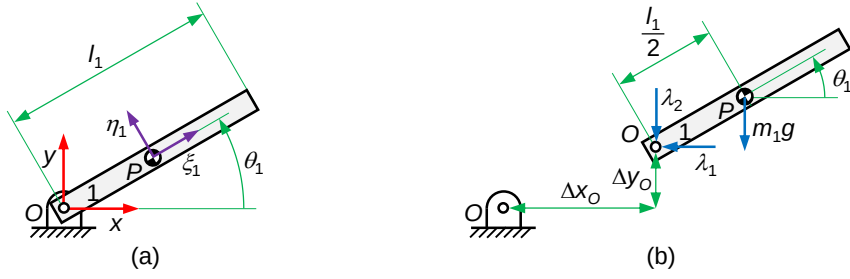


Fig. 4 (a) Simple pendulum multibody model; **(b)** Arm free-body diagram.

For the application of the Newton-Euler equations of motion to the simple pendulum multibody model, the free-body diagram of the arm has to be constructed, as Fig. 4b depicts. In this process, the hinge joint is substituted by the two appropriate constraint reaction forces, λ_1 and λ_2 . Thus, applying the Newton-Euler or translational and rotational equations of motion to the arm represented in Fig. 4b yields

$$m_1 \ddot{x}_1 = -\lambda_1 \quad (2)$$

$$m_1 \ddot{y}_1 = -\lambda_2 - m_1 g \quad (3)$$

$$I_1 \ddot{\theta}_1 = -\lambda_1 \frac{l_1}{2} \sin \theta_1 + \lambda_2 \frac{l_1}{2} \cos \theta_1 \quad (4)$$

This set of three differential equations contains five unknowns, three accelerations and two joint reaction forces, which makes a system with infinite number of solutions. Since, the simple pendulum exhibits a regular motion, additional information is required to solve the dynamics of the simple pendulum. Thus, with the purpose to have a unique solution for this multibody model, two extra equations are necessary, namely those associated with the two kinematic constraints of the hinge joint. To enforce the constraints Δx_O and Δy_O of zero displacement, it can be said that the position of the point O measured from the pendulum coordinate system must be equal, at all time, to the position of the point O measured from the ground body. Thus, from the analysis of the diagram in Fig. 4b, it can be observed that the two geometric constraint equations Φ_1 and Φ_2 associated with the revolute joint of the simple pendulum can be written as

$$\Phi_1 \equiv \Delta x_O = 0 \Leftrightarrow x_1 - \frac{l_1}{2} \cos \theta_1 = 0 \quad (5)$$

$$\Phi_2 \equiv \Delta y_O = 0 \Leftrightarrow y_1 - \frac{l_1}{2} \sin \theta_1 = 0 \quad (6)$$

The two algebraic constraint equations (5) and (6), which mathematically express how the arm should move with respect to the ground, can be written in the compact matrix form as

$$\Phi \equiv \Phi(\mathbf{q}) = \mathbf{0} \quad (7)$$

where $\mathbf{q}$ represents the array of absolute coordinates given by Eq. (1). In the most general case, position constraints given by Eq. (7) are nonlinear algebraic equations [8].

It should be noticed that Eqs. (5) and (6) are written at the position level and, therefore, they cannot be directly associated with the equations of motion (2)-(4). For that to be possible, the second time derivative of the constraint equations needed to be evaluated. Thus, the constraint equations at the velocity level can be obtained as the first time derivative of the position constraints (5) and (6), yielding

$$\dot{\Phi}_1 \equiv \dot{x}_1 + \frac{l_1}{2} \sin \theta_1 \dot{\theta}_1 = 0 \quad (8)$$

$$\dot{\Phi}_2 \equiv \dot{y}_1 - \frac{l_1}{2} \cos \theta_1 \dot{\theta}_1 = 0 \quad (9)$$

The velocity constraints equations (8) and (9) can be written in a matrix form as

$$\begin{bmatrix} 1 & 0 & \frac{l_1}{2} \sin \theta_1 \\ 0 & 1 & -\frac{l_1}{2} \cos \theta_1 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{\theta}_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (10)$$

or in a compact form as

$$\dot{\Phi} \equiv \Phi_q \dot{\mathbf{q}} = \mathbf{0} \quad (11)$$

where Φ_q represents the Jacobian matrix, that is, the matrix of the partial derivatives, $\partial\Phi/\partial\mathbf{q}$, and $\dot{\mathbf{q}}$ is the array of velocities. It must be noticed that velocity constraints given by (11) are algebraic equations linear in velocities [8]. The time derivative of the velocity constraints yields the kinematic constraints at the acceleration level. Thus, the time derivative of velocity constraints (8) and (9) results in

$$\ddot{\Phi}_1 \equiv \ddot{x}_1 + \frac{l_1}{2} \sin \theta_1 \ddot{\theta}_1 + \frac{l_1}{2} \cos \theta_1 \dot{\theta}_1^2 = 0 \quad (12)$$

$$\ddot{\Phi}_2 \equiv \ddot{y}_1 - \frac{l_1}{2} \cos \theta_1 \ddot{\theta}_1 + \frac{l_1}{2} \sin \theta_1 \dot{\theta}_1^2 = 0 \quad (13)$$

The acceleration constraints (12) and (13) can be rewritten in a matrix form as

$$\begin{bmatrix} 1 & 0 & \frac{l_1}{2} \sin \theta_1 \\ 0 & 1 & -\frac{l_1}{2} \cos \theta_1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\theta}_1 \end{Bmatrix} = \begin{Bmatrix} -\frac{l_1}{2} \cos \theta_1 \dot{\theta}_1^2 \\ -\frac{l_1}{2} \sin \theta_1 \dot{\theta}_1^2 \end{Bmatrix} \quad (14)$$

or in a compact form as

$$\ddot{\Phi} \equiv \Phi_q \ddot{\mathbf{q}} = \boldsymbol{\gamma} \quad (15)$$

where Φ_q represents the Jacobian matrix, $\ddot{\mathbf{q}}$ denotes the array of accelerations, and $\boldsymbol{\gamma}$ is the right-hand side vector of acceleration constraints that contains the quadratic terms in velocities in the equations of motion, namely the centrifugal parcel. It is worth noting that acceleration constraints given by (15) are algebraic equations linear in accelerations [8].

Finally, the Newton-Euler equations of motion of the arm, together with the constraint equations related to the hinge joint, can be written in the expanded form as

$$\begin{bmatrix} m_1 & 0 & 0 & 1 & 0 \\ 0 & m_1 & 0 & 0 & 1 \\ 0 & 0 & I_1 & \frac{l_1}{2} \sin \theta_1 & -\frac{l_1}{2} \cos \theta_1 \\ 1 & 0 & \frac{l_1}{2} \sin \theta_1 & 0 & 0 \\ 0 & 1 & -\frac{l_1}{2} \cos \theta_1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\theta}_1 \\ \lambda_1 \\ \lambda_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -m_1 g \\ 0 \\ -\frac{l_1}{2} \cos \theta_1 \dot{\theta}_1^2 \\ -\frac{l_1}{2} \sin \theta_1 \dot{\theta}_1^2 \end{Bmatrix} \quad (16)$$

or in a compact form as

$$\begin{bmatrix} \mathbf{M} & \Phi_q^T \\ \Phi_q & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{g} \\ \boldsymbol{\gamma} \end{Bmatrix} \quad (17)$$

where $\mathbf{M}$ is the system mass matrix, Φ_q represents the Jacobian matrix of the constraint equations, $\ddot{\mathbf{q}}$ denotes the vector containing the system accelerations, $\boldsymbol{\lambda}$ is Lagrange multipliers vector that contains the joint reaction forces, $\mathbf{g}$ is the generalized vector of applied forces, and $\boldsymbol{\gamma}$ denotes the right-hand side vector of the acceleration equations [1, 8].

Equation (17) represents a linear system of differential and algebraic equations (DAE) [84] that includes five equations with five unknowns. In fact, Eq. (17) constitutes a system of DAE of index-1 [85] that is linear in the accelerations and in the Lagrange multipliers, and can be solved using any numerical technique for this type of problem. At this stage, it must be noticed that there is a correspondence between the Newton-Euler equations of motion, the free-body diagrams and the algebraic constraint equations, which is materialized by the Jacobian matrix. This fact is especially relevant in the measure that the tedious free-body diagrams associated with the Newton-Euler method can be dispensable in the dynamics analysis of multibody systems [37].

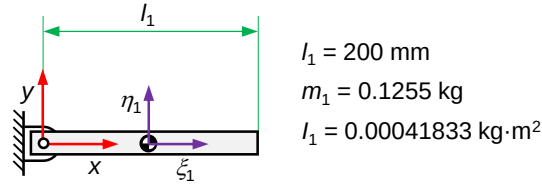


Fig. 5 Simple pendulum multibody model in its initial position.

In order to solve the dynamic equations of motion (17) it is first necessary to know the initial values of the positions and velocities of the pendulum system. The set of initial conditions must satisfy the constraint equations at both position and velocity levels [1, 8]. Thus, let consider that initially the simple pendulum under analysis rests in the horizontal position shown in Fig. 5. The system is then released from this initial configuration under the gravity action only, which is taken as acting in the negative y direction. The geometric and inertia properties of the arm are also included in the representation of Fig. 5. Consequently, for this particular configuration of the simple pendulum, the corresponding set of initial conditions for positions and velocities can easily be determined as

$$\mathbf{q}^0 = \{0.1 \quad 0.0 \quad 0.0\}^T$$

$$\dot{\mathbf{q}}^0 = \{0.0 \quad 0.0 \quad 0.0\}^T$$

Thus, after computing all the terms necessary to be introduced into the equations of motion, the numerical resolution of this set of linear equations yields the following solution

$$\ddot{x}_1 = 0.0000 \text{ m/s}^2$$

$$\ddot{y}_1 = -7.3575 \text{ m/s}^2$$

$$\ddot{\theta}_1 = -73.5751 \text{ rad/s}^2$$

$$\lambda_1 = 0.0000 \text{ N}$$

$$\lambda_2 = -0.3078 \text{ N}.$$

When performing a dynamic motion analysis of multibody systems, consisting of rigid bodies with kinematic constraints, it is, in general, desired to know the coordinates of the bodies as function of time. For the particular case of the simple pendulum under analysis, the coordinates to be computed during the dynamic analysis can be expressed as $x_1(t)$, $y_1(t)$ and $\theta_1(t)$. The velocities of the bodies may also be object of interest to be computed.

It should be noticed that in the above calculations, the accelerations obtained for the arm correspond to the initial instant of the analysis. Thus, based on the spirit of Newton's second law of motion, it is known that the velocities will change due to the existence of accelerations. The key issue in that process deals with the evaluation of the new positions and velocities. Since in most of the practical problems the analytical solution does not exist, a numerical procedure must be adopted to obtain the new positions and velocities. For this purpose, let consider the Euler's integration algorithm, solution of which can be written as

$$\mathbf{q}(t + \Delta t) = \mathbf{q}(t) + \dot{\mathbf{q}}\Delta t \quad (18)$$

$$\dot{\mathbf{q}}(t + \Delta t) = \dot{\mathbf{q}}(t) + \ddot{\mathbf{q}}\Delta t \quad (19)$$

Equations (18) and (19) can be utilized to compute the positions and velocities of the simple pendulum as the time analysis progresses. It must be noted that the Euler's integration scheme is quite popular due to its simplicity and computational efficiency, being the outcomes affected by the time step size, Δt [1].

Finally, it must be noticed that the Newton-Euler approach results in a set of differential and algebraic equations of motion (17). This system of equations can be solved for the linear and rotational accelerations, $\ddot{\mathbf{q}}$, and joint reaction forces, λ , by using any numerical algorithm available for handling systems of linear equations [86]. Subsequently, the resulting accelerations together with velocities are numerically integrated in time to obtain the corresponding velocities and positions at the next time step. This procedure must be repeated until the final time of the dynamic simulation is reached. Figure 6 shows a generic flowchart that summarizes the strategy to be adopted when solving the equations of motion of any multibody system, which can be condensed in the following stages [1]:

1. Start at instant of time t^0 with given initial condition for the positions, $\mathbf{q}^0$, and velocities, $\dot{\mathbf{q}}^0$,
2. Assemble the global mass matrix, $\mathbf{M}$, evaluate the Jacobian matrix, Φ_q , determine the right-hand side of the accelerations, $\mathbf{g}$, and calculate the generalized force vector, $\mathbf{g}$,
3. Solve the linear set of the equations of motion (17) for a constrained multibody system in order to obtain the accelerations, $\ddot{\mathbf{q}}$, and the Lagrange multipliers, λ , at instant of time t ,
4. Assemble the vector $\dot{\mathbf{y}}^t$ containing the generalized velocities, $\dot{\mathbf{q}}$, and accelerations, $\ddot{\mathbf{q}}$, for instant of time t ,
5. Integrate numerically the $\dot{\mathbf{q}}$ and $\ddot{\mathbf{q}}$ vectors for time step $t+\Delta t$ and obtain the new positions and velocities,
6. Update the time variable, go to step 2 and proceed with the process for a new time step, until the final time of the analysis is reached.

Since the equations of motion (17) do not account for the constraint equations at the position and velocity levels, it is necessary to incorporate any available technique to eliminate or minimize the constraints violation [87–89].

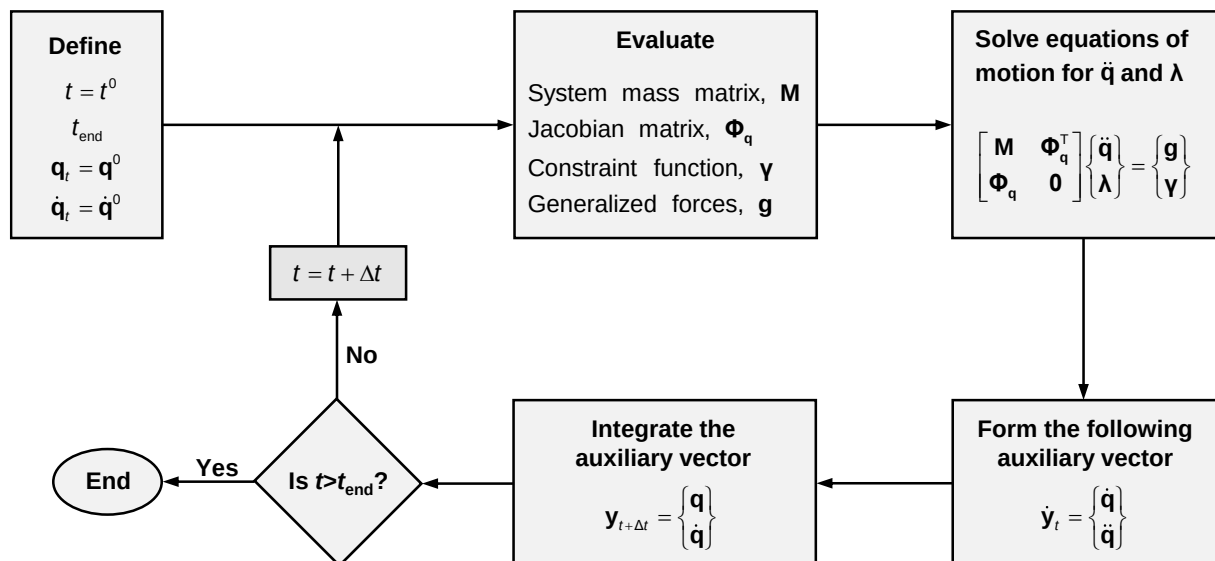


Fig. 6 Flowchart of computational procedure for dynamic analysis of constrained multibody systems.

3. Contact-impact kinematics

Contact-impact kinematics involves the evaluation of three fundamental parameters, namely the location of the potential contact points, the distance between those points, and the corresponding relative contact velocity [90]. These aspects play a key role in modeling and simulation of collisions in multibody systems, in the measure that the contact-impact forces are function of the contact kinematic analysis [91]. In order to better characterize the essence related to the contact-impact kinematics, let consider two generic rigid bodies that belong to a multibody system and that can experience an oblique eccentric collision. Figure 7a depicts two convex surfaces associated with bodies 1 and 2 in the state of separation, being the potential contact points denoted by P_1 and P_2 . Indeed, in most of the practical applications of multibody systems with collisions, the contact locus is considered to be punctual due to the convex boundary nature of the surfaces where the contact points might occur. The possible motion of each body in a multibody system can be described in terms of the distance and relative velocity of the potential contact points. Positive values of that distance represent a separation state, while negative values denote the existence of a relative deformation or penetration of the colliding bodies. These, two distinct scenarios are illustrated in the representations of Figs. 7a and b, respectively. Thus, it can be said that the change in sign of the distance indicates a transition from separation to contact situation, or vice versa. In turn, positive values of the relative normal velocity between the contact points, that is, the deformation or penetration velocity, indicate that the colliding bodies are approaching, which corresponds to the compression phase of the impact process. Negative values of the relative normal velocity denote that the colliding bodies are separating, which corresponds to the restitution phase of the impact process. The relative tangential velocity of the contact points determines whether the colliding surfaces are sliding or sticking [92], which are of paramount importance in the friction behavior in multibody systems.

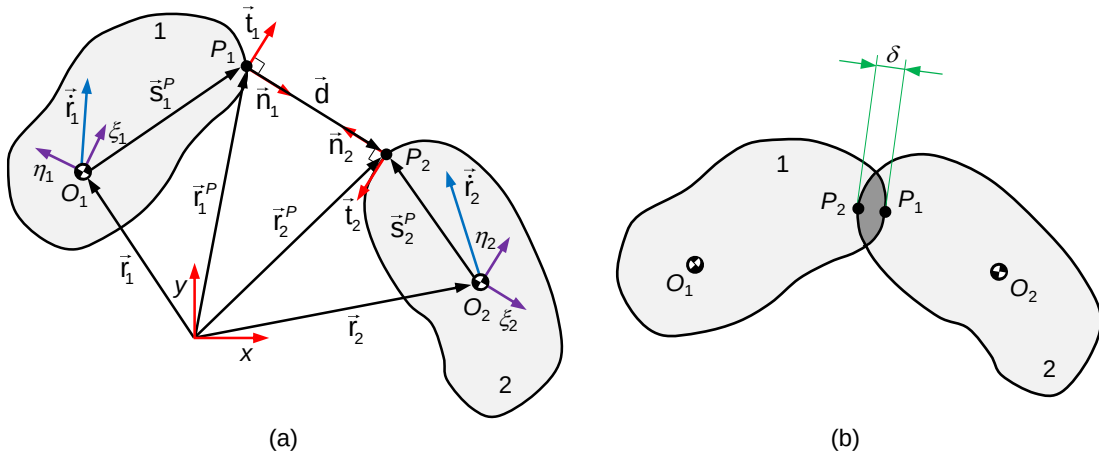


Fig. 7 (a) Two bodies in the state of separation; **(b)** Two bodies in the state of contact (penetration, δ).

With regard to Fig. 7a, the vector that connects the two potential contact points P_1 and P_2 is a gap function that can be expressed as

$$\mathbf{d} = \mathbf{r}_2^P - \mathbf{r}_1^P \quad (20)$$

where both $\mathbf{r}_1^P$ and $\mathbf{r}_2^P$ are described in terms of global coordinates with respect to the inertial reference frame xy , that is

$$\mathbf{r}_i^P = \mathbf{r}_i + \mathbf{A}_i \mathbf{s}_i'^P \quad (i = 1, 2) \quad (21)$$

in which $\mathbf{r}_1$ and $\mathbf{r}_2$ represent the global position vectors of the bodies 1 and 2, while $\mathbf{s}_1'^P$ and $\mathbf{s}_2'^P$ are the local components of the potential contact points with respect to local coordinates systems. In turn, $\mathbf{A}_1$ and $\mathbf{A}_2$ denote the rotational transformation matrices associated with bodies 1 and 2, respectively [1].

A normal vector to the plane of collision, materialized in Fig. 7a, can be established as

$$\mathbf{n}_1 = \frac{\mathbf{d}}{d} \quad (22)$$

where the magnitude of the distance vector $\mathbf{d}$ is given by

$$d = \delta = \mathbf{d}^T \mathbf{n}_1 \quad (23)$$

The tangential vector $\mathbf{t}$ represented in Fig. 7a can be obtained by simply rotating the normal vector in the anticlockwise direction by 90° .

The minimum distance condition expressed by Eq. (23) is not sufficient to determine the potential contact point between the colliding surfaces, since it does not cover all the possible scenarios that may occur in a contact problem [93, 94]. Therefore, the contact points are defined as those that correspond to maximum penetration, that is, the points associated with maximum relative deformation measured along the normal direction [95]. Thus, three geometric conditions for contact emerge as: (i) the distance between the potential contact points given by vector $\mathbf{d}$ corresponds to the minimum distance; (ii) the vector $\mathbf{d}$ has to be collinear with the normal vector $\mathbf{n}_1$; (iii) the normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ at the potential contact points have to be collinear. The last two conditions (ii) and (iii) can be written as two cross products as [96]

$$\mathbf{d} \times \mathbf{n}_1 = \mathbf{0} \quad (24)$$

$$\mathbf{n}_2 \times \mathbf{n}_1 = \mathbf{0} \quad (25)$$

The geometric conditions given by Eqs. (24) and (25) represent two nonlinear equations with two unknowns, solution of which can be found using any numerical algorithm for this type problem, such as the well-known Newton-Raphson iterative scheme [1, 8]. This nonlinear problem has to be solved at every time step of the resolution of the equations of motion of the multibody system under analysis. Thus, the obtained solution corresponds to the effective location of the potential contact points, which, consequently, allows for the calculation of the penetration using Eq. (23).

The remaining information relative to contact-impact kinematics can be established based on the computation of the velocities of the contact points, which can be written as [1]

$$\dot{\mathbf{r}}_i^P = \dot{\mathbf{r}}_i + \dot{\mathbf{A}}_i \mathbf{s}_i'^P \quad (i = 1, 2) \quad (26)$$

where the dot represents the derivative with respect to time. Then, the relative velocity of the contact points must be projected onto the normal and tangential directions of the contacting surfaces, since they play a key role in the determination of this kind of contact dynamics problem. The scalar normal and tangential velocities are given by

$$v_n = \dot{\delta} = (\dot{\mathbf{r}}_2^P - \dot{\mathbf{r}}_1^P)^T \mathbf{n}_1 \quad (27)$$

$$v_t = (\dot{\mathbf{r}}_2^P - \dot{\mathbf{r}}_1^P)^T \mathbf{t}_1 \quad (28)$$

This way of representing the relative normal and tangential contact velocities is quite convenient, in the measure that it is not necessary to deal with the derivation of the normal unit vector because these velocities are directly obtained by differentiating Eq. (21) [97]. The computational implementation of this methodology is very efficient. However, the above description is restricted to convex rigid bodies with a smooth surface at least in a neighborhood of the potential contact points such that the contact area reduces to a single point. This approach can be extended to more generalized contact geometries as long as a common tangent plane of the contacting bodies is uniquely defined [98–100].

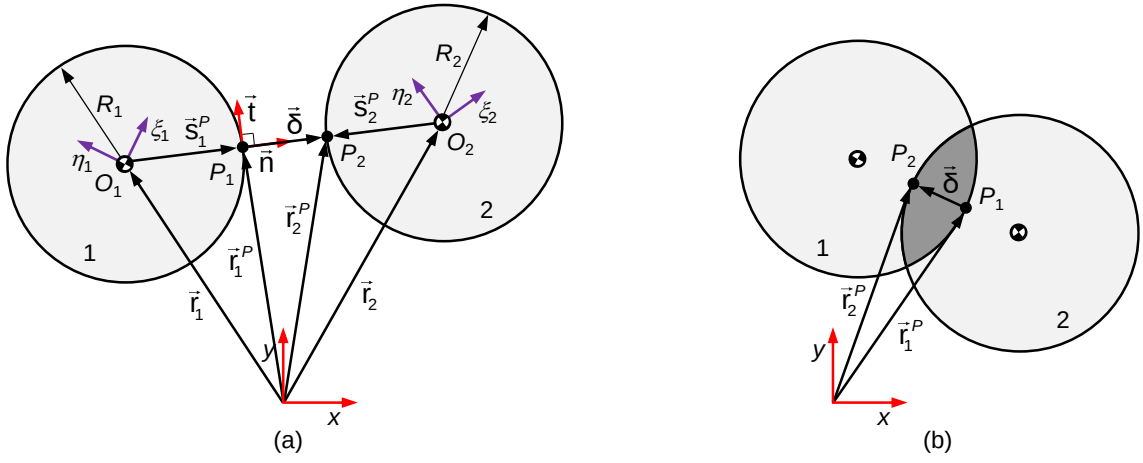


Fig. 8 (a) Two spheres in the state of separation; **(b)** Two spheres in the state of contact (penetration, δ).

In what follows, the case of the collision between two spheres is analyzed in terms of contact kinematics. With regard to Fig. 8a, the gap distance vector can be established as

$$\delta = \mathbf{r}_2^P - \mathbf{r}_1^P \quad (29)$$

which magnitude can be evaluated as

$$\delta = \sqrt{\delta^T \delta} \quad (30)$$

where δ^T is the transpose of vector δ .

A normal unit vector, perpendicular to plane of collision (see Fig. 8), can be defined as

$$\mathbf{n} = \frac{\delta}{\delta} \quad (31)$$

The global position vector of the contact points P_1 and P_2 can be expressed as

$$\mathbf{r}_1^P = \mathbf{r}_1 + R_1 \mathbf{n} \quad (32)$$

$$\mathbf{r}_2^P = \mathbf{r}_2 - R_2 \mathbf{n} \quad (33)$$

Thus, the gap distance vector between the contact points can be rewritten as

$$\delta = \mathbf{r}_2 - \mathbf{r}_1 - (R_1 + R_2) \mathbf{n} \quad (34)$$

The relative velocity of the contact points can be obtained by differentiating Eq. (34) with respect to time, yielding

$$\dot{\delta} = \dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_1 - (R_1 \dot{\theta}_1 + R_2 \dot{\theta}_2) \mathbf{t} \quad (35)$$

where the tangential unit vector $\mathbf{t}$ denotes the rotation of the vector $\mathbf{n}$ 90° in counter clockwise direction, and $\dot{\theta}_1$ and $\dot{\theta}_2$ are the angular velocities of spheres 1 and 2, respectively. The relative contact velocity can be projected onto the normal and tangential directions as

$$\dot{\delta} = v_n = (\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_1)^T \mathbf{n} \quad (36)$$

$$v_t = (\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_1)^T \mathbf{t} - (R_1 \dot{\theta}_1 + R_2 \dot{\theta}_2) \quad (37)$$

For the particular case of the one-dimensional central collision between two spheres, yields

$$\delta = x_2 - x_1 - (R_1 + R_2) \quad (38)$$

$$\dot{\delta} = \dot{x}_2 - \dot{x}_1 \quad (39)$$

4. Contact-impact dynamics

Contact-impact dynamics deals with the motion analysis of mechanical systems that experience contact and/or impact events. With the purpose of describing the main features associated with contact-impact events in multibody dynamics, let consider the one-dimensional central collision between two solid and isotropic spheres, as it is illustrated in Fig. 9a. For the sake of simplicity, let assume that both spheres are initially separated and moving with constant horizontal velocities without the action of any external force. Moreover, before the impact, the velocity of sphere 1 is higher than the velocity of sphere 2, meaning that sphere 1 is going to collide with sphere 2. After the impact, the velocity of sphere 2 is higher than the velocity of sphere 1, which implies that the two spheres separate from each other when the impact end. This scenario is represented by the velocity diagrams of Fig. 9b, where $t^{(-)}$ is the instant of time immediately before the impact, $t^{(+)}$ denotes the instant of time immediately after the impact, and Δt is the duration of the impact between the two spheres. Since there is variation of velocity of the spheres during the impact process, then, according to Newton's second law of motion, there exists force over the collision period. Figure 9c shows the acceleration diagrams before, during and after the impact of the two spheres. It should be noticed that the areas related to the acceleration diagrams are not equal, in the measure that the spheres 1 and 2 deaccelerate and accelerate differently. Furthermore, it is clear that the pre and post-impact accelerations of the two spheres are null, because the corresponding velocities are constant.

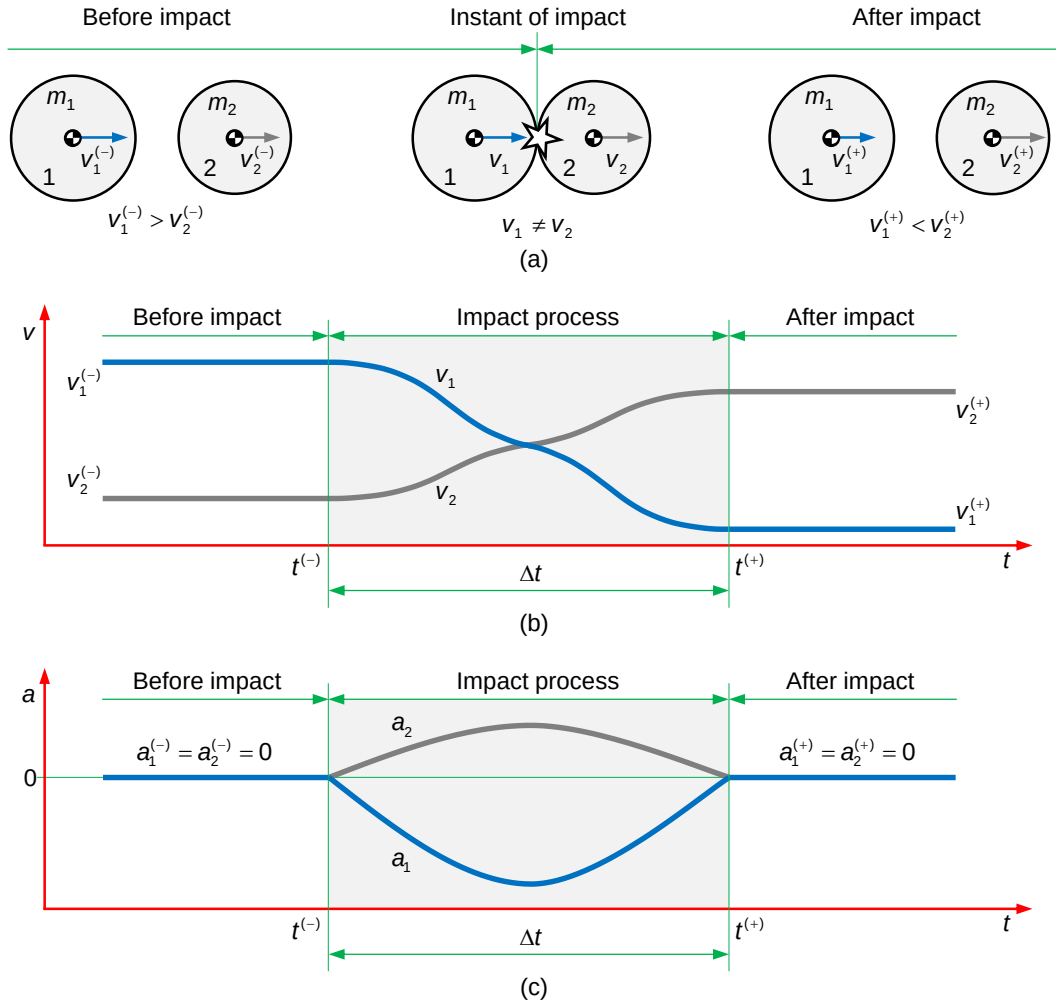


Fig. 9 (a) Representation of the one-dimensional central collision between two solid spheres; (b) Velocities of the spheres before, during, and after the impact process; (c) Accelerations of the spheres before, during, and after the impact process.

In order to study the dynamic interaction between the two spheres at the collision, let consider the free-body diagrams of each sphere, as Fig. 10 depicts.

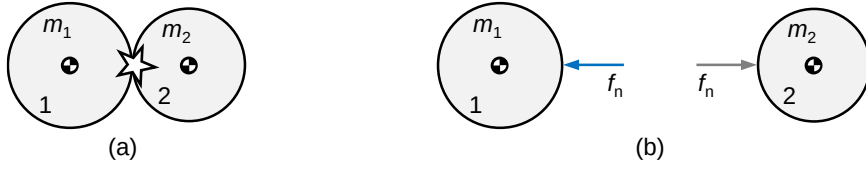


Fig. 10 (a) Impact between two spheres; **(b)** Free-body diagrams of the spheres at the impact.

According to Newton's third law of motion, there are an action and a reaction force during the impact, being the magnitude of those forces equal, but act in opposite directions. Figure 11a shows the force diagrams of the two spheres during the impact process. It should be noticed that the areas related to the force diagrams are equal, in the measure that the interaction force is equal in intensity on both spheres. Thus, applying the Newton's second law of motion to this frictionless impact process represented in Fig. 10 yields

$$-f_n = m_1 a_1 \quad (40)$$

$$f_n = m_2 a_2 \quad (41)$$

where f_n represents the normal contact force, m_1 and m_2 denote the masses of spheres 1 and 2, being a_1 and a_2 the corresponding linear accelerations.

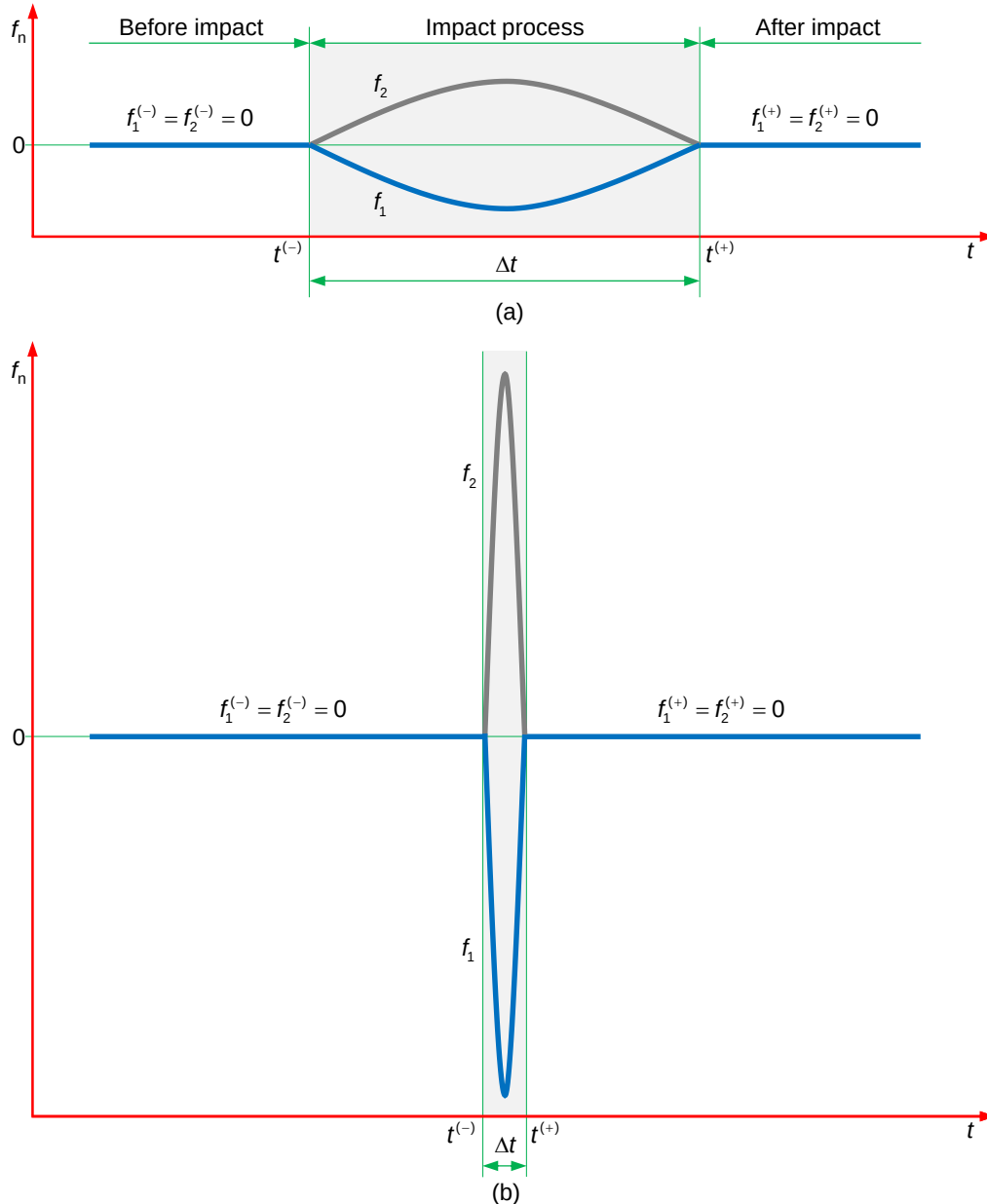


Fig. 11 (a) Normal contact force evolution; **(b)** Impact force acting instantaneously.

As it was mentioned, an impact is characterized by an high intensity force and a very short time duration; i.e., $f_n \rightarrow \infty$ and $\Delta t \rightarrow 0$. This means that the force diagrams are like shrunk, while keep the same area inside the force diagrams, as Fig. 11b depicts. Thus, by definition of impact, it is possible to write the following expression for the impulse

$$p = \int_{t^{(-)}}^{t^{(+)}} f_n dt \quad (42)$$

Applying Eq. (42) to the impact equations of motion (40) and (41) results in

$$p = \int_{t^{(-)}}^{t^{(+)}} -m_1 a_1 dt \Rightarrow p = -m_1 (v_1^{(+)} - v_1^{(-)}) \quad (43)$$

$$p = \int_{t^{(-)}}^{t^{(+)}} m_2 a_2 dt \Rightarrow p = m_2 (v_2^{(+)} - v_2^{(-)}) \quad (44)$$

Equations (43) and (44) represent a set of two equations with three unknowns, namely the impulse, p , and the post-impact velocities of the two spheres, $v_1^{(+)}$ and $v_2^{(+)}$, making a system with infinite solutions. Thus, in order to have a unique solution, the Newton's impact law must be considered, which was originally formulated in the following manner *"In bodies imperfectly elastic the velocity of the return is to be diminished together with the elastic force; because that force (except when the parts of bodies are bruised by their congress, or suffer some such extension as happens under the strokes of a hammer) is (as far as I can perceive) certain and determined, and makes the bodies ratio to the relative velocity with which they met."* [101]. This statement can be written as

$$c_r = -\frac{\text{relative velocity after the impact}}{\text{relative velocity before the impact}} \Rightarrow c_r = -\frac{v_1^{(+)} - v_2^{(+)}}{v_1^{(-)} - v_2^{(-)}} \quad (45)$$

Equation (45) represents the kinematic coefficient of restitution of the materials of the colliding spheres. The coefficient of restitution is an experimental parameter that depends on the materials involved, impact velocity among other factors [102–106]. Thus, combining Eqs. (43)-(45) it is possible to write the impact equations of motion in the expanded form as

$$\begin{bmatrix} m_1 & 0 & 1 \\ 0 & m_2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_1^{(+)} \\ v_2^{(+)} \\ p \end{bmatrix} = \begin{bmatrix} m_1 v_1^{(-)} \\ m_2 v_2^{(-)} \\ -c_r (v_1^{(-)} - v_2^{(-)}) \end{bmatrix} \quad (46)$$

which constitutes a linear system of three equations with three unknowns, namely the post-impact velocities, $v_1^{(+)}$, $v_2^{(+)}$, and the impulse, p . It must be noted that the impact equations (46) have a similar arrangement to the equations of motion for constrained multibody systems (16), which can be expressed as

$$\mathbf{M}\ddot{\mathbf{q}} + \Phi_q^T \boldsymbol{\lambda} - \mathbf{g} = \mathbf{0} \quad (47)$$

Equation (47) is written at the acceleration level and, therefore, it is not compatible with the impact equations (46), since the later is written as the velocity level. In order to overcome this difficulty, let integrate Eq. (47) over the impact period, that is

$$\int_{t^{(-)}}^{t^{(+)}} \mathbf{M}\ddot{\mathbf{q}} dt + \int_{t^{(-)}}^{t^{(+)}} \Phi_q^T \boldsymbol{\lambda} dt - \int_{t^{(-)}}^{t^{(+)}} \mathbf{g} dt = \mathbf{0} \quad (48)$$

yielding

$$\mathbf{M}(\dot{\mathbf{q}}^{(+)} - \dot{\mathbf{q}}^{(-)}) + \Phi_q^T \int_{t^{(-)}}^{t^{(+)}} \boldsymbol{\lambda} dt - \mathbf{g} \int_{t^{(-)}}^{t^{(+)}} dt = \mathbf{0} \quad (49)$$

In this process, based on the assumption that the impact duration is very short, the system mass matrix is constant, and the positions and orientations of the bodies do not change over the period of impact, and, hence, the Jacobian matrix is constant too. The integral of the Lagrange multipliers in Eq. (49) represents the impulses associated with the constraints. The last integral in Eq. (49) denotes the applied impulses to the system that are usually null, because the applied forces remain unchanged during the period of impact. Thus, Eq. (49) can be rewritten as

$$\mathbf{M}(\dot{\mathbf{q}}^{(+)} - \dot{\mathbf{q}}^{(-)}) + \Phi_{\mathbf{q}}^T \mathbf{p}_{\mathbf{q}} - \mathbf{g} \Delta t = \mathbf{0} \quad (50)$$

In summary, Eqs. (45) and (50) can be solved together to determine the post-impact velocities caused by a collision in multibody systems. Finally, it must be mentioned that the friction effect can also be incorporated in this piecewise formulation, and the interested reader in further details referred to the textbook by Nikravesh [8].

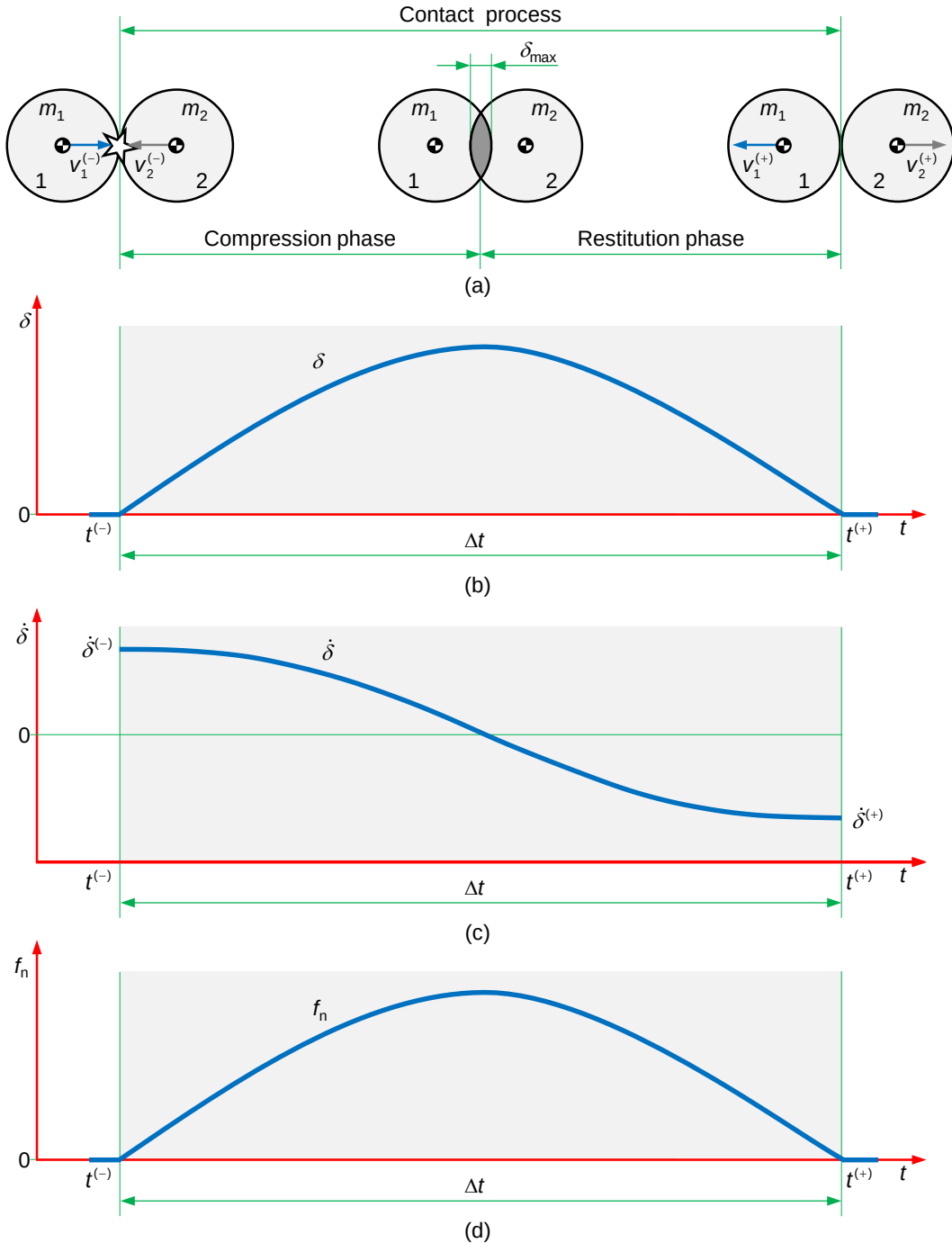


Fig. 12 (a) One-dimensional direct central collision between two spheres; (b) Deformation or penetration evolution with time; (c) Relative contact velocity evolution with time; (d) Normal contact force evolution with time.

In contrast to the piecewise analysis described above, which does not account for local deformations of the colliding bodies, in the continuous method of contact-impact problems the bodies are allowed to penetrate, or deform, at the contact zone. In order to better understand the continuous analysis approach, let consider the central collision between two spheres, as it is illustrated in Fig. 12a, which period of impact can be divided into two distinct phases, namely the compression phase, and the restitution phase. The compression period extends from the initial instant of impact to the instant of maximum deformation, $\delta_{\max}$. Thereafter, the restitution phase starts immediately at the end of the compression period, and finishes when the colliding spheres separate from each other. In the case of complete elasticity, an axis of symmetry exists about the instant of maximum deformation, while an unsymmetrical diagram is obtained for the case of partial restoration [61].

The geometric condition that allows for the evaluation of the penetration between the two spheres of Fig. 12a can be expressed as

$$\delta = x_2 - x_1 - (R_1 + R_2) \quad (51)$$

where x_1 and x_2 represent the x coordinate of the spheres 1 and 2, respectively, and R_1 and R_2 denotes the corresponding radii. Figure 12b shows the evolution of the penetration between the colliding spheres.

The pre- and post-impact relative velocities associated with the spheres are represented in the diagram of Fig. 12c. It should be noted that at the end of compression phase, both spheres move with a common instantaneous velocity. Thus, the initial impact velocity and the separating velocity can be expressed as

$$\dot{\delta}^{(-)} = v_1^{(-)} - v_2^{(-)} \quad (52)$$

$$\dot{\delta}^{(+)} = v_1^{(+)} - v_2^{(+)} \quad (53)$$

Figures 12b-d show the general diagrams for the contact penetration, relative contact velocity and contact force evolutions with time during the period of collision. In these plots, the instant of initial impact and the instant of separation are denoted by $t^{(-)}$ and $t^{(+)}$, respectively, while Δt represents the duration of the collision between the two spheres.

One of the simplest contact force models for representing the contact interaction between two spheres is the Hertz law, which can be expressed as [69]

$$f_n = K\delta^n \quad (54)$$

where K represents the generalized contact stiffness, δ is the contact penetration, and n denotes the exponent of nonlinearity. The contact stiffness can be determined as function of the material properties and contact geometry of the colliding bodies. The exponent n is equal to 3/2 for the case where there is a parabolic distribution of the contact stresses, as in the original work by Hertz [69]. The penetration, or deformation, can be determined from the configuration of the system and the corresponding relative locations of the colliding bodies using the formulation given by Eq. (51).

Figure 13a illustrates the Hertz contact force evolution with penetration, where it can be observed that the contact force varies in a nonlinear manner during both compression and restitution phases. The contact force initiates from zero at the beginning of the collision, reaches its maximum value, and then returns along the same line back to zero, where at this point the spheres separate from each other. In short, the Hertz contact force model mimics the impact process as a nonlinear spring along the direction of collision. It should be noted that the contact force law given by Eq. (54) does not account for any energy dissipation during the contact-impact event. In fact, the process of energy transfer is an extremely complex task of modeling contact-impact events in multibody dynamics [107–117].

Lankarani and Nikravesh [62] utilized the general trend of the Hertz contact law incorporated with a damping term to propose a continuous contact force model, which divides the contact force into elastic and dissipative components as

$$f_n = K\delta^n + D\dot{\delta} \quad (55)$$

where the first parcel of the right-hand side is the Hertz law, and the second one denotes the dissipative component associated with the damping of the colliding bodies. The quantity D represents the hysteresis parameter that is given by [62]

$$D = \chi\delta^n \quad (56)$$

where, in turn, the hysteresis factor χ is expressed as

$$\chi = \frac{3K(1-c_r^2)}{4\dot{\delta}^{(-)}} \quad (57)$$

Combining Eqs. (55)-(57) yields the continuous contact force model developed by Lankarani and Nikravesh as [62]

$$f_n = K\delta^n \left[1 + \frac{3(1-c_r^2)}{4} \frac{\dot{\delta}}{\dot{\delta}^{(-)}} \right] \quad (58)$$

in which K , δ and n have the same meaning as in the Hertz contact law, c_r represents the coefficient of restitution, $\dot{\delta}$ is the contact, or deformation, velocity and $\dot{\delta}^{(-)}$ is the contact velocity at the beginning of the collision. Figure 13b shows the force-penetration relation for the Lankarani and Nikravesh contact force model, where the compression and restitution paths do not coincide. It should be noted that the area inside the closed loop represented by the force-penetration diagram in Fig. 13b is the amount of energy dissipated during the contact process.

Finally, it must be stated that the continuous analysis is quite effective in handling contact-impact events in multibody systems, since the contact forces and deformations vary in a continuous manner. Thus, when contact between bodies is detected, contact forces perpendicular to the plane of collision are computed and applied. This method is sound and accurate both for contacts and impacts, in the measure that the instantaneous contact forces are simply introduced into the system's equation of the motion as external applied forces. Furthermore, the incorporation of the friction forces can easily be evaluated and introduced into the equations of motion in a similar manner [8].

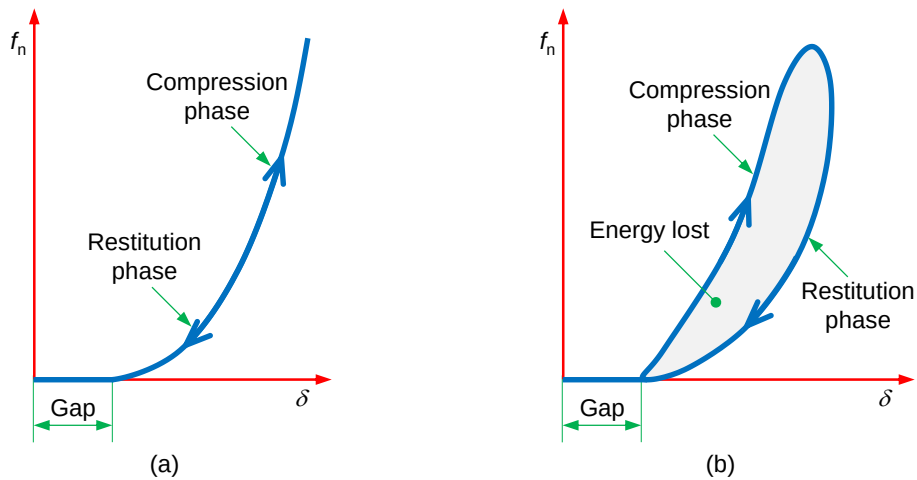


Fig. 13 (a) Force-penetration relation for the Hertz contact force law; **(b)** Force-penetration relation for the Lankarani and Nikravesh contact force model.

5. Contact force models

The continuous formulation for contact-impact dynamics in multibody systems is an explicit function of the local deformation of the colliding bodies and its rate. In this context, the contact force models are usually represented by a combination of spring and damper elements, which can be associated in series, as in the Maxwell model, or in parallel, as in the Kelvin-Voigt model, as illustrated in Fig. 14 [118]. The stiffness of the spring is typically obtained from the geometrical and material properties of the colliding bodies, while the damping parcel is often dependent of the coefficient of restitution. Brach [61] demonstrated that the coefficient of restitution is directly related to the energy lost during the contact process. In order to effectively simulate contact-impact events in multibody dynamics, it is quite important to identify and select the most appropriate contact force model. This task should be performed in such a way that realistic duration of the contact-impact event and correct energy dissipation is achieved. In the following, some of the existing contact force solutions are briefly analyzed.



Fig. 14 (a) Linear Maxwell contact approach; (b) Linear Kelvin-Voigt contact approach.

The simplest normal contact force model is the one represented by a linear spring element, which is due to Hooke [119]. This empirical law can be expressed as

$$f_n = k\delta \quad (59)$$

where k denotes the linear spring stiffness that embodies the elasticity of the colliding bodies, and δ represents the relative penetration between the contacting points. The spring stiffness of the Hooke contact force model can be determined analytically, numerically or experimentally [120–126]. In turn, the penetration is evaluated from the relative position of the colliding bodies and the geometric properties of the contacting surfaces [95]. The assumption of a linear relation between penetration and contact force is at best a coarse approximation, because the contact response is influenced by the shape and physical properties of the contacting surfaces, which clearly suggests the use of a constitutive law of more complex nature. Furthermore, the Hooke contact force model is limited to vary low impact velocities, and it does not account for any energy dissipation during the impact process. For this linear contact force model, Fig. 15a shows the force-penetration relation of two externally colliding aluminum spheres. The spheres are identical and have the same radius of 20 mm and the same mass of 0.092 kg. Both spheres have equal and opposite colliding velocities of 0.15 m/s, making the relative initial impact velocity equal to 0.3 m/s. The contact spring stiffness considered is equal to 2.5×10^7 N/m for the calculations used to generate the plot. By analyzing the graph of the Fig. 15a, it can be observed that the contact force varies in a continuous and linear manner with penetration, and it starts from zero and returns to zero while always remaining of compressive nature.

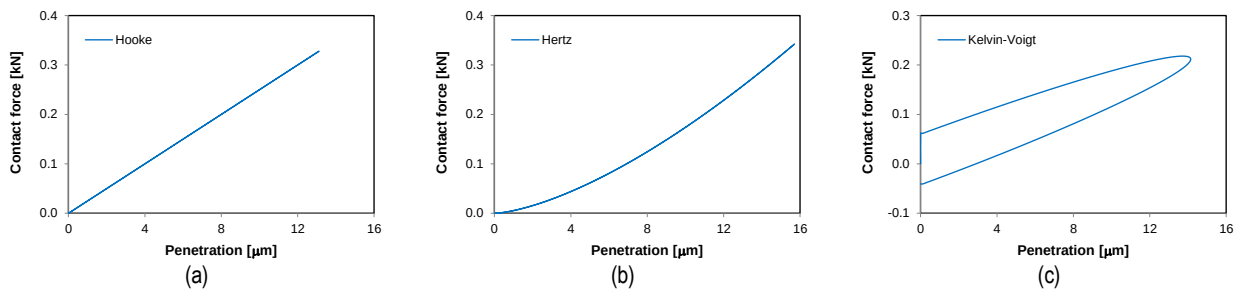


Fig. 15 Force-penetration relation for two externally colliding spheres: (a) Hooke contact force model; (b) Hertz contact force model; (c) Kelvin-Voigt contact force model.

The most popular contact force model to mimic the interaction between two colliding spheres is the one developed by Hertz [69], which is based on the classical theory of elasticity. The Hertz contact law relates the normal contact force with a nonlinear power function of penetration as

$$f_n = K\delta^n \quad (60)$$

in which K represents the contact stiffness, δ is the relative penetration, and n denotes the exponent of nonlinearity. According to Hertz approach, the contact stiffness for two solid and isotropic spheres in contact can be established as

$$K = \frac{4}{3(\sigma_1 + \sigma_2)} \sqrt{\frac{R_1 R_2}{R_1 + R_2}} \quad (61)$$

where the material properties, σ_1 and σ_2 , are defined as

$$\sigma_i = \frac{1 - \nu_i^2}{E_i} \quad (i = 1, 2) \quad (62)$$

and the quantities ν_i and E_i are, respectively, the Poisson's ratio and Young's modulus of each sphere. It should be noticed that, by definition, the radius is positive for convex surfaces, and negative for concave surfaces [127]. Hertz contact law describes the static compression of two elastic bodies whose surfaces, assumed to be perfectly smooth, can be approximated by two paraboloids in the vicinity of the contact point. It is apparent that the Hertz contact force model mimics the contact process as a nonlinear spring element along the direction of impact. This contact force model cannot be used during both phases of a contact event, namely, compression and restitution periods, since it does not take into account the energy dissipation during the contact process, Hertz law is a pure elastic model, that is, the mechanical energy stored during the compression period is exactly the same that is restored during the restitution period. Goldsmith [118] showed that the Hertz contact force model provides accurate results only for hard colliding materials and low impact velocities. In fact, when an elastic body is subjected to cyclic loads, the mechanical energy dissipated due internal damping causes hysteresis loop in the force-penetration relation. Thus, over the years several studies have been motivated and performed to extend Hertz contact approach to accommodate energy lost [77]. The use of the Hertz contact force model (60) to the collision of two externally spheres implies the outcome of Fig. 15b, where the force-penetration evolution is presented. The collision scenario is the same as in the case for the Hooke law described previously. A contact stiffness of $5.5 \times 10^9 \text{ N/m}^{3/2}$ has been considered for the calculations. It can be observed that the contact force starts from zero when the penetration is null, reaches the maximum value, and then returns along the same nonlinear curve back to zero, when at this point the colliding bodies separate from each other. The Hertz contact force model provides always compressive contact forces, meaning that the contacting bodies push each during the entire period of contact.

The first contact force model capable of accommodating energy dissipation during the contact process is the Kelvin-Voigt approach [128, 129]. This viscoelastic contact force model combines a linear spring and a linear damper in parallel (see Fig. 14b.) and can be formulated as

$$f_n = K\delta + D\dot{\delta} \quad (63)$$

where the first parcel on the right-hand side represents the elastic force component, and the second term accounts for the energy dissipation during the contact event. In this contact force model k denotes the linear spring stiffness, δ is the relative contact penetration, D represents the damping coefficient, and $\dot{\delta}$ denotes the relative normal contact velocity. One primary inconvenient associated with the Kelvin-Voigt approach is that the contact force exhibits a discontinuity at the instant of impact. Indeed, at the beginning of the contact, the force

component due to the spring elements is null because the relative penetration is null too, however, the damping force component has a finite value. Hence, this contact force model predicts the existence of a nonzero force at the beginning of the contact equal to $D\dot{\delta}$. This circumstance is not realistic from the physical point of view, because when the contact starts, both elastic and damping force components must be null. Another inconvenient of the Kelvin-Voigt model is that at the ending of contact the predict force is negative, due to the also negative contact velocity associated with the restitution phase. As a consequence of this fact, the contact force immediately before of the separation of the colliding bodies is equal to $-D\dot{\delta}$, which is a tensile force that holds the contacting bodies, instead of pushing them apart. This circumstance does not make sense from the physical point of view, since the bodies cannot attract or pull each other during a contact-impact event. The use of the contact force model given by Eq. (63) to model the collision between the two externally spheres results in the solution depicted in Fig. 15c, where the force-penetration relation is represented. A linear contact stiffness of 1.5×10^7 N/m and a linear damping coefficient of 200 Ns/m have been considered for the calculations. As it can be observed from Fig. 15c, the Kelvin-Voigt contact force model exhibits the weaknesses described above, namely the discontinuities at the beginning of the contact, and the negative contact force at the ending of the contact. Although this contact force model has some limitations, its simplicity and easiness to implement in a multibody code make it quite popular to handle contact-impact events in mechanical systems in different domains of application.

Herbert and McWhannell [130] showed that a linear viscoelastic approach to model collisions is not realistic, in the measure that the initial and final contact velocities cause discontinuities in the damping force component. Simon [131] and Hunt and Crossley [70] replaced the linear spring-damper parallel combination by a nonlinear one, resulting in a contact force equation of the form

$$f_n = K\delta^n + \chi\delta^n\dot{\delta} \quad (64)$$

where K represents the nonlinear contact stiffness parameter as in the Hertz law, δ is the relative contact penetration, $\dot{\delta}$ denotes the relative normal contact velocity, and χ is the hysteresis damping factor. In general, the nonlinear exponent n can be set to $3/2$ [132]. A primary advantage of this contact force formulation resides on its nonlinearity nature. Contact mechanics of bodies represented by nonlinear viscoelastic relations was investigated by Hunter [133]. It should be noted that the contact force given by Eq. (64) guarantees that the damping force component satisfies both boundary conditions at the instant of beginning of the contact and the instant of separation [134]. In other words, this contact formulation removes the force discontinuities associated with the Kelvin-Voigt approach. In the contact force model proposed by Hunt and Crossley (64), the spring force component is nonlinear, and the damping parcel is dependent on the relative contact penetration, which is sound from physical point of view, in the measure that the damping increases with penetration as the contact area increases over the contact process. The dependence of both elastic and damping parcels on the relative contact penetration implies that the contact force build up from zero upon contact and return to zero as separation approximates. Another advantage of this contact force model is that the tensile or pulling force between the colliding bodies does not arise. In fact, tensile force can occur only when an external force separates the colliding bodies [135]. In order to account for the mechanical energy dissipated during the contact-impact events, it is often convenient to relate the hysteresis factor χ with the coefficient of restitution, where the kinematic or Newton concept of coefficient of restitution, given by Eq. (45), is considered. In fact, over the last decades, several alternative solutions for the hysteresis damping factor have been proposed, namely those inspired in the cornerstone of the Hunt and Crossley work and summarized in Tab. 1. The interested reader in comprehensive comparison of the most relevant hunt-crossleian contact force models is referred to the work by Silva et al. [136].

Tab 1 Hysteresis damping factor for different hunt-crossleian contact force models.

Authors	Hysteresis damping factor	Contact force model
Hunt and Crossley (1975) [70]	$\chi = \frac{3(1-c_r)}{2} \frac{K}{\dot{\delta}^{(-)}}$	$f_n = K\delta^n \left[1 + \frac{3(1-c_r)}{2} \frac{\dot{\delta}}{\dot{\delta}^{(-)}} \right]$
Herbert and McWhannell (1977) [130]	$\chi = \frac{6(1-c_r)}{(2c_r-1)^2+3} \frac{K}{\dot{\delta}^{(-)}}$	$f_n = K\delta^{\frac{3}{2}} \left[1 + \frac{6(1-c_r)}{(2c_r-1)^2+3} \frac{\dot{\delta}}{\dot{\delta}^{(-)}} \right]$
Lee and Wang (1983) [137]	$\chi = \frac{3(1-c_r)}{4} \frac{K}{\dot{\delta}^{(-)}}$	$f_n = K\delta^{\frac{3}{2}} \left[1 + \frac{3(1-c_r)}{4} \frac{\dot{\delta}}{\dot{\delta}^{(-)}} \right]$
Lankarani and Nikravesh (1990) [62]	$\chi = \frac{3(1-c_r^2)}{4} \frac{K}{\dot{\delta}^{(-)}}$	$f_n = K\delta^{\frac{3}{2}} \left[1 + \frac{3(1-c_r^2)}{4} \frac{\dot{\delta}}{\dot{\delta}^{(-)}} \right]$
Gonthier et al. (2004) [138]	$\chi = \frac{1-c_r^2}{c_r} \frac{K}{\dot{\delta}^{(-)}}$	$f_n = K\delta^{\frac{3}{2}} \left[1 + \frac{1-c_r^2}{c_r} \frac{\dot{\delta}}{\dot{\delta}^{(-)}} \right]$
Flores et al. (2011) [139]	$\chi = \frac{8(1-c_r)}{5c_r} \frac{K}{\dot{\delta}^{(-)}}$	$f_n = K\delta^{\frac{3}{2}} \left[1 + \frac{8(1-c_r)}{5c_r} \frac{\dot{\delta}}{\dot{\delta}^{(-)}} \right]$

The use of the contact force models listed in Tab. 1 to model the collision between two externally spheres provides the outcomes illustrated in Figs. 16a-f. A contact stiffness value of $5.5 \times 10^9 \text{ N/m}^{3/2}$ and a coefficient of restitution of 0.8 have been utilized for the calculations. The comparative performance of the several contact force models considered is quantified by the plots of the contact force versus penetration. By analyzing the diagrams in Fig. 16, it can be observed that the force-penetration curves exhibit hysteresis loop, area of which represents the amount of mechanical energy dissipated during the collision. It is also visible that all the contact force models do not have discontinuities at the beginning and ending of the contact-impact event. In fact, for all the formulations considered, the contact force varies in a nonlinear and continuous manner, starting from zero, returning to zero, while always remaining of compressive nature. A more detailed analysis of the force-penetration diagrams permits to draw that Lee and Wang contact force model dissipated less energy, in virtue of a smaller hysteresis loop presented. In turn, both Hunt and Crossley, and Lankarani and Nikravesh approaches have a similar behavior in terms of hysteresis loop, since they dissipated a similar amount of energy. The remaining contact force models, namely, Herbert and McWhannell, Gonthier et al., and Flores et al., predict a larger quantity of energy dissipation during the contact process.

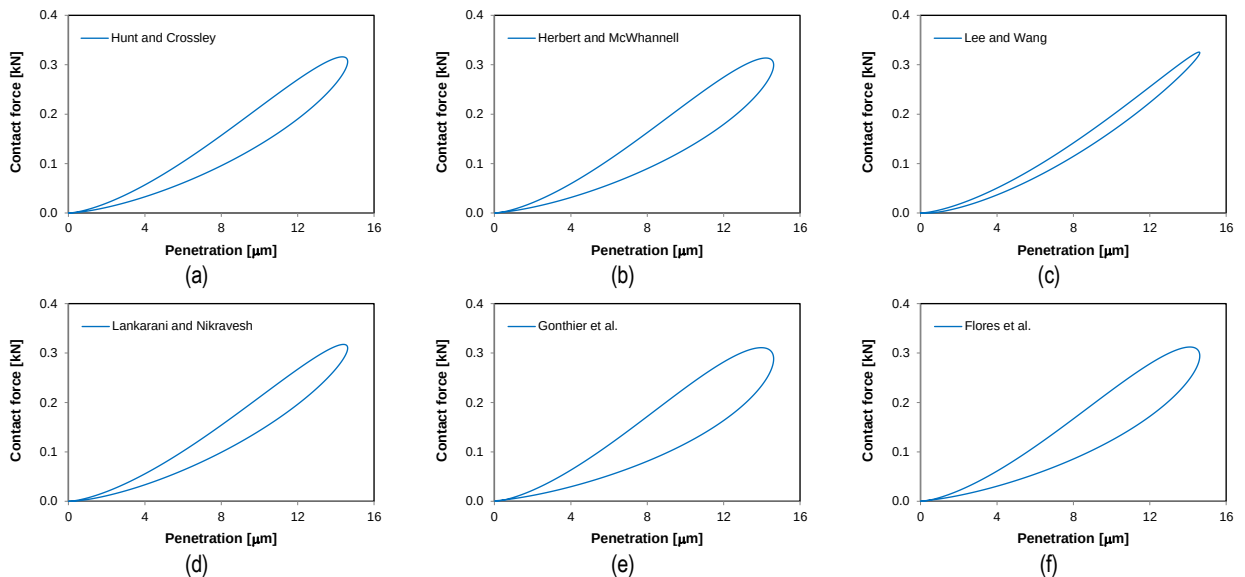


Fig. 16 Force-penetration relation for two externally colliding spheres: (a) Hunt and Crossley contact force model; (b) Herbert and McWhannell contact force model; (c) Lee and Wang contact force model; (d) Lankarani and Nikravesh contact force model; (e) Gonthier et al. contact force model; (f) Flores et al. contact force model.

6. Friction force models

In a broad sense, friction is the action of one object rubbing against another, or in other simple words, friction is the resistance an object encounters in moving over another. In fact, when two contacting bodies move or tend to move relative to each other, tangential forces are developed at the surfaces interaction. These forces are usually referred to as friction forces. Nikravesh [8] states that friction forces may be imposed between two contacting bodies to oppose their relative motion. In multibody dynamics, the existence of friction in the contact surfaces makes the contact-impact problems more complicated and demanding to handle, since friction may lead to different contact regimens, such as sticking and sliding. When the relative tangential velocity of two contacting surfaces approaches zero, stiction occurs, phenomenon of which must be predicted by the friction model considered in the dynamic analysis [140, 141]. In addition, friction behavior is dependent on many factors, namely the material and system properties, the shape and roughness of the contacting surfaces, the relative tangential velocity, the normal contact force, the existence of lubricant, etc. These features make the process of modeling frictional contact-impact events in multibody dynamics a task of complex nature. In general, four main types of friction can be present in mechanical systems, namely, (i) static or stiction, (ii) sliding; (iii) rolling; and (iv) viscous [142].

Prior to present and analyze some of most relevant friction force models suitable to be utilized in multibody dynamics, it is opportune to revisit the scientific roots and milestones of this subject. Leonardo da Vinci (1452-1519) was pioneer in the study of friction, namely in terms of the required weight to be applied to different objects placed on horizontal and inclined planes in order to start their motion [143–145]. Based on experimental observations, da Vinci wrote, in his notebooks, two basic friction laws. The first one says that the friction made by the same weight will be of equal resistance at the beginning of its movement although the contact may be of different breadths and lengths [146]. This hint can be considered as the embryogenesis of the well-known friction law, according to which the apparent area of contact does not have influence on the friction response. The second friction law establishes that friction produces double amount of effect if the weight be doubled [146]. In other words, this da Vinci outcome states that the friction is directly proportional to the weight of the object moving on the plane. Da Vinci also defined the concept of coefficient of friction, as the ration between friction force and weight of the moving object [145]. Lastly, da Vinci found a universal coefficient of friction equal to 1/4, independent of the material, and recognized the difference among sliding, rolling and viscous frictions [144]. Guillaume Amontons (1663-1705) developed his own experimental apparatus to investigate on friction and rediscovered the da Vinci friction laws [147]. The friction laws according to Amontons can be written as (i) the friction force is independent of the apparent area of contact between the two contacting surfaces; and (ii) the force of friction acting between two sliding surfaces is proportional to the load pressing the surfaces. Amontons also found a universal coefficient of friction equal to 1/3. Leonhard Euler (1707-1783) developed the first theoretical studies on friction, in particular about the sliding motion of a block on an inclined plane [148]. Euler, who introduced the symbol μ for the coefficient of friction, established a relation between the inclination angle, ϕ , of the plane and coefficient of friction as $\mu = \tan\phi$. Moreover, Euler distinguished between static and dynamic coefficient of friction, and found that the static friction is greater than the dynamic friction. Charles August Coulomb (1736-1806), who is probably the most renowned name in the friction domain, published the friction laws referring the work by Amontons [149]. In fact, Coulomb extended Amontons friction laws to a modern concept, as it still is used today. The two basic Coulomb's friction laws can be stated as (i) the friction force is independent of the nominal area of contact; and (ii) the friction is proportional to the normal contact force [150]. In particular, the second friction law, often called Amontons-Coulomb's law, holds for many different material combinations and geometries, and is of experimental nature, that is, nothing fundamental can be derived from it as, for example, the case of Newton's second law of motion.

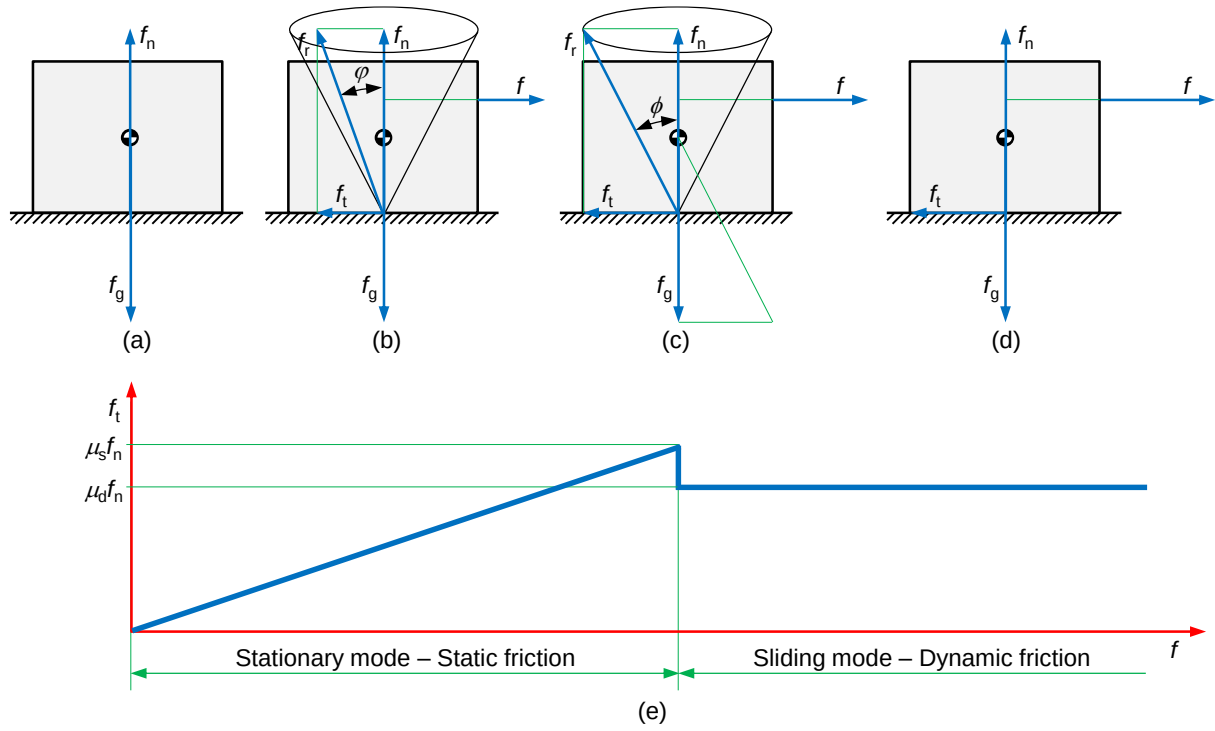


Fig. 17 (a) Stationary block on the ground; (b) Stationary block when a small external horizontal force is applied; (c) Block in the limiting static equilibrium condition; (d) Block in the sliding mode; (e) Friction force evolution from the stationary mode to the sliding mode.

The two friction force laws described above were experimentally established from the observation of the starting of the sliding motion of objects stationary placed on planes. Figure 17a shows the contact interaction of a block on the horizontal ground surface in the stationary mode. In this particular situation, the weight of the block is balanced by the normal contact force. Let consider an external horizontal force applied to the block, magnitude of which vary continuously from zero to the value sufficient to just move the block and then to maintain the motion. When the block is pulled by the external force, but the force is not large enough to initiate the motion of the block, then the stationary mode persists. This situation is illustrated in Fig. 17b, where the angle of adhesion, ϕ , that represents the direction of the resulting reaction force between the tangential friction force and normal contact force, is also shown. This resulting reaction force balances the external force and the weight. It is clear that in the static situation, the friction force prevents the motion of the block from occurring, that is, the tangential friction force and the applied force cancel out each other, and, therefore, the block remains in the stationary mode. For the static case, the tangential friction force can be evaluated using the Coulomb's dry friction law as [149]

$$f_t = \mu_s f_n \quad (65)$$

in which μ_s represents the static coefficient of friction, and f_n is the normal contact force. It should be noticed that the friction force adjusts itself in magnitude to equal to the external applied force until the limiting static equilibrium condition is reached, which corresponds to the maximum friction force. It is evident that the maximum friction force occurs at the end of the stationary phase, as it can be observed in the diagram of Fig. 17e. The limiting static equilibrium condition, in which the block is in the eminence to initiate the motion, is depicted in Fig. 17c. In the friction cone, the sides of which are the resulting reaction force between the tangential friction force and normal contact force, the radius of the base represents the tangential friction force (see Fig. 17c). The angle ϕ , represented in the friction cone of Fig. 17c, is often called friction angle. When the block is in the sliding mode, the magnitude of the tangential friction force is reduced, visible in the discontinuity of the diagram of Fig. 17e. It can be said there is no sliding motion while this resulting force is inside the friction cone. In fact, when

the external applied force is large enough to move the block, the static friction mode is altered to the dynamic friction regimen, as it is represented in Fig. 17d. In this case, the tangential friction force opposes the relative motion between the block and ground, but it is not sufficient to avoid the sliding motion of the block. During the sliding mode, the tangential friction force can be given by Coulomb's friction law as [149]

$$f_t = \mu_d f_n \quad (66)$$

where μ_d represents the dynamic coefficient of friction, and f_n is the normal contact force.

With regard to Fig. 17e, the Coulomb friction force model can be formulated as

$$f_t = \begin{cases} [-\mu_s f_n, \mu_s f_n] & \text{if } v_t = 0 \\ \mu_d f_n \operatorname{sgn}(v_t) & \text{if } v_t \neq 0 \end{cases} \quad (67)$$

where

$$\operatorname{sgn}(v_t) = \begin{cases} -1 & \text{if } v_t < 0 \\ 1 & \text{if } v_t > 0 \end{cases} \quad (68)$$

in which μ_s and μ_d are, respectively, the static and dynamic coefficients of friction, f_n represents the normal contact force, and v_t is the relative tangential velocity of the contacting surfaces. The first term in Eq. (67) is referred to as static or stiction friction, and the second term is named as dynamic or sliding friction. Figure 18a shows the representation of the Eq. (67) relative to Coulomb's solution, which represents the most fundamental model of friction between dry contacting surfaces, exhibits a discontinuity in the transition between stiction and sliding regimens. In fact, it has been experimentally showed that the force transition when relative tangential velocity changes from zero to non-zero does not happen instantaneously, but it occurs in a finite time interval instead. This phenomenon is known as Stribeck effect, which describes the decrease in the friction force as the relative tangential velocity increases [151]. The consequence of the Stribeck effect in the Coulomb friction approach is shown in the diagram of Fig. 18b. Another weakness associated with the Coulomb friction force model given by Eq. (67) is that the viscous effect is not considered, that is, the friction force is independent on the relative tangential velocity. In other words, the viscous action is characterized by a linear increase of friction with respect to velocity. The viscous effect can be established as a proportion of the relative tangential velocity of the sliding surfaces, that is [152]

$$f_v = \mu_v v_t \quad (69)$$

where μ_v represents the coefficient of viscous friction, and v_t is the magnitude of the relative tangential velocity. The incorporation of both Stribeck and viscous effects in the Coulomb friction approach results in the diagram of Fig. 18c, implementation of which leads to numerical and computational difficulties, since there is no specific values when the relative tangential velocity is null. In fact, this friction force model is undefined for the stiction mode. In short, the classical Coulomb friction force model presents some weaknesses, such as the independence of the relative tangential velocity. In practice, this is not true because friction force can be affected by many factors, namely the material properties, surfaces quality and cleanliness, temperature, wear, etc.; which all cannot be accommodated for by a constant coefficient of friction. Furthermore, the application of the Coulomb's solution in a computational program may lead to difficulties, since it is a discontinuous and nonlinear phenomenon that may involve switching between stiction and sliding scenarios. In face of the Coulomb friction shortcomings, over the years, there has been much interest on the subject of friction, which eventually resulted in more advanced friction force models, being some of the most representative listed in Tab. 2. Detailed information about other alternative friction force models can be found in the following references [74, 138, 153–171].

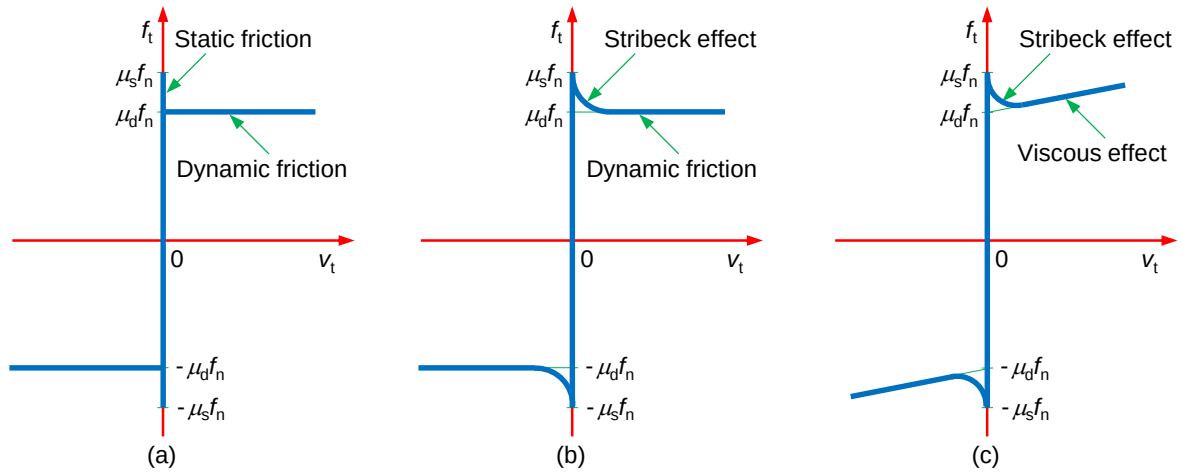


Fig. 18 (a) Coulomb friction model for static and dynamic regimens; (b) Coulomb friction model with Stribeck effect; (c) Coulomb friction model with both Stribeck and viscous effects.

Tab 2 Friction force models suitable to be used in multibody dynamics.

Authors	Friction force model
Bernard (1974) [172]	$f_t = \begin{cases} \mu_d f_n \min(k_\varepsilon v_t, 1) & \text{if } v_t \geq 0 \\ \mu_d f_n \max(k_\varepsilon v_t, -1) & \text{if } v_t < 0 \end{cases}$
Threlfall (1978) [173]	$f_t = \begin{cases} \mu_d f_n \left(1 - e^{-\frac{3v_t}{v_\varepsilon}} \right) & \text{if } v_t \leq v_\varepsilon \\ 0.95\mu_d f_n & \text{if } v_t > v_\varepsilon \end{cases}$
Rooney and Deravi (1982) [174]	$f_t = \begin{cases} \mu_d f_n \tanh\left(\frac{v_t}{v_\varepsilon}\right) & \text{if } v_t \leq v_\varepsilon \\ \mu_d f_n & \text{if } v_t > v_\varepsilon \end{cases}$
Bo and Pavelescu (1982) [175]	$f_t = \left(\mu_d f_n + (\mu_s f_n - \mu_d f_n) e^{-\left(\frac{v_t}{v_s}\right)^p} \right) + \mu_v v_t$
Andersson et al. (2007) [176]	$f_t = \left(\mu_d f_n + (\mu_s f_n - \mu_d f_n) e^{-\left(\frac{v_t}{v_s}\right)^p} \right) \tanh(k_\varepsilon v_t) + \mu_v v_t$
Brown and McPhee (2016) [177]	$f_t = \mu_d f_n \tanh\left(4 \frac{v_t}{v_\varepsilon}\right) + f_n (\mu_s - \mu_d) \frac{\frac{v_t}{v_\varepsilon}}{\left(\frac{1}{4} \left(\frac{v_t}{v_\varepsilon}\right)^2 + \frac{3}{4}\right)^2} + \mu_v v_t \tanh\left(4 \frac{f_n}{f_{nt}}\right)$

The representation of the friction force models listed in Tab. 2, against the relative tangential velocity, is shown in the diagrams illustrated in Figs. 19a-f. For the sake of simplicity, half portion of those diagrams is only considered. The Bernard friction force model [172] removes the discontinuity associated with the Coulomb's law by using a linear force-velocity relation, as Fig. 19a depicts. In this model, the parameter k_ε represents the slope of the linear transition at low relative tangential velocities, which characterizes the speed of the transition from $-\mu_d f_n$ to $+\mu_d f_n$, and vice-versa. This approach has the inconvenient of having a sharp gradient at the instants in which the friction force reaches its maximum value. With the purpose to eliminate the weakness related to the sharp gradient of the linear transition law, Threlfall [173] utilized an exponential function to smooth out the step function of the Coulomb's law, as Fig. 19b depicts. In this model, v_ε is the threshold, or transition velocity, at which the friction force is to be velocity independent. A very small value for the parameter v_ε leads to a friction behavior close to the Coulomb's law, penalizing the computational efficiency of the resolution of the equations

of motion of the system under analysis. The Threlfall friction model is not able to predict the stiction mode. Rooney and Deravi [174] used a trigonometric function to smooth the Coulomb's law, which ensures a better stability of the resolution of the equations of motion. Figure 19c illustrates the nature the Rooney and Deravi friction force model. The threshold velocity v_ϵ , that defines the width of the region, in which the friction force is dependent on the relative tangential velocity, is not necessarily the same as in the Threlfall approach. The choice of the value for v_ϵ is a compromise between a good system's response and computational efficiency. The Rooney and Deravi model has the weakness that at null relative tangential velocity, the corresponding friction force is also null. The friction solution proposed by Bo and Pavelescu [175] accounts for the three main friction phenomena, namely the stiction, viscous and Stribeck actions, as it can be observed in the representation of Fig. 19d. This friction force formulation has the same weakness as the Coulomb's law for null relative tangential velocity, since it can predict the stiction regimen, which ultimately causes numerical problems in the resolution of the equations of motion. The Bo and Pavelescu approach uses an exponential function to smooth the transition between static and dynamic friction, being the shape of the decay from these two regimens dependent on the v_s and p parameters. In this model, v_s represents the Stribeck velocity, while p represents the nature of the geometry of the contacting surfaces, being usually set to 2 [178]. Andersson et al. [176] presented an enhanced friction force model that overcomes the weakness of the Bo and Pavelescu model, by incorporating a hyperbolic tangent function to ensure a smooth transition of the friction force at very low relative tangential velocities. The parameters k_ϵ represents the rate of increase from no friction at null velocity to static friction. This model, represented in Fig. 19e, cannot predict the stiction mode because at zero relative tangential velocity, the friction force is null. Due to its continuous nature, the Andersson et al. approach has the merit to be computationally efficient. More recently, Brown and McPhee [177] proposed an advanced force-velocity friction model, which is function of several parameters with physical meaning. This model, illustrated in Fig. 19f, includes stiction, sliding and viscous effects. The parameter f_{nt} is the normal transition force that represents the required minimum normal force for viscous action occurs, being v_ϵ the transition velocity. Most of the above friction force models smooth the Coulomb's law, reason why they can be called as continuous or regularized solutions, which allows for a stable and efficiency resolution of the equations of motion of multibody systems. It must be stated that a simple and clear rule of thumb for the selection of appropriate values for the parameters involved in the friction models described is not available, and their choice is typically based on the trial and error method.

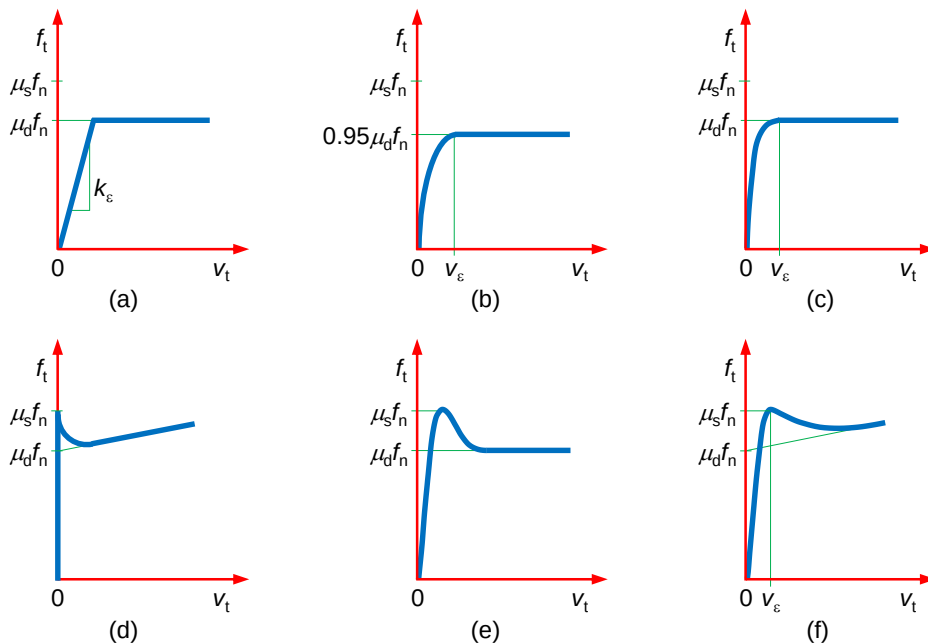


Fig. 19 (a) Bernard friction model; (b) Threlfall friction model; (c) Rooney and Deravi friction model; (d) Bo and Pavelescu friction model; (e) Andersson et al. friction model; (f) Brown and McPhee friction model.

7. Demonstrative application examples

This section presents some demonstrative application examples on the methodologies revisited in the previous sections, which allow assessing their effectiveness when modeling contact-impact events in multibody mechanical systems. The cases considered here range from simple systems, such as the classic bouncing ball problem, to more complex systems, such as the academic slider-crank mechanism with a revolute clearance joint. With the purpose of keeping the discussion simple, only few contact force models are utilized to simulate the collisions, namely those that can predict a reasonable system's dynamic response both in terms of computational accuracy and efficiency.

7.1. Bouncing ball system

The first application example presented here is the classical bouncing ball problem that involves normal collisions between the ball and the ground surface, as schematically illustrated in Fig. 20 [7, 179]. In this dynamic system, the ball has a mass of 1 kg, a radius of 100 mm, and generalized contact stiffness is equal to $140 \times 10^6 \text{ N/m}^{3/2}$. The ball is released from an initial height corresponding to 1 m, under the gravitational effect only, which is considered to act in the negative y direction. The system is characterized by a coefficient of restitution of 0.9. Hence, the ball describes a downward motion until a collision with the ground takes place, which is followed by a rebound directly dependent on the value of the coefficient of restitution. In the following discussion, the collision between the ball and ground is modeled with both continuous and piecewise methods.

As far as the continuous or regularized approach is concerned, applying the Newton's second law of motion to the ball yields

$$m\ddot{y} = f_g + f_n \quad (70)$$

where m represents the mass of the ball, $\ddot{y}$ is the linear acceleration of the ball, f_g denotes the gravitational force, which is given by $-mg$, and f_n is the normal contact force developed when the ball impacts with ground. The contact force model proposed by Lankarani and Nikravesh (58) is utilized to calculate the normal contact force during the impact events between the ball and ground. Thus, Eq. (70) is solved for the ball acceleration, being then numerically integrated to obtain the global dynamic behavior of the ball, in terms of velocity and position.

In turn, the ball motion analysis based on the piecewise or discrete approach is performed by integrating Eq. (70), that is

$$\int_{t^{(-)}}^{t^{(+)}} m\ddot{y}dt = \int_{t^{(-)}}^{t^{(+)}} (f_g + f_n)dt \quad (71)$$

which results in

$$m(\dot{y}^{(+)} - \dot{y}^{(-)}) = f_g\Delta t + p_n \quad (72)$$

where $\dot{y}^{(-)}$ and $\dot{y}^{(+)}$ represent the pre- and post-impact velocity, respectively, f_g is the gravitational force of the ball, and p_n denotes the impulse due the collision between the ball and ground. Combining Eqs. (43), (45) and (72) yields the equation of motion of the ball expressed at the velocity level, that is

$$\dot{y}^{(+)} = \dot{y}^{(-)} - g\Delta t - \dot{y}^{(-)}(1 - c_r) \quad (73)$$

in which c_r represents the coefficient of restitution. Therefore, Eq. (73) is utilized to determine the ball velocity in each instant of time, being then numerically integrated in time to obtain the ball position, using Eq. (18). Finally, it should be noticed that the last parcel of Eq. (73) is considered during the impacts only.

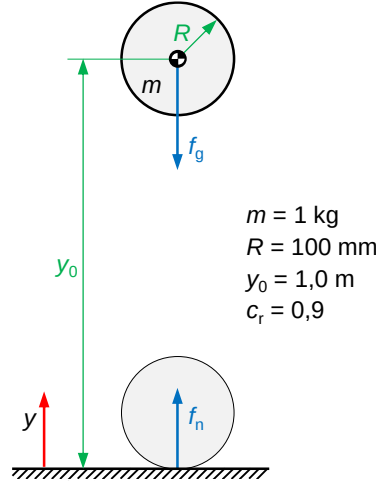


Fig. 20 Representation of the bouncing ball system.

Independently of the method used to handle the impact problem of the ball and ground, the geometric condition that permits to check if and when the ball touches the ground is given by

$$\delta = y - R \quad (74)$$

where δ represents the relative penetration between the ball and ground, y is the vertical position of the ball center of mass, and R denotes the radius of the ball. It should be noted that positive values of penetration represent effective contact scenarios [63].

Figure 21 shows the ball vertical position and velocity when both continuous and piecewise formulations are utilized to model the impacts between the ball and the ground. From the analysis of these two plots, it can be observed that when the ball impacts the ground, a rebound is produced, which is quite visible in the velocity jumps. Since the value of the coefficient of restitution is less than unity, the ball rebounds less in consecutive impacts, and, eventually, after a good number of impacts, the ball remains on the ground. Finally, it can be said that the continuous and piecewise methods predict similar ball response.

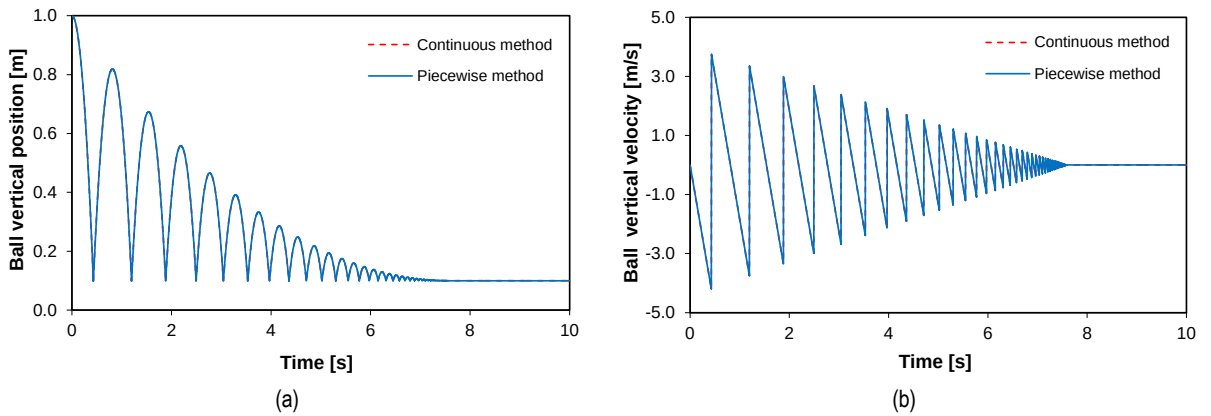


Fig. 21 Ball motion performance: **(a)** Ball center of mass position; **(b)** Ball center of mass velocity.

7.2. Slider-crank mechanism and a free block

With the purpose of analyzing the consequences of using different normal contact force models on the dynamic response of multibody mechanical systems, a slider-crank mechanism in which the slider body impacts with an external free sliding block is considered here as an application example [62, 77]. Figure 22 illustrates the multibody system, which is composed of five rigid bodies, namely the ground, the crank, the connecting rod, the slider and the free block. The slider-crank mechanism is kinematically constrained due to the existence of three ideal revolute joints and one ideal translational joint. The initial conditions of this mechanism are

those that result from an input crank angular velocity of 150 rad/s. The start of the dynamic analysis of this mechanism corresponds to the top dead point. In turn, the sliding block is moving with a constant linear velocity equal to 15 m/s to the left and is initially positioned at 0.914 m. Table 3 includes the geometric and inertia properties of this multibody model.

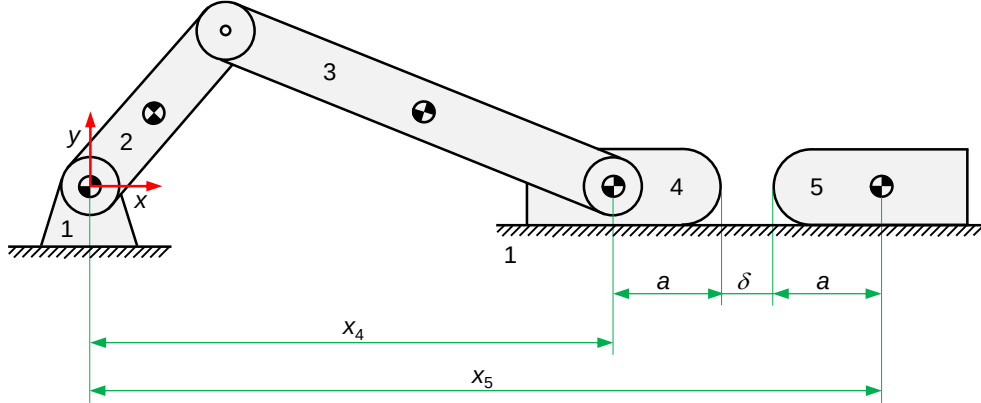


Fig. 22 Multibody system composed by a slider-crank mechanism and a free sliding block.

The potential contact surfaces of the sliding bodies 4 and 5 have a spherical shape, which radius is equal to 8.5 mm. The contact stiffness and the coefficient of restitution of the impacting surfaces are established as $9.5 \times 10^9 \text{ N/m}^{3/2}$ and 0.4, respectively. The mathematical relation that allows to evaluate the penetration of the impacting bodies is expressed as

$$\delta = x_5 - x_4 - 2a \quad (75)$$

where x_4 and x_5 are the x coordinates of the center of mass of bodies 4 and 5, respectively, and a is a characteristic dimension represented in Fig. 22, value of which is set to 16.93 mm.

The dynamic response of the multibody system is materialized by the diagrams of the free sliding block and the linear velocity of the constrained slider body, as it is presented in Figs. 23a and 23b. For this purpose, the six different contact force models included in Tab. 1 are considered to model the impact between the sliding bodies. It can be observed that the free slider experiences two impacts, which are visible in the slope changes in its position diagram, and also in the jumps in the velocity of the constrained slider. Moreover, it can be observed that the Lee and Wang contact force model causes rebounds of higher magnitude. This is sound because this particular contact force approach is the one that dissipates less energy, as it is shown in Fig. 16. Conversely, Gonthier et al. and Flores et al. models, which have a similar behavior, generates rebounds of less magnitude due to its dissipate nature. The remaining three contact force models predict an intermediate response. Similar conclusions can be extracted from the force-penetration plots for the two impacts between the sliders, as it is shown in the diagrams of Figs. 24a and 24b, which is valid for the first and second impacts.

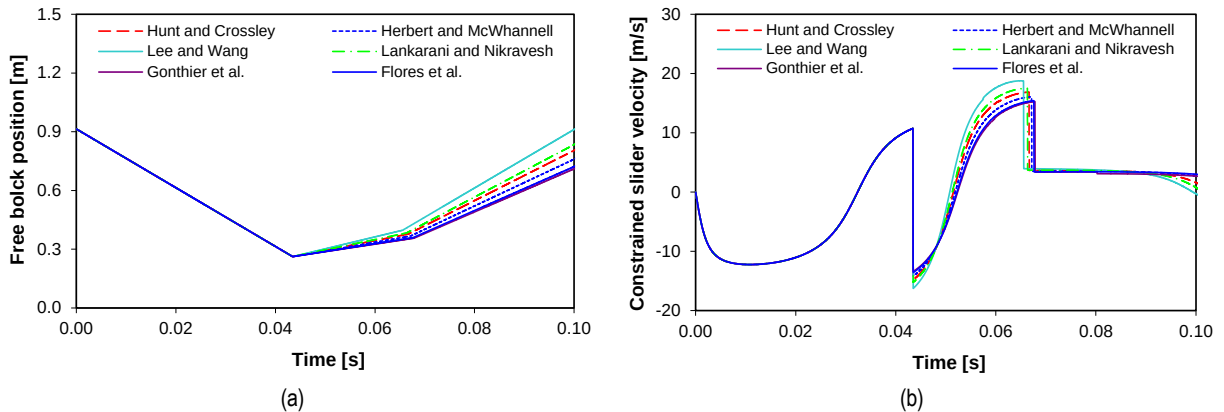


Fig. 23 (a) Position of the free sliding block; **(b)** velocity of the constrained slider.

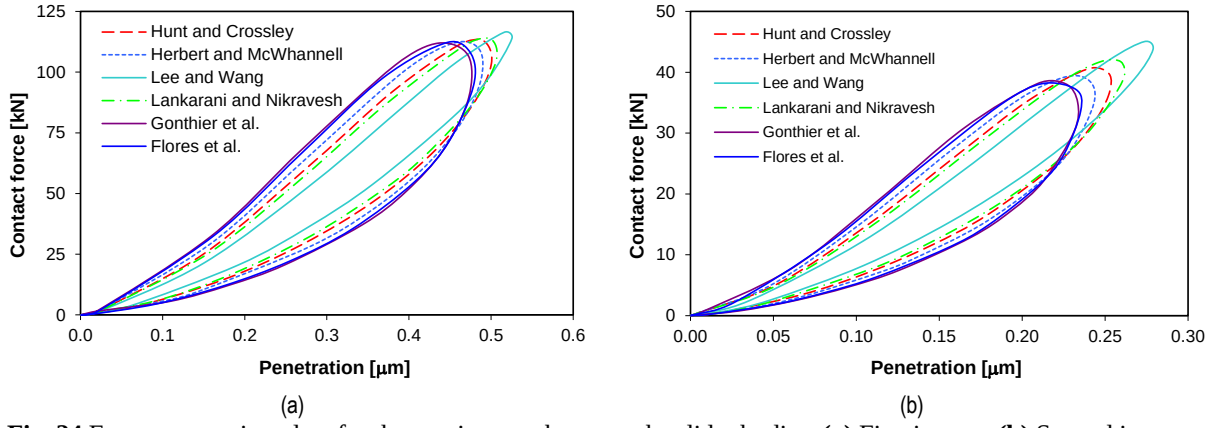


Fig. 24 Force-penetration plots for the two impacts between the slider bodies: **(a)** First impact; **(b)** Second impact.

Tab 3 Geometric and inertia properties for the slider-crank mechanism and free sliding block.

Body	Length [m]	Mass [kg]	Moment of inertia [kg·m ²]
Crank	0.153	0.038	7.4×10^{-5}
Connecting rod	0.306	0.076	5.9×10^{-5}
Slider	-	0.038	1.8×10^{-6}
Free block	-	0.190	2.7×10^{-5}

7.3. Slider-crank mechanism with a revolute clearance joint

A planar slider-crank mechanism with a revolute joint with clearance is chosen as an application example of the normal and tangential contact force models. This multibody mechanical system includes four rigid bodies, two ideal revolute joints, one ideal translational joint, and one revolute joint with clearance that connects the slider and connecting rod. A generic configuration of this mechanism is illustrated in Fig. 25 [127, 180]. The contact problem that occurs within the revolute joint with clearance is modeled as an eccentric collision between the journal and bearing elements, as Fig. 26 depicts. Thus, when an impact takes place, normal and tangential contact forces are generated at the contact points, which can be calculated based on the approaches described in Sections 5 and 6. Since the intra-joint contact forces do not act through the centers of mass of the bodies that contain the journal and bearing elements, the moment components for each body also have to be determined and incorporated into the equations of motion as external generalized forces [1].

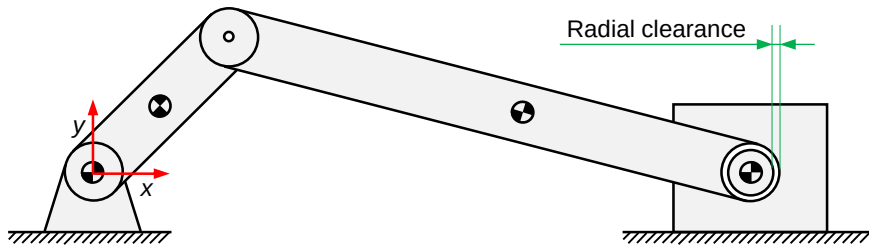


Fig. 25 Slider-crank mechanism with a revolute joint clearance between the connecting rod and slider.

Regarding the Fig. 26, the relative penetration vector at the contact points between the journal and bearing elements can be expressed as

$$\delta = \mathbf{e} - (R_B - R_J) \mathbf{n} \quad (76)$$

where $\mathbf{e}$ represents the eccentricity vector, that is the vector connecting the center of the bearing to the center of the journal, R_B and R_J denote the bearing and journal nominal radii, and $\mathbf{n}$ is the unit vector perpendicular to the plane of collision, as shown in Fig. 26. The magnitude of the penetration in each instant of time can be evaluated as

$$\delta = e - c \quad (77)$$

in which c is the radial clearance size, that is, $c = R_B - R_J$.

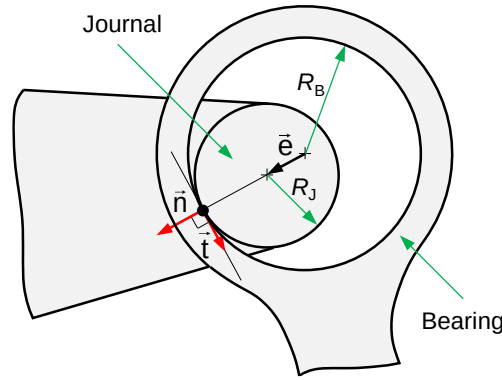


Fig. 26 Generic configuration of a revolute clearance joint in a multibody system.

The normal and tangential contact velocities of the contact points, required to determine the corresponding intra-joint contact forces, can be evaluated using Eqs. (27) and (28), respectively. In turn, the normal and tangential contact forces are determined employing Eqs. (58) and (66), respectively. The generalized contact stiffness relative to the bearing and journal pair is set to as $6.6 \times 10^{10} \text{ N/m}^{3/2}$, the system coefficients of restitution and friction are equal to 0.9 and 0.03, respectively, being the radial clearance size equal to 0.5 mm.

Tab 4 Geometric and inertia properties for the slider-crank mechanism with a revolute clearance joint.

Body	Length [m]	Mass [kg]	Moment of inertia [$\text{kg} \cdot \text{m}^2$]
Crank	0.15	0.03	1.0×10^{-5}
Connecting rod	0.13	0.21	2.5×10^{-4}
Slider	-	0.14	-

For the dynamic analysis of the slider-crank mechanism, the crank is the driving element that rotates at a constant angular velocity equal to 5000 rpm. The geometric and inertia properties of the bodies that compose this multibody system are presented in Tab. 4. In the computational simulations performed here, the initial configuration of the multibody system is with both crank and connecting rod bodies positioned horizontally, corresponding to the top dead point. The journal and bearing centers coincide at the initial instant of analysis. The dynamic behavior of this slider-crank mechanism is characterized by the plots of the crank moment relative to two complete crank rotations, as Fig. 27 illustrates, where the influence of modeling the intra-joint contact-impact events with and without friction can be observed. Overall, the effect of friction is to reduce the level of crank moment peaks associated with the collisions between the journal and bearing surfaces. The same conclusion can be drawn from the plots of the journal center trajectory inside the bearing boundaries, as the diagrams of Fig. 28 depicts. For the case without friction, the journal experiences three different operating regimens, namely the free flight mode, the impact immediately followed by rebounds and impact followed by continuous contact of the journal with the bearing wall. For the case with friction, the journal has a smooth behavior, in the measure that it tends to follow the bearing wall.

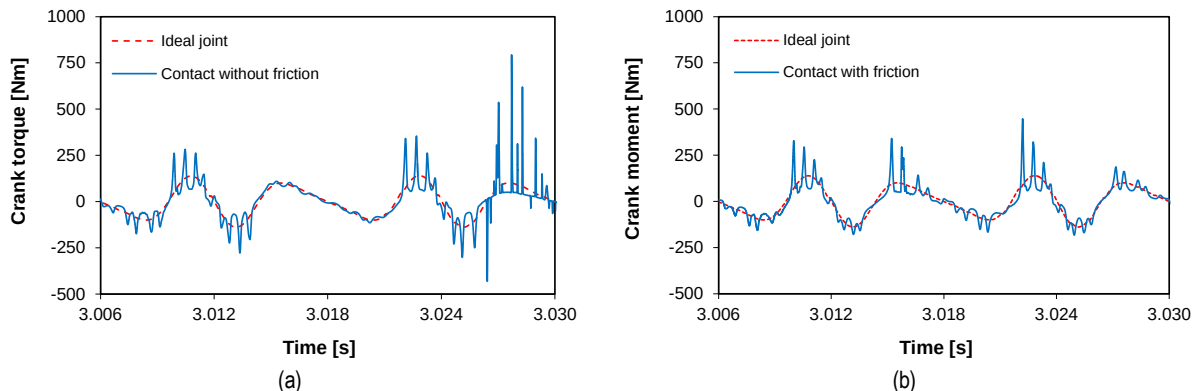


Fig. 27 Crank moment evolution for two full crank rotations: **(a)** Contact without friction; **(b)** Contact with friction.

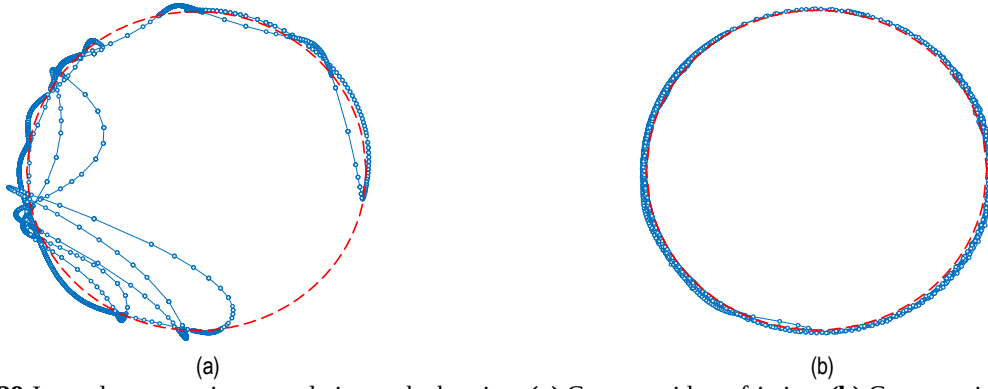


Fig. 28 Journal center trajectory relative to the bearing: **(a)** Contact without friction; **(b)** Contact with friction.

7.4. One degree of freedom mass-spring system

The single degree of freedom mass-spring, illustrated in Fig. 29, is utilized here as an application example to assess and compare the use of the several friction force models described in Section 6. This multibody mechanical system is also considered to examine the response of the standard Coulomb friction force model. This system consists of a mass connected to a wall by a linear spring that moves on a ground with friction [167, 181].

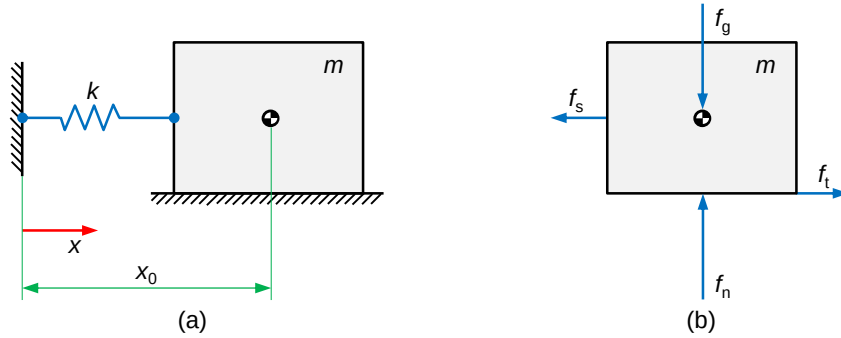


Fig. 29 **(a)** One degree of freedom mass-spring dynamic system; **(b)** Corresponding free body diagram.

With regard to Fig. 29b, the dynamic equation of motion of the mass-spring multibody model can be written as

$$m\ddot{x} = k_s (x_0 - x) - f_t \quad (78)$$

where m represents the mass of the moving body, k_s denotes the spring constant, x_0 is the initial position of the body, x is the instantaneous horizontal position of the body, and f_t is the friction force, which acts in the opposite direction to the relative tangential velocity. Equation (78) can easily be solved and the results integrated in time to obtain the dynamic behavior of the system. The physical and simulation values of the parameters used in this study are listed in Tab. 5.

Tab 5 Physical and simulation properties of the mass-spring multibody model.

Mass, m	1 kg	Initial position, x_0	1 m
Spring constant, k_s	10 N/m	Initial velocity, $\dot{x}_0$	0 m/s
Static friction coefficient, μ_s	0.2	Integration time step	0.001 s
Dynamic friction coefficient, μ_d	0.1	Simulation time	10 s

Figure 30 shows the dynamic response of the mass-spring system when the Coulomb friction force model is utilized. For this purpose, the position, velocity and acceleration of the moving body are the variables considered to generate the diagrams. In addition, the friction force is plotted against the relative tangential velocity. By and large, the diagrams of the mass position and velocity are sound and in line with the analytical solution [182, 183]. However, the

acceleration diagram presents an erratic behavior during the stiction mode, as Fig. 30c shows, in which the friction force oscillates because the coulomb's law cannot predict the friction force at zero relative tangential velocity. The zero relative tangential velocity occurs at the initial instant and during the change of direction of the motion. It should be noticed that the null initial velocity avoids the numerical difficulties associated with the Coulomb friction force model. Overall, the present analysis corroborates the outcomes available in the literature [72, 181].

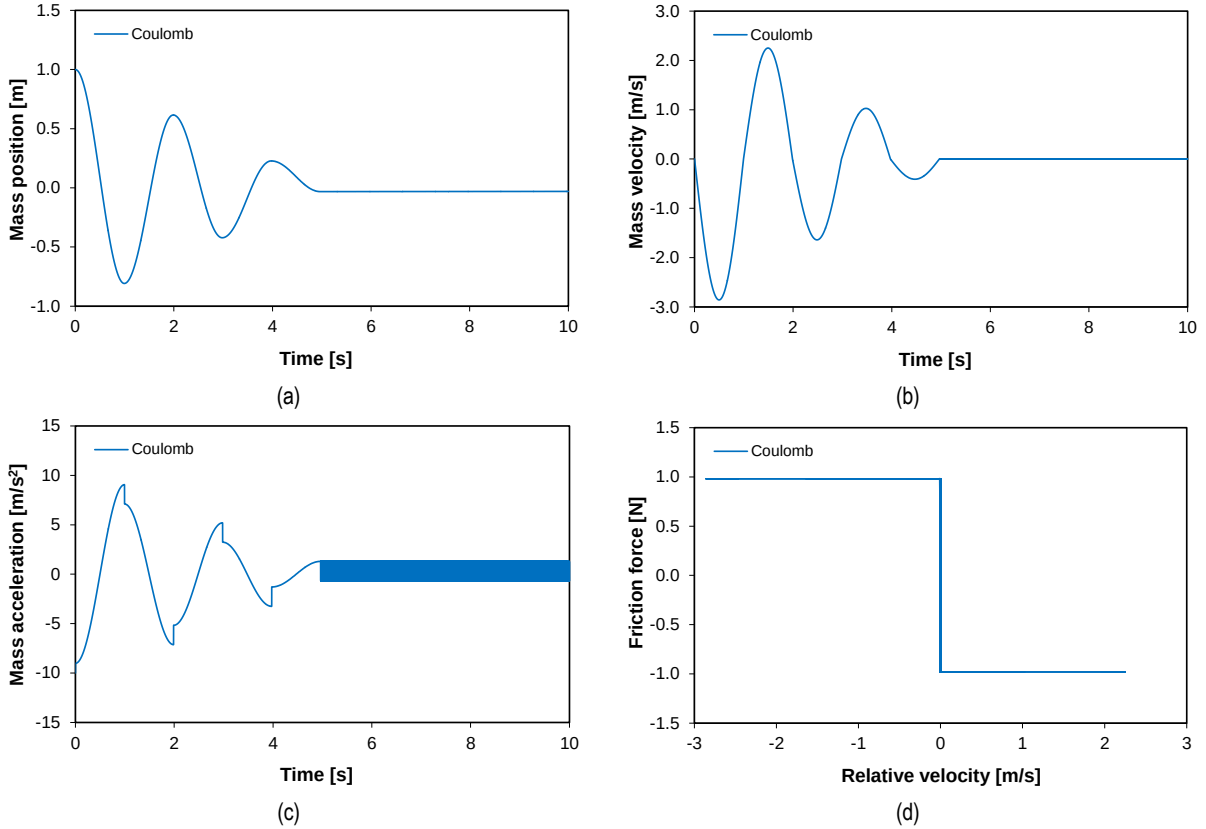
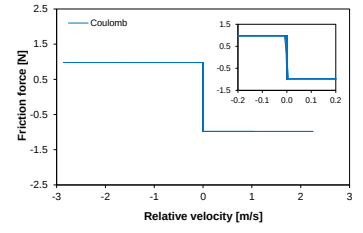
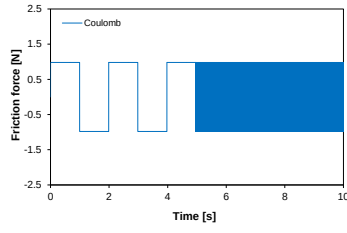
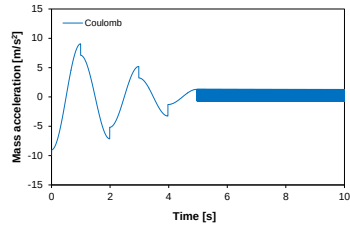
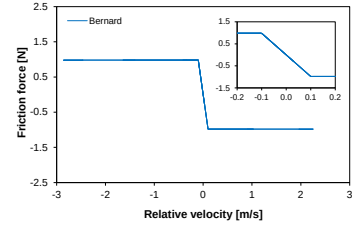
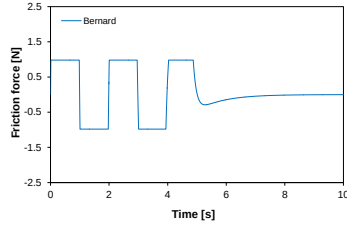
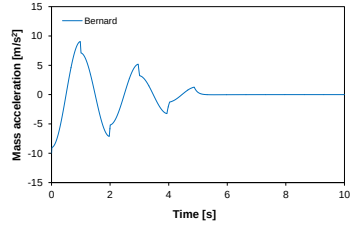


Fig. 30 Dynamic response of the mass-spring system from 0 to 10 s: **(a)** Mass position; **(b)** Mass velocity; **(c)** Mass acceleration; **(d)** Friction force versus relative tangential velocity.

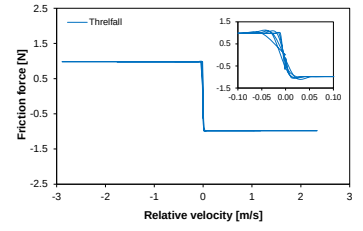
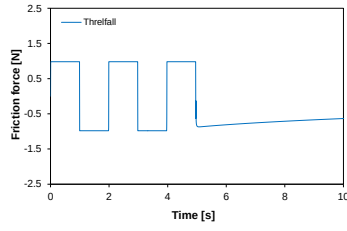
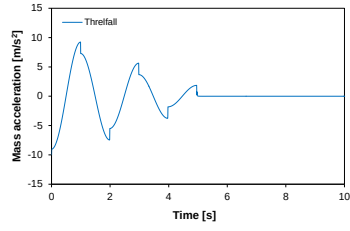
For the purpose of examining the dynamic behavior of the different friction force models described above, the mass-spring system is considered again. The values of the parameters utilized in each friction model are listed in Tab. 6, which were selected based on the information available in the literature and on the proper dynamic response of the system [72, 73, 183, 184]. The dynamic computational simulations were performed for 10 seconds, being the outcomes relative to the mass acceleration, friction force, and the friction force plotted against the relative tangential velocity, as shown in Fig. 31. As expected, the Coulomb and Bo and Pavelescu friction force models exhibit a high computational instability due to their own nature, which are not capable to predict friction force at null relative tangential velocity, corresponding to the stiction mode. The remaining friction force models present a smooth behavior in terms of the acceleration and friction force evolutions. In fact, the mass acceleration becomes zero at the beginning of the stiction phase, and there are no oscillations in the friction forces when the relative tangential velocity is null. It should be noticed that in the Bernard, Threlfall and Rooney and Deravi approaches, the friction force did not reach a null value for the stiction mode, as it is visible in the corresponding plots. Moreover, the diagrams of the friction force of the Andersson et al. and Brown and McPhee approaches incorporate a viscous component that persists during the stiction phase. The viscous parcel also contributes for the stability and continuity of the system's response. In a broad sense, it can be recognized that the Brown and McPhee friction force model is quite interesting due to the physical meaning of the parameters involved in their formulation, and also due to the computational stability and accuracy proportioned. Furthermore, this model is capable to account for the static, dynamic and viscous frictions.



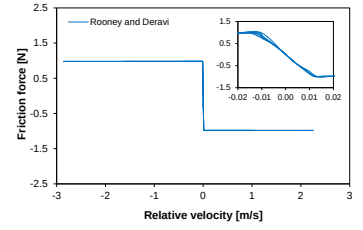
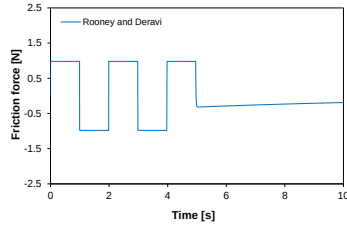
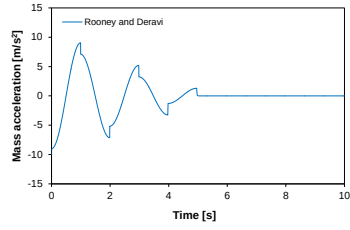
(a) Coulomb friction force model



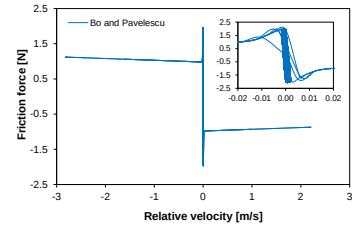
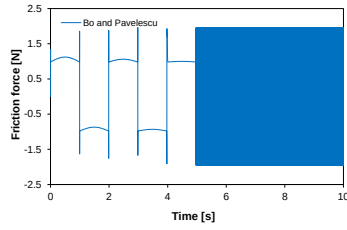
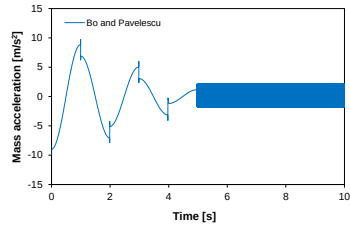
(b) Bernard friction force model



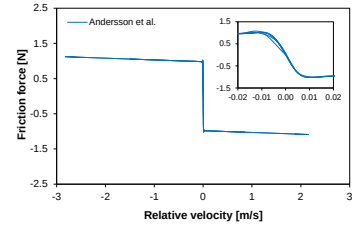
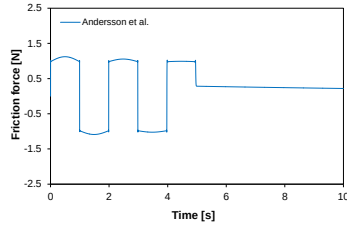
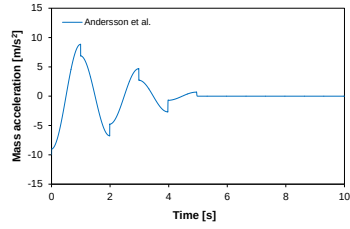
(c) Threlfall friction force model



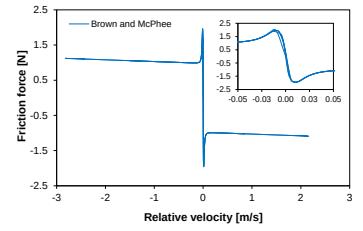
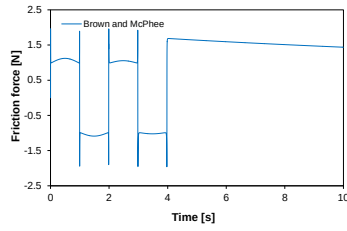
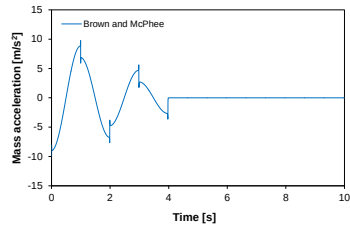
(d) Rooney and Deravi friction force model



(e) Bo and Pavelescu friction force model



(f) Andersson et al. friction force model



(g) Brown and McPhee friction force model

Fig. 31 Dynamic response of the mass-spring system for different friction force models: **(a)** Coulomb; **(b)** Bernard; **(c)** Threlfall; **(d)** Rooney and Deravi; **(e)** Bo and Pavelescu; **(f)** Andersson et al.; **(g)** Brown and McPhee.

Tab 6 Values of the parameters utilized in the several different friction force models.

Friction force model	Parameters considered
Bernard	$\mu_d = 0.1$; $k_e = 10 \text{ s/m}$
Threlfall	$\mu_d = 0.1$; $v_e = 0.01 \text{ m/s}$
Rooney and Deravi	$\mu_d = 0.1$; $v_e = 0.01 \text{ m/s}$
Bo and Pavelescu	$\mu_d = 0.1$; $\mu_s = 0.2$; $v_s = 0.01 \text{ m/s}$; $p = 2$; $\mu_v = 0.05 \text{ Ns/m}$
Andersson et al.	$\mu_d = 0.1$; $\mu_s = 0.2$; $v_s = 0.01 \text{ m/s}$; $p = 2$; $\mu_v = 0.05 \text{ Ns/m}$; $k_e = 100 \text{ s/m}$
Brown and McPhee	$\mu_d = 0.1$; $\mu_s = 0.2$; $v_e = 0.01 \text{ m/s}$; $\mu_v = 0.05 \text{ Ns/m}$; $f_{nt} = 10 \text{ N}$

8. Concluding remarks

A comprehensive discussion from the basis to the state-of-the-art on the subject of frictional collisions in multibody dynamics was presented in this paper. In the sequel of this review, the fundamental aspects related to the formulation of constrained multibody systems were revisited, in which the development and resolution of the Newton-Euler equations of motion were examined. The kinematic features associated with contact-impact events were assessed, in terms of penetration and normal and tangential contact velocity components of the potential contact points. The contact-impact dynamics, based on the piecewise and continuous methods, were also described. In the former, a collision is divided into two discrete stages, namely before and after impact, as the energy dissipated is accounted for via coefficient of restitution. The problems associated with discontinuities of the piecewise method can be overcome by employing a continuous formulation, in which the collision problems are solved continuously considering explicit relations between the contact force and penetration of the contacting surfaces. The continuous method constitutes a quite useful and effective method to model contact-impact events in dynamical systems due to its computational stability and efficiency when solving the equations of motion of constrained multibody systems. In addition, it provides a more suitable method for contact-impact problems in which the duration of the contact is larger enough to that the system's configuration might change during the period of contact. Some of the most relevant normal dissipative and tangential friction contact force models were analyzed and compared. Finally, several appropriate examples of application were utilized to assess the effectiveness of the use of different techniques to handle contact-impact frictional events in multibody dynamics.

Overall, it is evident that the topic of contact-impact with friction has been quite a fertile area of research. Despite many advances from the pioneering works of giants such as Hertz and Coulomb, to the latest advances in model developments and simulation techniques over the last few decades, the topic of contact-impact with friction is still of significant interest to the multibody dynamics community and beyond, and is expected to produce major advances in the coming years. Future research may include developments on parameters identification and estimation, using the input data from physical experiments to drive the simulation of uncertain model parameters.

Open issues include: set up a collection of benchmark problems to compare and validate techniques available in the literature to assess their suitability to predict the physical behavior; and need for benchmark problems in particular rough surface collisions with friction. Other important aspects related to contact dynamics problems comprise: non-hertzian contacts with large contact area, systems with “many” multiple contacts in simultaneous, models to accelerate the contact detection in complex systems; together with studies on interaction with non-rigid structures, multiscale problems (nano, micro etc.), among others. Numerical methods able properly handle the resolution of the equations of motion in the presence of contact-impact events are still desired to be investigated. Numerical approaches to handle instabilities and discontinuities commonly encountered in some friction force models are also required to better represent the mechanical nature of frictional contact-impact events.

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