The normalized expected utility – entropy and variance model for decisions under risk

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Abstract

The normalized expected utility - entropy and variance (NEU-EV) decision model and associated risk measure are proposed for the analysis and modelling of decisions under risk. The model depends on normalized entropy and variance and expected utility, can be applied to decision problems depending on actions, where the states have different numbers of outcomes. Several properties of risk perception and examples of certainty effects, common ratio effects and common consequence effects are analysed using this model, considering actions with non-negative outcomes. The results show that the NEU-EV model is an adequate model for explaining these risk decision problems.

Keywords: risk decision model, expected utility, entropy and variance, common consequence effect, common ratio effect, certainty effect

1. Introduction

In the context of decision making under risk several models and risk measures have been proposed which are based on expected utility theory by von Neumann and Morgenstern [28]. Due to inconsistencies in modelling empirical experiments using expected utility alone, as reported e.g. in [13], [18], modifications and generalizations of the expected utility model have been developed. Kahnemann and Tversky [13] proposed prospect theory as an alternative model for decision making under risk, assigning value to gains and losses and replacing

probabilities by decision weights. Marley and Luce [20] presented a theoretical analysis of five different classes of utility representations. Some of the new models include entropy as additional risk decision factor, e.g. the expected utility and entropy model (EU-E model) by Yang and Qiu [30], their normalized EU-E model presented in [31], and the entropy-modified expected utility models in [15] and [16], or in [23], which extends the earlier expected utility and entropy approach of Meginniss [21]. According to [1], entropy can be employed as measure of risk. There, the authors proposed an adaptive decision model, where choices are made via entropy minimization. The importance of entropy in decision making was also shown in [8] and [9], where a generalized maximum entropy formulation was used as a basis for estimation and inference in a range of econometric-statistical models.

Recently, a generalization of the EU-E model was obtained in [4], the expected utility-entropy and variance (EU-EV) model, which is based also on variance as risk factor. Variance was also used in other decision models. Pollatsek and Tversky [24] showed that under the assumption of a psychological theory of perceived risk, the risk of an option is expressible as a linear combination of its mean and variance. Other examples are the mean-variance model for Modern Portfolio Theory [19] or the expectation-variance-indeterminacy models [27].

The models depending on expected utility and entropy developed in [30], [31], [16], [4] were shown to be compatible with results obtained in empirical experiments concerning certain decision problems, which result in paradoxes, when trying to explain them solely with expected utility theory. The Allais paradox [2] is one decision paradox that was successfully interpreted with those models. This paradox serves to depict a phenomenon called the certainty effect, a tendency of people's behaviour to favour a certain or a risk-free option relative to an uncertain one. The common ratio effect and the common consequence effect are other phenomena, where people typically make decisions violating expected utility theory. These effects were analysed in [31], where the authors demonstrate an adequate application of the normalized EU-E model to overcome

the inconsistency problems and describe the observed phenomenon.

In the present paper a new normalized expected utility-entropy and variance (NEU-EV) decision model is proposed. Examples of decision problems, including certainty effects, common ratio and common consequence effects, are analysed with this model. Properties and applications of the model are considered for actions representing gambles or lotteries with non-negative outcomes.

The NEU-EV model improves and extends the earlier EU-EV model in the following aspect.

Since decision problems may have actions where the corresponding states have different numbers of outcomes and since the normalized entropy is a standardized measure permitting a better comparison of these kind of actions than the simple entropy, it is reasonable to integrate normalized entropy as risk decision factor in the model. In fact, as it will be seen, the normalization of entropy is crucial and makes a significant difference in the results for risk decision between actions with different numbers of outcomes. The normalized entropy is obtained by dividing the simple entropy through the maximum entropy and ranges from 0 to 1, independently of the number of outcomes. The NEU-EV model and risk measure uses thus normalized entropy and normalized variance as uncertainty risk factors and normalized expected utility as subjective preference factor. This generalizes the applicability of the earlier EU-EV model which used the simple entropy, along with normalized variance and normalized expected utility.

The novelty in the NEU-EV model, besides extending the earlier EU-EV model by introducing normalized entropy allowing a wider applicability to problems with different state numbers, is that it also extends the earlier NEU-E model including the variance as additional risk factor. The reason for including the variance is that there exist decision problems, where actions have equal expected utilities or expected values and equal entropies or normalized entropies however different variances (see the investment decision problem of Nawrocki and Harding [22] analysed with the EU-EV model in [4] or the example of Yang and Qiu [31] considered in section 4 of the present paper) and it is therefore

necessary to solve also this kind of decision problems.

The paper is organized as follows. In section 2, the NEU-EV decision model and the associated risk measure are defined. Some properties of the model are explored in section 3 and an application for understanding the appropriateness of the model is presented in section 4, where an example considered in [31] is studied and the results obtained with the NEU-EV model are compared with those obtained with the normalized EU-E model. In section 5, the NEU-EV model is applied to several decision problems known as common ratio effects (in subsection 5.1) and common consequence effects (in subsection 5.2), including also the certainty effects. The conclusions are presented in section 6. The appendix contains information about the decision models applied in this paper.

2. The normalized expected utility-entropy and variance measure of risk

The EU-EV model proposed in [4] depends on the uncertainty risk factors entropy and variance and on the expected utility, where the utility function transforms the objective outcomes into subjectively scaled values, while the probabilities remain unchanged. The model is an extension of the EU-E model [30], since it uses the variance as additional risk factor. The inclusion of expected utility, entropy and variance in the decision model is justified by the following reasons. As stated in [30], in decision analysis the decision maker's choice of an action is determined by the uncertainty of outcomes and by the decision maker's expected utility, which are related to risk as follows: the higher the uncertainty of an outcome is, the higher the risk, and the higher the expected utility of an action is, the lower the risk. This is the intuitive interpretation behind the two-attribute models for perceived risk developed in [11], or the risk-value models based on the idea of risk-value tradeoffs (or risk-return tradeoffs) (see [12]), according to which people judge risk considering the effect of the expected return of a risky outcome and also its variation or uncertainty. The use of entropy alone to model uncertainty has the advantage that it can be computed for quantitative

and qualitative data. Considering the inclusion of entropy in models for quantitative data, it has the advantage that it is free from an assumption concerning the underlying distribution, however, it has the disadvantage that it ignores the dispersion of data, which can be measured by the variance (or standard deviation). Primarily, the risk of an option has been associated with the dispersion of the corresponding random variable and, therefore, risk has been measured by variance (e.g. in portfolio selection models [19]). Independently, entropy and variance have been proposed in the literature to measure risk, see Brachinger and Weber [3] (and references therein), who called them naive risk measures. As shown in [4], see also [22], it is possible to have actions (e.g. securities with different levels of risk) with the same entropy and with the same expected utility, but with different variances, so that the variance may also play a decisive role for risk classification. Therefore, in the NEU-EV model, uncertainty is related to the state's entropy and variance and the perception of risk is that risk increases with entropy and with variance. However, risk can also not be assessed measuring only uncertainty. It is reasonable to take into account both, the perception of risk given by uncertainty and also the preference based on expected utility. This approach for assessing risk is motivated by the interpretation of the risk-value tradeoff models in [11], [12] and by the successful applications of the EU-EV model in [4] and the EU-E model in [30]. In the context of expected utility, the decision maker's attitude towards risk is defined implicitly through the shape (characterized by the second derivative) of the utility function, whose purpose is to model preferences. In the proposed NEU-EV decision model, risk is also linked with preference through the expected utility and as it will be seen, the shape of the utility function can influence the risk decisions. The normalized quantities, expected utility, on the one hand, and entropy and variance, on the other hand, are then combined using a trade-off parameter. The role of the trade-off parameter and the NEU-EV model will be explained in detail in the following paragraphs.

In order to define the NEU-EV risk measure and associated decision model, the following notations will be used. Let $G(\Theta, A, X)$ be a decision model under

risk, where Θ is the state space, A the action space and $X = X(a, \theta)$ the payoff function, where $a \in A$ and $\theta \in \Theta$. Let $u(X) = u(X(a, \theta))$ be the utility function of the decision maker. Then, the decision model under risk can be written as $G(\Theta, A, u)$. For a finite action space with finite state space, consider $A = \{a_1, a_2, \ldots, a_m\}$ and let θ_i be the state corresponding to action a_i for $i = 1, \ldots, m$. Suppose that each state θ_i has n_i outcomes: $\theta_i = \{\theta_{i1}, \theta_{i2}, \ldots, \theta_{in_i}\}$, the state space being then $\Theta = \{\theta_1, \ldots, \theta_m\}$. Let p_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n_i$, denote the distribution law of θ_i, where $\sum_{j=1}^{n_i} p_{ij} = 1$ and $p_{ij} \geq 0$. Then $p_{ij} = P(\theta = \theta_{ij} | a = a_i)$ denotes the probability that state θ_{ij} occurs when taking action a_i. The corresponding payoff, when taking action a_i while state θ_{ij} occurs, is given by $X = X(a_i, \theta_{ij}) = x_{ij}$, $i = 1, \ldots, m$, $j = 1, \ldots, n_i$. The action a_i can be represented in the form

$$a_i = (x_{i1}, p_{i1}; x_{i2}, p_{i2}; \ldots; x_{in}, p_{in}), \tag{1}$$

where here n is used to denote the number of outcomes of a_i. Given an action a_i defined by (1), one can characterize the distribution of its outcomes using a simplified notation, where the outcomes can be represented by a discrete random variable X taking values $x_{i1}, \ldots, x_{in}$ with probabilities $P(X = x_{ij}) = p_{ij}$, $j = 1, \ldots, n$.

The entropy, defined originally by Shannon [26] as information measure, is also a measure of uncertainty (see e.g. [25], [29]). Consider the action $a = (x_1, p_1; \ldots; x_n, p_n)$, where the state θ has n outcomes $\theta_1, \ldots, \theta_n$ with payoffs $X = X(a, \theta_i) = x_i$, $i = 1, \ldots, n$, which occur with probabilities $p_i = P(\theta = \theta_i | a)$, $i = 1, \ldots, n$, respectively. Then, the entropy of the state θ corresponding to action a is

$$H_a(\theta) = -\sum_{i=1}^{n} p_i \ln p_i. \tag{2}$$

The normalized entropy is defined by

$$NH_a(\theta) = \begin{cases} \dfrac{-\sum_{i=1}^{n} p_i \ln p_i}{\ln(n)}, & \text{if } n > 1, \\ 0, & \text{if } n = 1. \end{cases} \tag{3}$$

Note, that since the entropy is maximum if $p_i = \frac{1}{n}$ for all $i = 1, \ldots, n$, one has

$$H_a(\theta) = -\sum_{i=1}^{n} p_i \ln p_i \leq -\sum_{i=1}^{n} \frac{1}{n} \ln\left(\frac{1}{n}\right) = \ln(n),$$

where $\ln(n)$ is the maximum entropy. Consequently, it holds that $NH_a(\theta) \leq 1$ and therefore $NH_a(\theta) \in [0, 1]$, where $NH_a(\theta) = 0$ means that there is no uncertainty in terms of entropy, which is obtained if $p_i = 1$ for one i and $p_j = 0$ for all $j \neq i$, where $i = 1, \ldots, n$. The normalized entropy is a standardized measure and has the advantage that it permits a better comparison of entropies obtained for actions, where the states have different numbers of outcomes.

The NEU-EV risk measure and the associated NEU-EV decision model are presented in Definition 1 and Definition 2, respectively.

Definition 1. *Let $G(\Theta, A, u)$ be a decision analysis model, where the utility function $u = u(X(a, \theta))$ is increasing and $X(a, \theta)$ is the outcome corresponding to state $\theta \in \Theta$ when taking action $a \in A$. The normalized expected utility-entropy and variance (NEU-EV) measure of risk for an action a is defined by*

$$R(a) = \begin{cases} \dfrac{\lambda}{2}\left[NH_a(\theta) + \dfrac{\mathrm{Var[X(a,\theta)]}}{\max\limits_{a\in A}\{\mathrm{Var[X(a,\theta)]}\}}\right] - (1-\lambda)\dfrac{\mathbb{E}[u(X(a,\theta))]}{\max\limits_{a\in A}\{|\mathbb{E}[u(X(a,\theta))]|\}}, & \text{if } n > 1, \\ -(1-\lambda)\dfrac{\mathbb{E}[u(X(a,\theta))]}{\max\limits_{a\in A}\{|\mathbb{E}[u(X(a,\theta))]|\}}, & \text{if } n = 1, \end{cases} \tag{4}$$

where λ is a real constant satisfying $0 \leq \lambda \leq 1$, $NH_a(\theta)$ is the normalized entropy of the state distribution corresponding to action a and n is the number of outcomes.

The difference between the EU-EV and the NEU-EV models is that in the NEU-EV model the entropy (2) is substituted by the normalized entropy (3), so that in this new model all terms are normalized.

For simplicity, $\mathbb{E}[u(X(a, \theta))]$ will be denoted by $\mathbb{E}[u(a)]$, $\mathrm{Var[X(a, \theta)]}$ by $\mathrm{Var[a]}$ and $NH_a(\theta)$ by $NH(a)$ (and $H_a(\theta)$ by $H(a)$). The arithmetic mean of the normalized entropy and variance will be represented by

$$NEV(a) := \frac{1}{2}\left[NH(a) + \frac{\mathrm{Var[a]}}{\max\limits_{a\in A}\{\mathrm{Var[a]}\}}\right].$$

With these notation simplifications the risk for an action a can be represented by

$$R(a) = \begin{cases} \lambda NEV(a) - (1-\lambda)\dfrac{\mathbb{E}[u(a)]}{\max\limits_{a\in A}\{|\mathbb{E}[u(a)]|\}}, & \text{if } n > 1, \\ -(1-\lambda)\dfrac{\mathbb{E}[u(a)]}{\max\limits_{a\in A}\{|\mathbb{E}[u(a)]|\}}, & \text{if } n = 1, \end{cases} \tag{5}$$

where $0 \leq \lambda \leq 1$.

One says that $R(a)$ is the risk of action a. The constant λ is called tradeoff coefficient, since it is used in the definition to balance the decision maker's normalized expected utility of an action and the uncertainty reflected by the state's normalized entropy and normalized variance associated with the action. In this risk measure, the normalized entropy and variance are combined as arithmetic mean. If $\lambda = 0$, then the risk measure is based only on the expected utility and if $\lambda = 1$ the risk measure depends only on the uncertainty given by normalized entropy and variance. If $\lambda \in (0, 1)$, then the effect of the expected utility on the risk measure is bigger if λ approaches 0 and if λ approaches 1, the risk measure is more influenced by the uncertainty than by the expected utility. If $\lambda = 0.5$, then an equal weight is given to the uncertainty component and to the expected utility component.

This article is concerned with applications of the NEU-EV risk measure to risky actions representing gambles or lotteries with non-negative outcomes, where a decision maker must choose between two or more risky actions.

Given two actions a_1 and a_2, calculating their NEU-EV risks $R(a_1)$ and $R(a_2)$, one can conclude the following in terms of the NEU-EV risk: if $R(a_1) < R(a_2)$, then a_2 is riskier than a_1; if $R(a_1) > R(a_2)$, then a_1 is riskier than a_2; if $R(a_1) = R(a_2)$, then a_1 and a_2 are equally risky. Given two or more actions, they can be ordered with respect to the NEU-EV risk.

If a decision maker must choose between a_1 and a_2, then he compares $R(a_1)$ with $R(a_2)$ and chooses the action with lower NEU-EV risk. Thus, the decision maker chooses a_1 if $R(a_1) < R(a_2)$, a_2 if $R(a_1) > R(a_2)$ and if $R(a_1) = R(a_2)$, then the decision maker is indifferent between a_1 and a_2.

Here it is assumed that there exists a preference relation on the set of actions denoted by $\succeq$ (also designated as weak preference relation), where $\succ$ denotes the strict preference and the indifference is denoted by $\sim$, which is related and compatible with the risk ordering (or risk relation) described above, as given in Definition 2.

We now define the preference relation based on the risk measure. We assume this is the relation used by a person in choosing between two actions representing gambles or lotteries with non-negative outcomes.

Definition 2. *Let $G(\Theta, A, u)$ be a decision analysis model with increasing utility function u and non-negative outcomes.*

1. *Consider two actions $a_1, a_2 \in A$ with corresponding NEU-EV risk measures $R(a_1)$ and $R(a_2)$. Then:*
   (i) a_1 is strictly preferred over a_2, $a_1 \succ a_2$, if $R(a_1) < R(a_2)$.
   (ii) a_1 is weakly preferred over a_2, $a_1 \succeq a_2$, if $R(a_1) \leq R(a_2)$.
   (iii) a_1 is indifferent to a_2, $a_1 \sim a_2$, if $R(a_1) = R(a_2)$.
2. *Consider various actions, $A = \{a_1, a_2, \ldots, a_m\}$, then they can be ordered using the NEU-EV measure of risk. The optimal action is the one with minimum NEU-EV risk. In that case, one chooses a_i if*

$$R(a_i) = \min_{a_k \in A} R(a_k).$$

It will be explored to what extent the risk preference ordering implied by the NEU-EV risk measure in Definition 2 is compatible with the risk ordering derived from judgements about perceived risk. For that purpose, properties of the NEU-EV risk measure are derived in the next section. Then, in section 5, it will be analysed if the NEU-EV risk measure can represent numerically the risk preference ordering obtained from empirical studies.

3. Properties of the normalized expected utility-entropy and variance measure of risk

In this section, some properties of the NEU-EV risk measure and the associated decision model will be explored.

First, observe that for a given action $a \in A$ having non-negative outcomes, the NEU-EV risk has the following properties:

i) For $\lambda \in [0,1]$, one has $R(a) \in [-1,1]$.

ii) $R(a)$ increases when λ increases.

The first property $i)$ is a consequence of the following inequalities: since $0 \leq NEV(a) \leq 1$, because $0 \leq NH(a) \leq 1$ and $0 \leq \frac{\text{Var[a]}}{\max\limits_{a \in A}\{\text{Var[a]}\}} \leq 1$, and due to $0 \leq \frac{\mathbb{E}[u(a)]}{\max\limits_{a \in A}\{\mathbb{E}[u(a)]\}} \leq 1$, then $-1 \leq R(a) \leq 1$. Note that in this paper all results are derived for non-negative outcomes; for outcomes representing losses or for mixed outcomes the normalized entropy would range differently, from -1 to 0 or from -1 to 1, respectively, (see (4)), and the range for the normalized entropy and normalized variance would remain equal. When λ increases, then the influence of the $NEV(a)$ term and thus of uncertainty increases, while the expected utility term decreases, so that the second property $ii)$ shows that the choice of the trade-off factor gives the risk measure a further flexibility to modelling risk, besides the choice of the utility function.

The NEU-EV model is related to other models proposed in the literature. Consider two actions a_1 and a_2. Then:

1. If $\lambda = 0$, the model reduces to the expected utility model: $R(a_1) < R(a_2) \Leftrightarrow \mathbb{E}[u(a_1)] > \mathbb{E}[u(a_2)]$.
2. If $\lambda = 0$ and $u(x) = ax + b$, $a > 0$, decisions are taken according to the expected value principle: $R(a_1) < R(a_2) \Leftrightarrow \mathbb{E}[a_1] > \mathbb{E}[a_2]$.
3. If $\lambda = 1$, risk is determined only by normalized entropy and variance: $R(a_1) < R(a_2) \Leftrightarrow NEV(a_1) < NEV(a_2)$.
4. If $\lambda = 1$ and $NH(a) = 0$, risk is determined only by variance: $R(a_1) < R(a_2) \Leftrightarrow \text{Var}(a_1) < \text{Var}(a_2)$.
5. If $\lambda = 1$ and $\text{Var(a)} = 0$, risk is determined only by entropy: $R(a_1) < R(a_2) \Leftrightarrow NH(a_1) < NH(a_2)$.

See the Appendix A for understanding the relation between the NEU-EV model and the other models which are applied later in this article, where also the acronyms of the models are defined.

The risk decisions in the cases of Proposition 1 are direct consequences of the definition and are in accordance with people's perception of risk.

Proposition 1. *Consider a decision analysis model $G(\Theta, A, u)$, where $A = \{a_1, a_2\}$, with increasing utility function u and with NEU-EV risk measure defined in (4). Then, the following properties hold:*

(a) If $\mathbb{E}[u(a_1)] \geq \mathbb{E}[u(a_2)]$ and $NEV(a_1) < NEV(a_2)$, then $R(a_1) < R(a_2)$.

(b) If $\mathbb{E}[u(a_1)] > \mathbb{E}[u(a_2)]$ and $NEV(a_1) = NEV(a_2)$, then $R(a_1) < R(a_2)$.

The next theorem provides risk decisions in the special situation of comparing an action with a certain outcome with an action depending on probable outcomes, under the hypothesis that both outcomes have equal expected values.

Theorem 1. *Consider a decision analysis model $G(\Theta, A, u)$, where $u(x) \geq 0$, $u'(x) \geq 0$ and the outcomes are non-negative, and the NEU-EV risk measure defined in (4). Let $A = \{a_1, a_2\}$, where the action a_1 has a certain outcome c, $a_1 = (c, 1)$ and the action a_2 has the distribution $a_2 = (x_1, p_1; \ldots; x_n, p_n)$, where $\sum_{i=1}^{n} p_i = 1$ and $p_i < 1$ for at least one i, where $i = 1, \ldots, n$. Suppose that $\mathbb{E}[a_1] = \mathbb{E}[a_2]$. Then:*

(a) If $u''(x) \leq 0$, then $R(a_1) < R(a_2)$.

(b) If $u''(x) > 0$, then $R(a_1) < R(a_2)$ for $\frac{1-\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]}}{1+NEV(a_2)-\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]}} < \lambda \leq 1$.

PROOF. (a) Since $\mathbb{E}[a_1] = \mathbb{E}[a_2] \Leftrightarrow c = \sum_{i=1}^{n} x_i p_i$, one has

$$\mathbb{E}[u(a_1)] = u(c) = u\left(\sum_{i=1}^{n} x_i p_i\right) = u(\mathbb{E}[a_2]). \tag{6}$$

Given that $u''(x) \leq 0$, which includes the risk-averse and risk-neutral cases, Jensen's inequality implies $u(\mathbb{E}[a_2]) \geq \mathbb{E}[u(a_2)]$, so that, taking into account (6), $\mathbb{E}[u(a_1)] \geq \mathbb{E}[u(a_2)]$. The risk measures are then given by

$$R(a_1) = -(1-\lambda),$$
$$R(a_2) = \lambda NEV(a_2) - (1-\lambda)\frac{\mathbb{E}[u(a_2)]}{\mathbb{E}[u(a_1)]},$$

and, since $\lambda NEV(a_2) \geq 0$ and $0 \leq \frac{\mathbb{E}[u(a_2)]}{\mathbb{E}[u(a_1)]} \leq 1$, it follows that $R(a_1) < R(a_2)$.

(b) From the fact that $\mathbb{E}[u(a_1)] = u(c) = u\left(\sum_{i=1}^{n} x_i p_i\right) = u(\mathbb{E}[a_2])$ and $\mathbb{E}[u(a_1)] < \mathbb{E}[u(a_2)]$, due to Jensen's inequality for risk-seeking utility functions, it follows that

$$R(a_1) = -(1-\lambda)\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]}$$

$$R(a_2) = \lambda NEV(a_2) - (1-\lambda).$$

Now,

$$R(a_1) < R(a_2) \Leftrightarrow \lambda > \frac{1 - \frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]}}{1 + NEV(a_2) - \frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]}}$$

and since both the numerator and the denominator are positive and $NEV(a_2) > 0$, then the term $\frac{1-\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]}}{1+NEV(a_2)-\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]}}$ lies in the interval $[0, 1]$. Therefore, there exist intervals for λ contained in $[0, 1]$ given by

$$\frac{1 - \frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]}}{1 + NEV(a_2) - \frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]}} < \lambda \leq 1,$$

where the inequality $R(a_1) < R(a_2)$ holds. The lower limit for λ decreases and tends to 0, when $NEV(a_2)$ increases. □

Thus, by (a) of Theorem 1, with the NEU-EV model, given two actions with equal expected values, where one action has a certain outcome, the action with the certain outcome is always preferred for individuals with risk-averse and risk-neutral utility functions. Considering individuals with risk-seeking utilities, then by (b) of Theorem 1, the preference of the certain option is also expressed through the model for particular intervals of the tradeoff factor, although there is also the possibility for these individuals to prefer the uncertain option. However, this possibility decreases with increasing entropy and/or variance, as it can be seen from the lower limit of the tradeoff factor. Dropping the assumption of equal expected values $\mathbb{E}[a_1] = \mathbb{E}[a_2]$ in Theorem 1, one can state the result in Theorem 2.

Theorem 2. *Consider a decision analysis model $G(\Theta, A, u)$, where $u(x) \geq 0$, $u'(x) \geq 0$ and the outcomes are non-negative, and the NEU-EV risk measure*

defined in (4). Let $A = \{a_1, a_2\}$*, where the action* a_1 *has a certain outcome* c*,* $a_1 = (c, 1)$ *and the action* a_2 *has the distribution* $a_2 = (x_1, p_1; \ldots; x_n, p_n)$*, where* $\sum_{i=1}^{n} p_i = 1$ *and* $p_i < 1$ *for at least one* i*, where* $i = 1, \ldots, n$*. Then:*

(a) If $\mathbb{E}[u(a_1)] \geq \mathbb{E}[u(a_2)]$*, then* $R(a_1) < R(a_2)$*.*

(b) If $\mathbb{E}[u(a_1)] < \mathbb{E}[u(a_2)]$*, then* $R(a_1) < R(a_2)$ *for* $\frac{1-\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]}}{1+NEV(a_2)-\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]}} < \lambda \leq 1$*.*

PROOF. (a) Considering the case $\mathbb{E}[u(a_1)] = \mathbb{E}[u(a_2)]$, the risk measures for both actions read

$$R(a_1) = -(1 - \lambda), \tag{7}$$

$$R(a_2) = \lambda NEV(a_2) - (1 - \lambda), \tag{8}$$

and therefore $R(a_1) < R(a_2)$. In the case $\mathbb{E}[u(a_1)] > \mathbb{E}[u(a_2)]$, one has (7) and

$$R(a_2) = \lambda NEV(a_2) - (1 - \lambda)\frac{\mathbb{E}[u(a_2)]}{\mathbb{E}[u(a_1)]},$$

yielding $R(a_1) < R(a_2)$.

(b) In the case $\mathbb{E}[u(a_1)] < \mathbb{E}[u(a_2)]$, one obtains

$$R(a_1) = -(1 - \lambda)\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]}$$

and (8). Analysing the inequality $R(a_1) < R(a_2)$, one concludes that

$$R(a_1) < R(a_2) \Leftrightarrow \lambda > \frac{1 - \frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]}}{1 + NEV(a_2) - \frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]}}$$

and, since both the numerator and the denominator are positive and $NEV(a_2) > 0$, then the term $\frac{1-\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]}}{1+NEV(a_2)-\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]}}$ lies in the interval $[0, 1]$.

Therefore, there exist intervals for λ contained in $[0, 1]$ given by

$$\frac{1 - \frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]}}{1 + NEV(a_2) - \frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]}} < \lambda \leq 1,$$

where $R(a_1) < R(a_2)$ holds. The lower limit for λ decreases and tends to 0, when $NEV(a_2)$ increases. □

The results of the last theorem show that also in more general situations, where both actions can have different expected values, even if the expected value or expected utility of a_1 is lower than that of a_2 ($\mathbb{E}[a_1] < \mathbb{E}[a_2]$ corresponds to case (b) in Theorem 2 for $u(x) = x$), the NEU-EV model predicts and permits the modelling of the preference of the action with certain outcome over the action with uncertain outcome. Note that the results in Theorem 1 depend on the knowledge about the second derivative of the utility function, since the restriction of equal expected values $\mathbb{E}[a_1] = \mathbb{E}[a_2]$ allows then using Jensen's inequality to derive the results. However, in Theorem 2, when dropping the hypothesis, it is not possible to apply Jensen's inequality and therefore one must consider the three cases $\mathbb{E}[u(a_1)] > \mathbb{E}[u(a_2)]$, $\mathbb{E}[u(a_1)] = \mathbb{E}[u(a_2)]$ and $\mathbb{E}[u(a_1)] < \mathbb{E}[u(a_2)]$ in order to obtain the results.

The following two properties in Theorem 3 have an interpretation in the context of risk perception in lotteries. The first property states that perceived risk decreases if a constant positive amount is added to all outcomes of a lottery (see [5], [14]), and the second property, that perceived risk decreases if in a lottery with two outcomes the probability of the maximum outcome increases. These properties give evidence for the perception that greater gain and greater gain probabilities reduce the risk (see [7], [3]).

Given a lottery $a = (x_1, p_1; \ldots; x_n, p_n)$, the notation $a + k$, where k is a constant, will be used to denote the lottery, where k is added to all outcomes of a: $a+k = (x_1+k, p_1; \ldots; x_n+k, p_n)$. The notation ka will be used to denote the lottery, where all outcomes of a are multiplied by k: $ka = (kx_1, p_1; \ldots; kx_n, p_n)$.

Theorem 3. *Let $G(\Theta, A, u)$ be a decision analysis model with non-negative outcomes and increasing utility function u. Consider the NEU-EV risk measure defined in (4).*

(a) Let $A = \{a, a + k\}$, where k is a positive constant. Then, one has

$$R(a + k) < R(a). \tag{9}$$

(b) Let $A = \{a_1, a_2\}$, where $a_1 = (x, p; y, 1 - p)$ and $a_2 = (x, q; y, 1 - q)$, with $x > y$ and $0 < q < p < 1$. Then:

(i) If $q < p < \frac{1}{2}$, *then* $R(a_1) < R(a_2)$ *for* $0 \leq \lambda < \frac{1-\frac{\mathbb{E}[u(a_2)]}{\mathbb{E}[u(a_1)]}}{1-\frac{\mathbb{E}[u(a_2)]}{\mathbb{E}[u(a_1)]}+NEV(a_1)-NEV(a_2)}$.

(ii) If $\frac{1}{2} < q < p$, *then* $R(a_1) < R(a_2)$.

PROOF. (a) Since $NH(a) = NH(a+k)$, $\mathrm{Var[a]} = \mathrm{Var[a+k]}$, due to

$$\mathrm{Var[a+k]} = \sum_{\mathrm{i=1}}^{\mathrm{n}} (\mathrm{x_i}+\mathrm{k})^2 \mathrm{p_i} - \left[\sum_{\mathrm{i=1}}^{\mathrm{n}} (\mathrm{x_i}+\mathrm{k})\mathrm{p_i}\right]^2 = \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{x_i^2 p_i} - \left(\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{x_i p_i}\right)^2 = \mathrm{Var[a]},$$

which imply $NEV(a) = NEV(a+k)$, and $\mathbb{E}[u(a)] < \mathbb{E}[u(a+k)]$, one has

$$R(a) = \lambda NEV(a) - (1-\lambda)\frac{\mathbb{E}[u(a)]}{\mathbb{E}[u(a+k)]},$$
$$R(a+k) = \lambda NEV(a) - (1-\lambda).$$

Comparing both expressions, one concludes that $R(a+k) < R(a)$ (cf. Proposition 1 (b)).

(b) Calculating the expected utilities, the normalized entropies and the variances for a_1 and a_2 gives

$$\mathbb{E}[u(a_1)] = p[u(x)-u(y)] + u(y), \mathbb{E}[u(a_2)] = q[u(x)-u(y)] + u(y),$$
$$NH(a_1) = \frac{1}{\ln 2}[-p\ln p - (1-p)\ln(1-p)],$$
$$NH(a_2) = \frac{1}{\ln 2}[-q\ln q - (1-q)\ln(1-q)],$$
$$\mathrm{Var[a_1]} = \mathrm{p(1-p)(x-y)^2}, \mathrm{Var[a_2]} = \mathrm{q(1-q)(x-y)^2}.$$

(i) Let $q < p < \frac{1}{2}$. In this case, one has $NH(a_1) > NH(a_2)$, $\mathrm{Var[a_1]} > \mathrm{Var[a_2]}$, $\mathbb{E}[u(a_1)] > \mathbb{E}[u(a_2)]$ and

$$R(a_1) = \lambda NEV(a_1) - (1-\lambda),$$
$$R(a_2) = \lambda NEV(a_2) - (1-\lambda)\frac{\mathbb{E}[u(a_2)]}{\mathbb{E}[u(a_1)]}.$$

Analysing the inequality $R(a_1) < R(a_2)$, one concludes that

$$R(a_1) < R(a_2) \Leftrightarrow \lambda < \frac{1-\frac{\mathbb{E}[u(a_2)]}{\mathbb{E}[u(a_1)]}}{1-\frac{\mathbb{E}[u(a_2)]}{\mathbb{E}[u(a_1)]}+NEV(a_1)-NEV(a_2)}$$

and, since the numerator is positive and $NEV(a_1) - NEV(a_2) > 0$, then the upper limit for λ lies in the interval $[0, 1]$. Therefore, there exist intervals for λ contained in $[0, 1]$ given by

$$0 \leq \lambda < \frac{1 - \frac{\mathbb{E}[u(a_2)]}{\mathbb{E}[u(a_1)]}}{1 - \frac{\mathbb{E}[u(a_2)]}{\mathbb{E}[u(a_1)]} + NEV(a_1) - NEV(a_2)},$$

where $R(a_1) < R(a_2)$ holds.

(ii) Let $\frac{1}{2} < q < p$. In this case, one has $NH(a_1) < NH(a_2)$, $\mathrm{Var}[\mathrm{a}_1] < \mathrm{Var}[\mathrm{a}_2]$, so that $NEV(a_1) < NEV(a_2)$, and $\mathbb{E}[u(a_1)] > \mathbb{E}[u(a_2)]$. Then, it follows from Proposition 1 (a) that $R(a_1) < R(a_2)$.

□

The following result in Theorem 4 (a) indicates that a change in scale does not affect the risk ordering if the utility function satisfies a compatibility condition. The result in Theorem 4 (b) states that if a scale has a lower risk for a given action than another scale, then the same risk order will hold for any other action for the given scales if the utility function satisfies a compatibility condition. These results are compatible with an independence condition required for the decision-maker's risk ordering presented in [17].

Theorem 4. *Let $G(\Theta, A, u)$ be a decision analysis model with non-negative outcomes and increasing utility function u. Consider the NEU-EV risk measure defined in (4). Let $A = \{a_1, a_2\}$ and let k, l be two different positive constants. Then:*

(a) $R(a_1) < R(a_2) \Leftrightarrow R(ka_1) < R(ka_2)$, *whenever* $\dfrac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]} = \dfrac{\mathbb{E}[u(ka_1)]}{\mathbb{E}[u(ka_2)]}$.

(b) $R(ka_1) < R(la_1) \Leftrightarrow R(ka_2) < R(la_2)$, *whenever* $\dfrac{\mathbb{E}[u(la_1)]}{\mathbb{E}[u(la_2)]} = \dfrac{\mathbb{E}[u(ka_1)]}{\mathbb{E}[u(ka_2)]}$.

Proof. (a) ($\Rightarrow$): Suppose that $R(a_1) < R(a_2)$. Since

$$R(a_1) = \lambda NEV(a_1) - (1 - \lambda)\frac{\mathbb{E}[u(a_1)]}{\max\left\{\mathbb{E}[u(a_1)], \mathbb{E}[u(a_2)]\right\}}$$

$$R(a_2) = \lambda NEV(a_2) - (1 - \lambda)\frac{\mathbb{E}[u(a_2)]}{\max\left\{\mathbb{E}[u(a_1)], \mathbb{E}[u(a_2)]\right\}},$$

then

$$R(a_1) < R(a_2) \Leftrightarrow \lambda[NEV(a_1) - NEV(a_2)] < (1-\lambda)\frac{\mathbb{E}[u(a_1)] - \mathbb{E}[u(a_2)]}{\max\left\{\mathbb{E}[u(a_1)], \mathbb{E}[u(a_2)]\right\}}. \tag{10}$$

Due to the fact that $NH(ka_i) = NH(a_i)$ and $\mathrm{Var[ka_i]} = \mathrm{k^2 Var[a_i]}$ for $i = 1, 2$, which implies

$$\frac{\mathrm{Var[ka_i]}}{\max\{\mathrm{Var[ka_1], Var[ka_2]}\}} = \frac{\mathrm{Var[a_i]}}{\max\{\mathrm{Var[a_1], Var[a_2]}\}},$$

one obtains that $R(ka_1) < R(ka_2)$ is equivalent to the condition

$$\lambda[NEV(a_1) - NEV(a_2)] < (1-\lambda)\frac{\mathbb{E}[u(ka_1)] - \mathbb{E}[u(ka_2)]}{\max\left\{\mathbb{E}[u(ka_1)], \mathbb{E}[u(ka_2)]\right\}},$$

which is implied by (10) whenever

$$\frac{\mathbb{E}[u(a_1)] - \mathbb{E}[u(a_2)]}{\max\left\{\mathbb{E}[u(a_1)], \mathbb{E}[u(a_2)]\right\}} \leq \frac{\mathbb{E}[u(ka_1)] - \mathbb{E}[u(ka_2)]}{\max\left\{\mathbb{E}[u(ka_1)], \mathbb{E}[u(ka_2)]\right\}}. \tag{11}$$

It turns out that for both cases $\mathbb{E}[u(a_1)] > \mathbb{E}[u(a_2)] \Rightarrow \mathbb{E}[u(ka_1)] > \mathbb{E}[u(ka_2)]$ and $\mathbb{E}[u(a_1)] < \mathbb{E}[u(a_2)] \Rightarrow \mathbb{E}[u(ka_1)] < \mathbb{E}[u(ka_2)]$, condition (11) reduces to

$$\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]} \leq \frac{\mathbb{E}[u(ka_1)]}{\mathbb{E}[u(ka_2)]}. \tag{12}$$

($\Leftarrow$): Now, suppose that $R(ka_1) < R(ka_2)$. In this case, given that

$$\lambda[NEV(a_1) - NEV(a_2)] < (1-\lambda)\frac{\mathbb{E}[u(ka_1)] - \mathbb{E}[u(ka_2)]}{\max\left\{\mathbb{E}[u(ka_1)], \mathbb{E}[u(ka_2)]\right\}}$$

and we want to prove that $R(a_1) < R(a_2)$, which corresponds to the inequality given in (10), it is clear that this condition holds whenever

$$\frac{\mathbb{E}[u(ka_1)] - \mathbb{E}[u(ka_2)]}{\max\left\{\mathbb{E}[u(ka_1)], \mathbb{E}[u(ka_2)]\right\}} \leq \frac{\mathbb{E}[u(a_1)] - \mathbb{E}[u(a_2)]}{\max\left\{\mathbb{E}[u(a_1)], \mathbb{E}[u(a_2)]\right\}}, \tag{13}$$

yielding

$$\frac{\mathbb{E}[u(ka_1)]}{\mathbb{E}[u(ka_2)]} \leq \frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]} \tag{14}$$

for both cases $\mathbb{E}[u(a_1)] > \mathbb{E}[u(a_2)] \Rightarrow \mathbb{E}[u(ka_1)] > \mathbb{E}[u(ka_2)]$ and $\mathbb{E}[u(a_1)] < \mathbb{E}[u(a_2)] \Rightarrow \mathbb{E}[u(ka_1)] < \mathbb{E}[u(ka_2)]$.

Then, one can conclude from (12) and (14) that $R(a_1) < R(a_2) \Leftrightarrow R(ka_1) < R(ka_2)$ whenever

$$\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]} = \frac{\mathbb{E}[u(ka_1)]}{\mathbb{E}[u(ka_2)]}.$$

(b) ($\Rightarrow$): Suppose that $R(ka_1) < R(la_1)$ and consider first the case $k > l$. Then, since $NH(ka_1) = NH(la_1)$, $\mathrm{Var[ka_1]} > \mathrm{Var[la_1]}$ and $\mathbb{E}[u(ka_1)] > \mathbb{E}[u(la_1)]$, it follows that

$$R(ka_1) < R(la_1) \Leftrightarrow \lambda\left(1 - \frac{l^2}{k^2}\right) < (1-\lambda)\left(1 - \frac{\mathbb{E}[u(la_1)]}{\mathbb{E}[u(ka_1)]}\right). \tag{15}$$

On the other hand, $R(ka_2) < R(la_2)$ is equivalent to

$$\lambda\left(1 - \frac{l^2}{k^2}\right) < (1-\lambda)\left(1 - \frac{\mathbb{E}[u(la_2)]}{\mathbb{E}[u(ka_2)]}\right), \tag{16}$$

which is implied by (15) if

$$\frac{\mathbb{E}[u(la_2)]}{\mathbb{E}[u(ka_2)]} \leq \frac{\mathbb{E}[u(la_1)]}{\mathbb{E}[u(ka_1)]}. \tag{17}$$

Considering the case $k < l$, one can check in an analogous way that also in this case $R(ka_1) < R(la_1) \Rightarrow R(ka_2) < R(la_2)$ if (17) holds.

($\Leftarrow$): In order to prove that $R(ka_2) < R(la_2) \Rightarrow R(ka_1) < R(la_1)$, one just can interchange a_1 and a_2 in the previous proof ($\Rightarrow$). The corresponding inequality for the implication to be satisfied in this case reads

$$\frac{\mathbb{E}[u(la_1)]}{\mathbb{E}[u(ka_1)]} \leq \frac{\mathbb{E}[u(la_2)]}{\mathbb{E}[u(ka_2)]}. \tag{18}$$

It follows from (17) and (18) that $R(ka_1) < R(la_1) \Leftrightarrow R(ka_2) < R(la_2)$ whenever

$$\frac{\mathbb{E}[u(la_1)]}{\mathbb{E}[u(ka_1)]} = \frac{\mathbb{E}[u(la_2)]}{\mathbb{E}[u(ka_2)]}.$$

□

Remark 1. *The conditions* $\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]} = \frac{\mathbb{E}[u(ka_1)]}{\mathbb{E}[u(ka_2)]}$ *and* $\frac{\mathbb{E}[u(la_1)]}{\mathbb{E}[u(la_2)]} = \frac{\mathbb{E}[u(ka_1)]}{\mathbb{E}[u(ka_2)]}$ *in Theorem 4 are satisfied e.g. by the utility functions* $u(x) = x$, $u(x) = x^2$ *and* $u(x) = \sqrt{x}$ *as one can easily verify.*

Lemma 1 shows that with the NEU-EV risk measure it is possible to describe the property that says that the riskiness of a gamble increases, if the scale value increases. This property was considered e.g. in [17].

Lemma 1. *Let $G(\Theta, A, u)$ be a decision analysis model with non-negative outcomes and increasing utility function u. Consider the NEU-EV risk measure defined in (4), $a \in A$ and let k, l be two different positive constants. If $k > l$, then $R(ka) > R(la)$ for $\frac{1-\frac{\mathbb{E}[u(la)]}{\mathbb{E}[u(ka)]}}{1-\frac{\mathbb{E}[u(la)]}{\mathbb{E}[u(ka)]}+\frac{1}{2}\left(1-\frac{l^2}{k^2}\right)} < \lambda \leq 1$.*

Proof. Let $k > l$. Then, since $NH(ka) = NH(la) = NH(a)$, Var[ka] = k^2Var[a] > l^2Var[a] = Var[la] and $\mathbb{E}[u(ka)] > \mathbb{E}[u(la)]$, one has

$$R(ka) > R(la) \Leftrightarrow \lambda > \frac{1-\frac{\mathbb{E}[u(la)]}{\mathbb{E}[u(ka)]}}{1-\frac{\mathbb{E}[u(la)]}{\mathbb{E}[u(ka)]}+\frac{1}{2}\left(1-\frac{l^2}{k^2}\right)}.$$

Analysing the term on the right hand side of the last inequality, since both, the numerator and the denominator are positive, and since the denominator is greater than the numerator, one concludes that there exist intervals for λ contained in $[0, 1]$, given by $\frac{1-\frac{\mathbb{E}[u(la)]}{\mathbb{E}[u(ka)]}}{1-\frac{\mathbb{E}[u(la)]}{\mathbb{E}[u(ka)]}+\frac{1}{2}\left(1-\frac{l^2}{k^2}\right)} < \lambda \leq 1$, where $k > l \Rightarrow R(ka) > R(la)$. □

Consider now the property of first order stochastic dominance, where given two actions a_1 and a_2, the action a_1 first order stochastically dominates a_2 if for all outcomes x_i, $i = 1, \ldots, n$: $\mathbb{P}(a_1 \geq x_i) \geq \mathbb{P}(a_2 \geq x_i)$, and for one outcome x_k, $k \in \{1, \ldots, n\}$: $\mathbb{P}(a_1 \geq x_k) > \mathbb{P}(a_2 \geq x_k)$. Although, in general, first order stochastic dominance can not directly be related with the ordering in terms of the NEU-EV risk, since for example the entropy only depends on the outcomes' probabilities, one can establish a relation between first order stochastic dominance and the NEU-EV risk in the following particular cases.

Proposition 2. *Let $G(\Theta, A, u)$ be a decision analysis model with non-negative outcomes and increasing utility function u and $A = \{a_1, a_2\}$. Consider the NEU-EV risk measure defined in (5).*

(a) If $\lambda = 0$, then a_1 first order stochastically dominates a_2 if and only if $R(a_1) \leq R(a_2)$.

(b) If $NEV(a_1) \leq NEV(a_2)$, then a_1 first order stochastically dominates a_2 if and only if $R(a_1) \leq R(a_2)$.

PROOF. The results follow from the definition of the NEU-EV risk and from the known result that a_1 first order stochastically dominates a_2 if and only if $\mathbb{E}[u(a_1)] \geq \mathbb{E}[u(a_2)]$ for increasing utility functions. □

In prospect theory [13], a decision maker, when confronted with various prospects, chooses the prospect with highest value, where the value of the prospect depends on decision weights, instead of probabilities, and on values assigned to outcomes. In order to compare the NEU-EV model with prospect theory, first a regular prospect with non-negative outcomes and then a stricly positive prospect will be considered.

A regular prospect can be written as $a_1 = (x, p; y, q; 0, 1-p-q)$, or simply $a_1 = (x, p; y, q)$, with $p+q < 1$, $x \geq 0 \geq y$ or $y \geq 0 \geq x$, whose value is given by

$$V(a_1) = \pi(p)v(x) + \pi(q)v(y),$$

where v denotes the value function and π the weighting function and $v(0) = 0$, $\pi(0) = 0$, $\pi(1) = 1$. Here we will assume that $x \geq 0$ and $y = 0$ (for non-negative outcomes), so that $a_1 = (x, p; 0, 1-p)$ and

$$V(a_1) = \pi(p)v(x). \tag{19}$$

Calculating the NEU-EV risk for a_1, assuming that $u(0) = 0$ and $u(x) > 0$ for $x > 0$, then $R(a_1)$ can be expressed as

$$R(a_1) = \pi_0(p) + \pi_1(p)v_1(x) + \pi_2(p)v_2(x), \tag{20}$$

with

$$\pi_0(p) = c_0[-p \ln p - (1-p)\ln(1-p)],$$

$$\pi_1(p)v_1(x) = \begin{cases} c_{11}, & \text{if} \quad \max\limits_{a \in A}\{\mathrm{Var[a]}\} = \mathrm{Var[a_1]} \\ c_{12}\ (p - p^2)\ x^2, & \text{if} \quad \max\limits_{a \in A}\{\mathrm{Var[a]}\} \neq \mathrm{Var[a_1]}, \end{cases}$$

$$\pi_2(p)v_2(x) = \begin{cases} -c_{21}, & \text{if} \quad \max\limits_{a \in A}\big\{\mathbb{E}[u(a)]\big\} = \mathbb{E}[u(a_1)] \\ -c_{22}\ p\ u(x), & \text{if} \quad \max\limits_{a \in A}\big\{\mathbb{E}[u(a)]\big\} \neq \mathbb{E}[u(a_1)], \end{cases}$$

where c_0, c_{11}, c_{12}, c_{21}, c_{22} are positive constants, depending on $\lambda \in [0,1]$. In prospect theory a decision maker chooses a prospect with maximum value. Using the NEU-EV model, the decision maker chooses the prospect with minimum risk and since maximizing value corresponds to minimizing risk, one can compare $V(a_1)$ with $-R(a_1)$, so that $-R(a_1)$ would then correspond to the value function, and the terms $-\pi_0(p)$, $-\pi_1(p)v_1(x)$ and $-\pi_2(p)v_2(x)$ should be increasing, which happens for $p \geq \frac{1}{2}$ and $u'(x) \geq 0$. An analysis of $V(a_1)$ in (19) and $-R(a_1)$ reveals that the NEU-EV model is not a particular model of prospect theory, because it includes an entropy term representing an additional weighting function and it consists of a sum of two products of different weighting and value functions. Additional conditions should be imposed for the value and weighting functions to be increasing ($p \geq \frac{1}{2}$ and $u'(x) \geq 0$), in order to have a similar form of value as defined in prospect theory.

Now, consider a strictly positive prospect $a_2 = (x, p; y, q)$, where $p + q = 1$ and $x > y > 0$, whose value is given by (see [13])

$$V(a_2) = v(y) + \pi(p)[v(x) - v(y)]. \tag{21}$$

Calculating the NEU-EV risk for a_2, assuming that $u(0) = 0$ and $u(x) > 0$ for $x > 0$, then $R(a_2)$ can be expressed as

$$R(a_2) = \pi_0(p) + v_1(y) + \pi_1(p)[v_1(x) - v_1(y)] + \pi_2(p)v_2(x, y), \tag{22}$$

with

$$\pi_0(p) = c_0[-p \ln p - (1-p)\ln(1-p)],$$

$$v_1(y) + \pi_1(p)[v_1(x) - v_1(y)] = \begin{cases} -c_{11}, & \text{if} \quad \max\limits_{a \in A}\{\mathbb{E}[u(a)]\} = \mathbb{E}[u(a_2)] \\ -c_{12}\, u(y) - c_{12}\, p[u(x) - u(y)], & \text{if} \quad \max\limits_{a \in A}\{\mathbb{E}[u(a)]\} \neq \mathbb{E}[u(a_2)], \end{cases}$$

$$\pi_2(p)v_2(x, y) = \begin{cases} c_{21}, & \text{if} \quad \max\limits_{a \in A}\{\mathrm{Var[a]}\} = \mathrm{Var[a_2]} \\ c_{22}\, (p - p^2)\, (x - y)^2, & \text{if} \quad \max\limits_{a \in A}\{\mathrm{Var[a]}\} \neq \mathrm{Var[a_2]}, \end{cases}$$

where c_0, c_{11}, c_{12}, c_{21}, c_{22} are positive constants, depending on $\lambda \in [0,1]$. Comparing $-R(a_2)$ with $V(a_2)$ in (21), one concludes that also in this case the

NEU-EV model is not a particular model of prospect theory. Although the term $-\{v_1(y)+\pi_1(p)[v_1(x)-v_1(y)]\}$ is similar to $V(a_2)$, the NEU-EV model includes the entropy term in $\pi_0(p)$, representing an additional weighting function, and it contains an additional product $\pi_2(p)v_2(x,y)$, which can not be written in a separated form as sum of two products, one depending on the value function of x and the other depending on the value function of y, due to the variance term in the NEU-EV model. Also for this prospect, the property of increasing value would hold if $p \geq \frac{1}{2}$ and $u'(\cdot) \geq 0$.

The results indicate that the NEU-EV model deviates from being a model of prospect theory, although it can be expressed similarly and even containing terms that can be considered products of value and weighting functions (for example in the regular prospect).

Comparing the NEU-EV model with the axiomatized model by Pollatsek and Tversky [24], where risk is expressed as linear combination of mean and variance, one concludes that the NEU-EV model, which could also be considered as a generalization of that model, due to the inclusion of entropy and using expected utility, is not consistent with the axioms motivating that model. For example the cancellation axiom is in general not valid with the NEU-EV model, since one can find counterexamples where the axiom fails for certain ranges of λ.

4. An application of the normalized expected utility-entropy and variance measure of risk

In this section, the NEU-EV model is applied to an example considered and analysed by Yang and Qiu [31]. The example is interesting since it shows, on the one hand, the appropriateness of the normalized EU-E or NEU-EV model over the non-normalized ones and, on the other hand, it shows the convenience of using also the variance as additional factor in the decision model. The definitions of the here used models are listed in Appendix A.

Example

The example presented in [31] consists of two risky actions a_1 and a_2, where

a_1 has six return state values θ_{1j}, $j = 1, \ldots, 6$, with equal probabilities $p_{1j} = \frac{1}{6}$, $j = 1, \ldots, 6$, and a_2 has two return state values θ_{2j}, $j = 1, 2$, with equal probabilities $p_{2j} = 0.5$, $j = 1, 2$ (see Table 1).

Table 1: Outcomes of a_1 and a_2.

		θ_{i1}	θ_{i2}	θ_{i3}	θ_{i4}	θ_{i5}	θ_{i6}	$\mathbb{E}[a]$	$\mathbb{E}[a^2]$	$\mathbb{E}[\sqrt{a}]$	Var[a]	$H(a)$	$NH(a)$
	x_{1j}	9	9.6	9.8	10.2	10.4	11						
a_1	p_{1j}	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	10	100.4	3.161	0.4	1.79	1
	x_{2j}	9	11										
a_2	p_{2j}	0.5	0.5					10	101	3.158	1	0.69	1

One can observe that a_1 and a_2 have the same mean, which is the expected utility for the risk neutral utility function $u(x) = x$: $\mathbb{E}[a_1] = \mathbb{E}[a_2] = 10$. For the risk-seeking utility function $u(x) = x^2$, the expected utility, denoted in Table 1 by $\mathbb{E}[a^2]$, is higher for a_2: $\mathbb{E}[a_1^2] = 100.4 < \mathbb{E}[a_2^2] = 101$. For the risk-averse utility function $u(x) = \sqrt{x}$ the inequality is inverted and the expected utility, denoted by $\mathbb{E}[\sqrt{a}]$, of a_1 is higher: $\mathbb{E}[\sqrt{a_1}] = 3.161 > \mathbb{E}[\sqrt{a_2}] = 3.158$.

Regarding the entropy measures, the normalized entropies coincide, $NH(a_1) = NH(a_2) = 1$, as expected, since both in a_1 and a_2 the probabilities for all outcomes are $p_{ij} = \frac{1}{n_i}$, where n_i is the number of states in a_i, namely $n_1 = 6$ and $n_2 = 2$. However, the non-normalized entropies differ: $H(a_1) = 1.79 > H(a_2) = 0.69$. Concerning the variances, a_2 has a higher variance: $\text{Var}[\text{a}_1] = 0.4 < \text{Var}[\text{a}_2] = 1$.

In the following analysis the different decisions obtained with the EU-E and EU-EV models and with the corresponding normalized models (NEU-E and NEU-EV models) will be investigated. In particular, the influence of the type of utility function, of the variance and of the normalized entropy on the decisions will be analysed.

Table 2 contains the EU-E and EU-EV risk measures for a_1 and a_2, where a_1 and a_2 have the values assumed in Table 1, considering the risk-neutral utility function $u(x) = x$, the risk-seeking utility function $u(x) = x^2$ and the risk-averse

utility function $u(x) = \sqrt{x}$.

Table 2: Risk measures EU-E and EU-EV for a_1 and a_2.

Utility function	$R(a_i)$	EU-E	EU-EV
$u(x) = x$	$R(a_1)$	$2.79\lambda - 1$	$2.095\lambda - 1$
	$R(a_2)$	$1.79\lambda - 1$	$1.845\lambda - 1$
$u(x) = x^2$	$R(a_1)$	$2.784\lambda - 0.994$	$2.089\lambda - 0.994$
	$R(a_2)$	$1.69\lambda - 1$	$1.845\lambda - 1$
$u(x) = \sqrt{x}$	$R(a_1)$	$2.79\lambda - 1$	$2.095\lambda - 1$
	$R(a_2)$	$1.689\lambda - 0.999$	$1.844\lambda - 0.999$

Table 3 contains intervals for the tradeoff factors corresponding to the decisions $R(a_1) < R(a_2)$, $R(a_1) = R(a_2)$ and $R(a_2) < R(a_1)$ for the risk-measures EU-E and EU-EV considering the three different utility functions. The dash in the tables (e.g. in Table 3 and Table 5) means that no value of $\lambda \in [0, 1]$ gives the stated relation.

Table 3: Tradeoff factors for decisions with EU-E and EU-EV.

Utility function	Risk decision	EU-E	EU-EV
$u(x) = x$	$R(a_1) < R(a_2)$	—	—
	$R(a_1) = R(a_2)$	—	—
	$R(a_1) > R(a_2)$	$0 \leq \lambda \leq 1$	$0 \leq \lambda \leq 1$
$u(x) = x^2$	$R(a_1) < R(a_2)$	—	—
	$R(a_1) = R(a_2)$	—	—
	$R(a_1) > R(a_2)$	$0 \leq \lambda \leq 1$	$0 \leq \lambda \leq 1$
$u(x) = \sqrt{x}$	$R(a_1) < R(a_2)$	$0 \leq \lambda < 0.001$	$0 \leq \lambda < 0.003$
	$R(a_1) = R(a_2)$	$\lambda = 0.001$	$\lambda = 0.003$
	$R(a_1) > R(a_2)$	$0.001 < \lambda \leq 1$	$0.003 < \lambda \leq 1$

Analysing these results, one concludes that with the risk-neutral and the risk-seeking utility functions, one chooses always a_2 with the EU-E and EU-EV decision models. Action a_1 is only chosen with a risk-averse utility func-

tion and with an almost vanishing tradeoff factor, that is, almost ignoring the risk factors entropy and variance. However, as mentioned in [31], the EU-E representation may not provide the proper description for risky choices, since people may perceive a same relative uncertainty, regarding entropy, and choose a_1. In this case, also the EU-EV decision model is not an appropriate one. The people's preference of a_1 over a_2 can perhaps also be explained due to the fact that, since in both actions there is a chance to receive an outcome that is greater than or equal to 9 and that the probability in a_1 to yield a value greater than 9 is higher than the corresponding probability in a_2, that is $\mathbb{P}(a_1 > 9) = \frac{5}{6} > \mathbb{P}(a_2 > 9) = \frac{3}{6} = 0.5$ (here also the simplified notation is used, where a_1 and a_2 represent the corresponding outcome variables $X(a_1, \theta)$ and $X(a_2, \theta)$, respectively). This is related with the expected utility, where the choice of the utility function may have a decisive role. Also the people's perception of variance may play an important role in the decisions they take. In fact, a_1 has a less variance than a_2.

Now, this example will be analysed using the normalized decision models, NEU-E and NEU-EV.

Table 4 contains the NEU-E and NEU-EV risk measures for a_1 and a_2 considering the risk-neutral utility function $u(x) = x$, the risk-seeking utility function $u(x) = x^2$ and the risk-averse utility function $u(x) = \sqrt{x}$.

Table 4: NEU-E and NEU-EV risk measures for a_1 and a_2.

Utility function	$R(a_i)$	NEU-E	NEU-EV
$u(x) = x$	$R(a_1)$	$2\lambda - 1$	$1.7\lambda - 1$
	$R(a_2)$	$2\lambda - 1$	$2\lambda - 1$
$u(x) = x^2$	$R(a_1)$	$1.994\lambda - 0.994$	$1.694\lambda - 0.994$
	$R(a_2)$	$2\lambda - 1$	$2\lambda - 1$
$u(x) = \sqrt{x}$	$R(a_1)$	$2\lambda - 1$	$1.7\lambda - 1$
	$R(a_2)$	$1.999\lambda - 0.999$	$1.999\lambda - 0.999$

The intervals for the tradeoff factors corresponding to the decisions $R(a_1) <$

$R(a_2)$, $R(a_1) = R(a_2)$ and $R(a_2) < R(a_1)$ for the NEU-E and NEU-EV risk-measures are presented in Table 5 .

Table 5: Tradeoff factors for decisions with NEU-E and NEU-EV.

Utility function	Risk decision	NEU-E	NEU-EV
$u(x) = x$	$R(a_1) < R(a_2)$	—	$0 < \lambda \leq 1$
	$R(a_1) = R(a_2)$	$0 \leq \lambda \leq 1$	$\lambda = 0$
	$R(a_1) > R(a_2)$	—	—
$u(x) = x^2$	$R(a_1) < R(a_2)$	—	$0.019 < \lambda \leq 1$
	$R(a_1) = R(a_2)$	$\lambda = 1$	$\lambda = 0.019$
	$R(a_1) > R(a_2)$	$0 \leq \lambda < 1$	$0 \leq \lambda < 0.019$
$u(x) = \sqrt{x}$	$R(a_1) < R(a_2)$	$0 \leq \lambda \leq 1$	$0 \leq \lambda \leq 1$
	$R(a_1) = R(a_2)$	—	—
	$R(a_1) > R(a_2)$	—	—

Analysing the results obtained for the normalized models, one can see from Table 5, that the NEU-EV model privileges the choice of a_1 over a_2 for all three utility functions, where this is more evident for the risk-averse utility, where one always chooses a_1. In the risk-neutral case there is the chance to be indifferent between a_1 and a_2 for a vanishing tradeoff factor, where the uncertainty risk factors are ignored. In the risk-seeking case there is a further possibility to prefer a_2 over a_1, however for almost vanishing tradeoff factors, $0 \leq \lambda < 0.019$, which means that one almost ignores again the entropy and variance risk factors. Thus, only with a risk-seeking utility function there is the possibility to choose a_2 instead of a_1. Comparing these results with those obtained from the NEU-E model, one can observe that with this model one only prefers a_1 over a_2 in the risk-averse case. The risk-neutral utility leads to an indifference between a_1 and a_2 and the risk-seeking utility privileges the choice of a_2 over a_1 and leading also to the possibility to be indifferent between a_1 and a_2 if the expected utility is ignored (for $\lambda = 1$).

Comparing the results obtained for the non-normalized models and for the

normalized ones, one observes that the normalized models are in fact more adequate to model the preference of a_1 over a_2. One concludes that in cases where actions have states with different numbers of outcomes, the NEU-E and NEU-EV models should be used. These are more appropriate than the non-normalized ones, since they consider the relative entropy and therefore the relative uncertainty.

One can also conclude that in certain situations, the NEU-EV model may perhaps be more appropriate to model the decisions and the people's perceived risk, since it also takes into account the risk factor variance, which may be decisive for assessing risk, as it can be seen in this example. Note that in the risk-neutral case, a_1 and a_2 have the same expected utilities and the same normalized entropies yielding the indifference in the NEU-E model, however the variances are different, a_1 having a smaller variance than a_2. Thus, the variance penalizes the risk $R(a_2)$ leading to the preference of a_1. This example shows that there exist situations where actions have the same expected utility and the same normalized entropy, however different variances, and therefore it is reasonable and relevant to include also the variance in the decision model. This justifies the suitability of modelling and taking decisions with the NEU-EV risk model.

Another interesting point in this analysis is the importance of the choice of the utility function. In this example, one notices that the utility functions influence the intervals of the tradeoff factors and thus the decisions with both models. Since other classes of utility functions could be used and also parameter-dependent utility functions, this opens the possibility for both models to adapt in a better way the risk-measures to the different decision maker's behaviours. Therefore, in the EU-E and EU-EV models (in the normalized and non-normalized models), not only the choice of the tradeoff factor, but also the choice of the utility function permits flexibly adapting the decision models to the decision maker's preference behaviours.

5. Modelling certainty effects, common ratio effects and common consequence effects using the NEU-EV model

The phenomenon, where people overweight outcomes that are considered certain relative to outcomes that are merely probable, was labelled certainty effect by Kahneman and Tversky [13]. This phenomenon leads to preferences of choices which are inconsistent with the preferences predicted by the expected utility principle. The Allais paradox [2] is one known counter-example to expected utility theory, where the certainty effect becomes evident. In [31] and [4] it has been shown that the Allais paradox can be well described with the EU-E model and EU-EV model, respectively. The common ratio effect and the common consequence effect are other examples of paradoxes which show that people tend to favour risk-free options. Usually the examples include one risk-free prospect (a certain prospect) or one prospect which differs from the others in a very high outcome probability. In the following two subsections, examples of common ratio effect and common consequence effect are considered and analysed using the NEU-EV model. These effects include also an example of certainty effect.

5.1. Common ratio effects

The common ratio effect is a phenomenon, where people typically make decisions violating expected utility theory. It was described by Machina [18] as follows. The common ratio effect involves decisions between two alternatives and consists of two pairs of prospects (a_1, a_2) and (a_3, a_4), which are offered in the following form:

$$\begin{aligned} a_1 &= (x, p; 0, 1-p) \quad \text{versus} \quad a_2 = (y, q; 0, 1-q), \\ a_3 &= (x, rp; 0, 1-rp) \quad \text{versus} \quad a_4 = (y, rq; 0, 1-rq), \end{aligned}$$

where the probabilities p and q are such that $p > q$ and the outcomes satisfy $0 < x < y$ (or reversing the roles of p and q and x and y it would be $q > p$ and $0 < y < x$) and $0 < r < 1$. The term common ratio comes from comparing the probabilities of x and y in both pairs of prospects by a ratio, which is common:

$\frac{p}{q} = \frac{rp}{rq}$. The common ratio effect includes the certainty effect of Kahneman and Tversky [13], by setting $p = 1$, and the Bergen paradox of Hagen [10], by setting $y = 2x$ and $p = 2q$.

The expected utility principle predicts choices of a_1 and a_3 or else a_2 and a_4. However, according to [18], experimental studies showed a different arrangement of preferred choices, namely the preference of a_1 and a_4 over a_2 and a_3, respectively: for example, for tests with $q = 0.45$, $p = 2q$, $y = 2x = 6000$ and $r = \frac{1}{450}$, Kahneman and Tversky [13] found that while 86% of 66 subjects preferred a $0.9 : 0.1$ chance of 3000 or 0 to a $0.45 : 0.55$ chance of 6000 or 0, 73% preferred a $0.001 : 0.999$ chance of 6000 or 0 to a $0.002 : 0.998$ chance of 3000 or 0. Before analysing this particular problem of common ratio effect and Bergen paradox using the NEU-EV model (Problem 1 in this subsection), we will investigate the Bergen paradox in its general form with pairs of prospects (a_1, a_2) and (a_3, a_4), given by:

$$\begin{array}{lll} a_1 = (x, 2q; 0, 1-2q) & \text{versus} & a_2 = (2x, q; 0, 1-q), \\ a_3 = (x, 2rq; 0, 1-2rq) & \text{versus} & a_4 = (2x, rq; 0, 1-rq), \end{array} \tag{23}$$

where $x > 0$ and $0 < r < 1$.

Theorem 5. *Consider the NEU-EV risk measure defined in (4), with increasing utility function $u(x)$, and the Bergen paradox with pairs of prospects (a_1, a_2) and (a_3, a_4) defined in (23).*

(a) Let $u''(x) = 0$. Then, $R(a_1) < R(a_2)$ and $R(a_4) > R(a_3)$.

(b) Let $u''(x) > 0$. Then, $R(a_1) < R(a_2)$ for

$$\frac{\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]} - 1}{\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]} - 1 + NEV(a_1) - NEV(a_2)} < \lambda \leq 1,$$

and $R(a_4) < R(a_3)$ for

$$0 \leq \lambda < \frac{1 - \frac{\mathbb{E}[u(a_3)]}{\mathbb{E}[u(a_4)]}}{1 - \frac{\mathbb{E}[u(a_3)]}{\mathbb{E}[u(a_4)]} + NEV(a_4) - NEV(a_3)}.$$

(c) Let $u''(x) < 0$. Then, $R(a_1) < R(a_2)$ and $R(a_4) > R(a_4)$.

PROOF. Before proving the results separately for the three considered cases, some more general results will be listed. Since $2q < 1$, it follows that $q < \frac{1}{2}$. For a_1, a_2, a_3 and a_4 defined in (23), one has

$$\begin{aligned}
&\mathrm{Var[a_1]} = 2qx^2(1-2q) < \mathrm{Var[a_2]} = 4\mathrm{qx}^2(1-\mathrm{q}),\\
&\mathrm{Var[a_3]} = 2rqx^2(1-2rq) < \mathrm{Var[a_4]} = 4\mathrm{rqx}^2(1-\mathrm{rq}),
\end{aligned}$$

so that

$$\begin{aligned}
NEV(a_1) - NEV(a_2) &= \frac{1}{2\ln 2}\left[-2q\ln(2q) - (1-2q)\ln(1-2q) + q\ln(q) + (1-q)\ln(1-q)\right],\\
&\quad + \frac{1}{4(q-1)}\\
NEV(a_4) - NEV(a_3) &= \frac{1}{2\ln 2}\left[-rq\ln(rq) - (1-rq)\ln(1-rq) + 2rq\ln(2rq) + (1-2rq)\ln(1-2rq)\right]\\
&\quad + \frac{1}{4(1-rq)}.
\end{aligned}$$

Analysing the expressions $NEV(a_1) - NEV(a_2)$ and $NEV(a_4) - NEV(a_3)$ graphically, see Figure 1, it becomes evident that

$$NEV(a_1) - NEV(a_2) < 0, \tag{24}$$

for $0 < q < \frac{1}{2}$, and

$$NEV(a_4) - NEV(a_3) > 0, \tag{25}$$

for $0 < q < \frac{1}{2}$ and $0 < r < 1$.

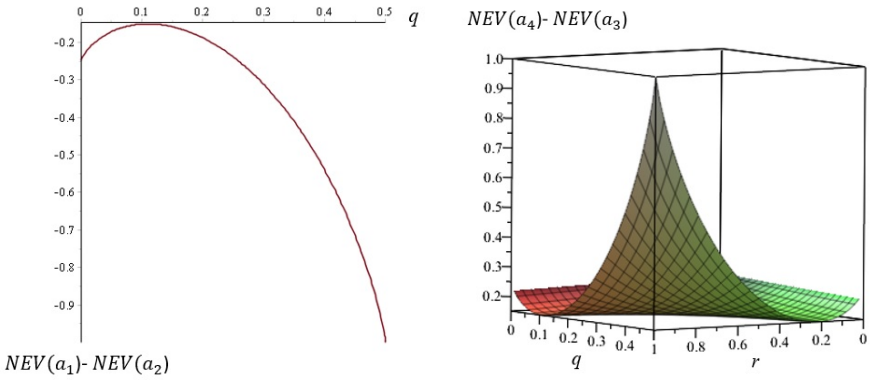


Figure 1: *Left panel*: Graph of $NEV(a_1) - NEV(a_2)$ for $0 < q < \frac{1}{2}$. *Right panel*: Graph of $NEV(a_4) - NEV(a_3)$, for $0 < q < \frac{1}{2}$ and $0 < r < 1$.

In order to analyse the differences

$$\mathbb{E}[u(a_1)] - \mathbb{E}[u(a_2)] = q(2u(x) - u(2x) - u(0)), \tag{26}$$

$$\mathbb{E}[u(a_4)] - \mathbb{E}[u(a_3)] = rq(u(2x) - 2u(x) + u(0)), \tag{27}$$

the result of Lemma 2 will be used.

Lemma 2. *Consider an at least twice differentiable function $u(x)$. Let k be a real constant satisfying $k \geq 1$.*

(i) If $u(x)$ is a convex function, then $ku(x) \leq u(kx) + (k-1)u(0)$.

(ii) If $u(x)$ is a concave function, then $ku(x) \geq u(kx) + (k-1)u(0)$.

PROOF. (i) Consider the constant $\alpha \in [0,1]$. If $u(x)$ is a convex function, then

$$u(\alpha x + (1-\alpha)0) \leq \alpha u(x) + (1-\alpha)u(0).$$

Consider $k \geq 1$. Since $\frac{1}{k} \in (0,1]$, one has

$$u\left(\frac{1}{k}x\right) \leq \frac{1}{k}u(x) + \left(1 - \frac{1}{k}\right)u(0)$$

and substituting x by kx gives

$$u(x) \leq \frac{1}{k}u(kx) + \left(1 - \frac{1}{k}\right)u(0) \Leftrightarrow ku(x) \leq u(kx) + (k-1)u(0). \tag{28}$$

(ii) Consider a concave function $u(x)$. Since $-u(x)$ is convex, the result follows from (i), substituting $u(x)$ by $-u(x)$ in (28).

□

The results of the theorem will now be proved, considering separately the cases where $u(x)$ is linear, convex and concave.

(a) Let $u''(x) = 0$, so that the utility function is linear $u(x) = ax + b$, where a and b are real constants and $a > 0$. From (26) and (27), one gets $\mathbb{E}[u(a_1)] - \mathbb{E}[u(a_2)] = 0$ and $\mathbb{E}[u(a_4)] - \mathbb{E}[u(a_3)] = 0$. In this case, $R(a_1) < R(a_2)$ is equivalent to the condition $\lambda\,[NEV(a_1) - NEV(a_2)] < 0$, which due to (24) is valid for the entire range of $\lambda \in [0, 1]$. The inequality $R(a_4) < R(a_3)$ is equivalent to the condition $\lambda\,[NEV(a_4) - NEV(a_3)] < 0$, which due to (25) is not satisfied for $\lambda \in [0, 1]$. In this case, the inequality $\lambda\,[NEV(a_4) - NEV(a_3)] > 0$ is valid for $\lambda \in [0, 1]$, which means that $R(a_4) > R(a_3)$.

(b) Consider the case where $u''(x) > 0$, so that the utility function is convex. Applying the result (i) of Lemma 2 with $k = 2$ to (26), it turns out that $\mathbb{E}[u(a_1)] - \mathbb{E}[u(a_2)] < 0$. As a consequence, taking also into account (24),

$$R(a_1) < R(a_2) \Leftrightarrow \lambda > \frac{\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]} - 1}{\left[NEV(a_1) - NEV(a_2) + \frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]} - 1\right]},$$

where the term on the right hand side of the last inequality lies in the interval $[0, 1]$. Therefore, one concludes that there exist intervals for λ contained in $[0, 1]$ given by

$$\frac{\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]} - 1}{\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]} - 1 + NEV(a_1) - NEV(a_2)} < \lambda \leq 1,$$

where $R(a_1) < R(a_2)$ is satisfied. Now, analysing $R(a_4) - R(a_3) < 0$, since $\mathbb{E}[u(a_4)] - \mathbb{E}[u(a_3)] > 0$, by Lemma 2 (i) applied to (27), and due to (25), one concludes that

$$R(a_4) < R(a_3) \Leftrightarrow \lambda < \frac{1 - \frac{\mathbb{E}[u(a_3)]}{\mathbb{E}[u(a_4)]}}{\left[NEV(a_4) - NEV(a_3) + 1 - \frac{\mathbb{E}[u(a_3)]}{\mathbb{E}[u(a_4)]}\right]}.$$

The term on the right hand side of the last inequality is positive and less than 1. Therefore, one concludes that there exist intervals for λ contained in $[0,1]$, given by

$$0 \leq \lambda < \frac{1 - \frac{\mathbb{E}[u(a_3)]}{\mathbb{E}[u(a_4)]}}{\left[NEV(a_4) - NEV(a_3) + 1 - \frac{\mathbb{E}[u(a_3)]}{\mathbb{E}[u(a_4)]}\right]},$$

where $R(a_4) < R(a_3)$ holds.

(c) Consider the case $u''(x) < 0$. Then, the result (ii) of Lemma 2 for a concave function with $k = 2$ applied to (26), leads to $\mathbb{E}[u(a_1)] - \mathbb{E}[u(a_2)] > 0$. Taking into account that $NEV(a_1) - NEV(a_2) < 0$, from (24), there are two possibilities for $R(a_1) < R(a_2)$ and its equivalent condition

$$\lambda \left[NEV(a_1) - NEV(a_2) + 1 - \frac{\mathbb{E}[u(a_2)]}{\mathbb{E}[u(a_1)]}\right] < 1 - \frac{\mathbb{E}[u(a_2)]}{\mathbb{E}[u(a_1)]}$$

to hold. If $NEV(a_1) - NEV(a_2) + 1 - \frac{\mathbb{E}[u(a_2)]}{\mathbb{E}[u(a_1)]} < 0$, then, one obtains

$$R(a_1) < R(a_2) \Leftrightarrow \lambda > \frac{1 - \frac{\mathbb{E}[u(a_2)]}{\mathbb{E}[u(a_1)]}}{\left[NEV(a_1) - NEV(a_2) + 1 - \frac{\mathbb{E}[u(a_2)]}{\mathbb{E}[u(a_1)]}\right]},$$

where the term on the right hand side of the last inequality is negative, the numerator being positive and the denominator negative. Therefore, $R(a_1) < R(a_2)$ is satisfied in this case for the entire range of $\lambda \in [0,1]$.
If $NEV(a_1) - NEV(a_2) + 1 - \frac{\mathbb{E}[u(a_2)]}{\mathbb{E}[u(a_1)]} > 0$, then, one obtains

$$R(a_1) < R(a_2) \Leftrightarrow \lambda < \frac{1 - \frac{\mathbb{E}[u(a_2)]}{\mathbb{E}[u(a_1)]}}{\left[NEV(a_1) - NEV(a_2) + 1 - \frac{\mathbb{E}[u(a_2)]}{\mathbb{E}[u(a_1)]}\right]},$$

where the term on the right hand side is greater than 1, the denominator being less than the numerator and both being positive. Therefore, also in this case $R(a_1) < R(a_2)$ holds for $\lambda \in [0,1]$. Consequently, one has $R(a_1) < R(a_2)$ for a concave utility function.
Analysing the condition $R(a_4) < R(a_3)$, since $NEV(a_4) - NEV(a_3) > 0$, from (25), and since $\mathbb{E}[u(a_4)] - \mathbb{E}[u(a_3)] < 0$ by Lemma 2 (ii) for a concave utility function applied to (27), there are also two possibilities for the

following equivalent condition

$$\lambda \left[NEV(a_4) - NEV(a_3) + \frac{\mathbb{E}[u(a_4)]}{\mathbb{E}[u(a_3)]} - 1 \right] < \frac{\mathbb{E}[u(a_4)]}{\mathbb{E}[u(a_3)]} - 1$$

to hold. If $NEV(a_4) - NEV(a_3) + \frac{\mathbb{E}[u(a_4)]}{\mathbb{E}[u(a_3)]} - 1 < 0$, then, one obtains

$$R(a_4) < R(a_3) \Leftrightarrow \lambda > \frac{\frac{\mathbb{E}[u(a_4)]}{\mathbb{E}[u(a_3)]} - 1}{\left[NEV(a_4) - NEV(a_3) + \frac{\mathbb{E}[u(a_4)]}{\mathbb{E}[u(a_3)]} - 1 \right]},$$

where the term on the right hand side of the last inequality is greater than 1, both numerator band denominator being negative. Therefore, $R(a_4) < R(a_3)$ is not satisfied for $\lambda \in [0, 1]$. In this case, the opposite inequality $R(a_4) > R(a_3)$ is valid.
Considering the other case, where $NEV(a_4) - NEV(a_3) + \frac{\mathbb{E}[u(a_4)]}{\mathbb{E}[u(a_3)]} - 1 > 0$, one obtains

$$R(a_4) < R(a_3) \Leftrightarrow \lambda < \frac{\frac{\mathbb{E}[u(a_4)]}{\mathbb{E}[u(a_3)]} - 1}{\left[NEV(a_4) - NEV(a_3) + \frac{\mathbb{E}[u(a_4)]}{\mathbb{E}[u(a_3)]} - 1 \right]},$$

where the term on the right hand side is negative, the denominator being positive and the numerator negative. Therefore, also in this case, $R(a_4) < R(a_3)$ is not satisfied for $\lambda \in [0, 1]$. Instead, the opposite inequality is valid. Consequently, one has $R(a_4) > R(a_3)$ for a concave utility function.

□

Remark 2. *The last theorem shows that the preferences of a_1 over a_2 and a_4 over a_3, as observed in empirical experiments, which correspond to $R(a_1) < R(a_2)$ and $R(a_4) < R(a_3)$, are simultaneously satisfied if the utility function is convex and if*

$$\frac{\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]} - 1}{\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]} - 1 + NEV(a_1) - NEV(a_2)} \leq \frac{1 - \frac{\mathbb{E}[u(a_3)]}{\mathbb{E}[u(a_4)]}}{1 - \frac{\mathbb{E}[u(a_3)]}{\mathbb{E}[u(a_4)]} + NEV(a_4) - NEV(a_3)}.$$

Now the particular problem presented in [13] and analysed in [31], mentioned previously, will be analysed. This example is a particular case of the common ratio effect and of the Bergen paradox.

Problem 1 (Common ratio effect and Bergen paradox)

Consider the common ratio effect, where $x = 3000$, $q = 0.45$, $y = 6000$, $p = 0.9$, $rq = 0.001$, $rp = 0.002$ with conditions $p > q$ and $0 < x < y$ (see Table 6). The common ratio is given by $\frac{p}{q} = \frac{rp}{rq} = 2$ and $r = \frac{rp}{p} = \frac{rq}{q} = \frac{1}{450}$.

Table 6: Outcomes of a_1, a_2, a_3 and a_4 for Problem 1.

		θ_{i1}	θ_{i2}	$\mathbb{E}[a]$	$\mathbb{E}[a^2]$	Var[a]	$H(a)$	$NH(a)$
	x_{1j}	3000	0					
a_1	p_{1j}	0.9	0.10	2700	8100000	810000	0.3251	0.4690
	x_{2j}	6000	0					
a_2	p_{2j}	0.45	0.55	2700	16200000	8910000	0.6881	0.9928
	x_{3j}	3000	0					
a_3	p_{3j}	0.002	0.998	6	18000	17.964	0.0114	0.0208
	x_{4j}	6000	0					
a_4	p_{4j}	0.001	0.999	6	36000	35.964	0.0079	0.0114

According to empirical studies [13], the majority of people prefer a_1 over a_2 and a_4 over a_3. Concerning the choice problem between a_1 and a_2 where both probabilities of winning are substantial, people choose the prospect with higher probability, namely a_1. In the choice problem between a_3 and a_4, where the probabilities are almost zero, people choose the prospect with higher gain, which is a_4. According to the expected utility model, the preference $\mathbb{E}[u(a_1)] > \mathbb{E}[u(a_2)]$ is equivalent to $2u(3000) > u(6000) + u(0)$. However, the preference $\mathbb{E}[u(a_4)] > \mathbb{E}[u(a_3)]$ leads to the opposite inequality $2u(3000) < u(6000) + u(0)$. Therefore, these preferences contradict the expected utility principle. In [31] the authors show that the preferences obtained with the NEU-E model using a linear utility function $u(x) = x$ are exactly the same as those of the empirical studies. Here we will analyse if it is also possible to describe the preferences with the NEU-EV model.

Considering the linear utility function $u(x) = x$ the results (see Table 7 and Table 8) reveal that with the NEU-EV model a risk-neutral decision-maker

prefers a_1 over a_2 and a_3 over a_4 and only for $\lambda = 0$ the decision-maker is indifferent for both pairs of choices. The empirical preference of a_4 over a_3 can therefore not be described with the linear utility function. Due to the higher variance of a_4, the variance of a_4 being approximately the double of the variance of a_3 (see Table 6), the risk becomes higher for a_4. The uncertainty components of the NEU-EV risk measures for a_3 and a_4 are related as follows

$$NEV(a_4) = 0.5057 > NEV(a_3) = 0.2602.$$

This shows that, in order to model the empirical preference of a_4 over a_3, the only possibility is to decrease the risk $R(a_4)$ using an utility function yielding a higher expected utility for a_4 that compensates the penalization of the high uncertainty component. Given the outcomes for a_3 and a_4, this objective can be achieved with a convex utility function which increases the expected utility very more for a_4 than for a_3. In fact, this is possible using for example the convex utility function $u(x) = x^2$, (the corresponding expected utilities for the four actions are listed in Table 6). Using the results of Table 7, one can observe from Table 8 that the preference of a_1 over a_2 is achieved for $\lambda \in (0.4110, 1]$ and the preference of a_4 over a_3 is obtained for $\lambda \in [0, 0.6706]$.

Thus, one concludes that the NEU-EV model with the risk-seeking utility function $u(x) = x^2$ can capture well the people's attitude towards risk with $\lambda \in (0.4110, 0.6706]$. With these values of λ an approximate equal weight is given to the uncertainty component and to the expected utility component, with the possibility to give a slightly higher weight to the uncertainty component.

One could also obtain this interval from Theorem 5 (b) (see also Remark 2), by calculating directly the upper and lower limits for λ. Using the results in Table 6, one can confirm that

$$\frac{\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]} - 1}{\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]} - 1 + NEV(a_1) - NEV(a_2)} = 0.4110,$$

$$\frac{1 - \frac{\mathbb{E}[u(a_3)]}{\mathbb{E}[u(a_4)]}}{1 - \frac{\mathbb{E}[u(a_3)]}{\mathbb{E}[u(a_4)]} + NEV(a_4) - NEV(a_3)} = 0.6706.$$

Note that the convex utility function must satisfy the condition in Remark 2 in order for the results to be compatible with the empirical preferences. For example the convex utility function $u(x) = e^{0.1x}$ could also be used to model the preferences, leading to the following lower and upper limits for λ

$$\frac{\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]} - 1}{\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]} - 1 + NEV(a_1) - NEV(a_2)} = 0.5826,$$

$$\frac{1 - \frac{\mathbb{E}[u(a_3)]}{\mathbb{E}[u(a_4)]}}{1 - \frac{\mathbb{E}[u(a_3)]}{\mathbb{E}[u(a_4)]} + NEV(a_4) - NEV(a_3)} = 0.8029.$$

This example shows further the flexibility of the NEU-EV model (and also of the NEU-E model) in adapting the risk decisions to the observed behaviour by choosing an adequate utility function and also the tradeoff factors belonging to a certain interval. The NEU-EV model applied to this problem, when compared with the NEU-E model (see [31]), has the advantage that it captures also the uncertainty caused by variance. This has the consequence that a risk-seeking utility function is used to model e.g. the choice of a_4 over a_3 with the NEU-EV model, where in the NEU-E model the linear risk-neutral utility was used.

Table 7: Risk measures for a_1, a_2, a_3 and a_4 for Problem 1.

Utility function	$R(a_i)$	NEU-EV
$u(x) = x$	$R(a_1)$	$1.2799\lambda - 1$
	$R(a_2)$	$1.9964\lambda - 1$
	$R(a_3)$	$1.2602\lambda - 1$
	$R(a_4)$	$1.5057\lambda - 1$
$u(x) = x^2$	$R(a_1)$	$0.7799\lambda - 0.5$
	$R(a_2)$	$1.9964\lambda - 1$
	$R(a_3)$	$0.7601\lambda - 0.5$
	$R(a_4)$	$1.5057\lambda - 1$

Table 8: Tradeoff factors for decisions with NEU-EV for Problem 1.

Utility function	Risk decision	NEU-EV
$u(x) = x$	$R(a_1) < R(a_2)$	$0 < \lambda \leq 1$
	$R(a_1) = R(a_2)$	$\lambda = 0$
	$R(a_1) > R(a_2)$	—
	$R(a_3) < R(a_4)$	$0 < \lambda \leq 1$
	$R(a_3) = R(a_4)$	$\lambda = 0$
	$R(a_3) > R(a_4)$	—
$u(x) = x^2$	$R(a_1) < R(a_2)$	$0.4110 < \lambda \leq 1$
	$R(a_1) = R(a_2)$	$\lambda = 0.4110$
	$R(a_1) > R(a_2)$	$0 \leq \lambda \leq 0.4110$
	$R(a_3) < R(a_4)$	$0.6706 < \lambda \leq 1$
	$R(a_3) = R(a_4)$	$\lambda = 0.6706$
	$R(a_3) > R(a_4)$	$0 \leq \lambda \leq 0.6706$

Problem 2 (Common ratio effect and certainty effect)

The following example presented in [13] is another particular case of the common ratio effect, where $x = 3000$, $p = 1$, $y = 4000$, $q = 0.8$, $rp = 0.25$, $rq = 0.2$, $r = 0.25$ with conditions $p > q$ and $0 < x < y$ (see Table 9). The common ratio is given by $\frac{p}{q} = \frac{rp}{rq} = \frac{5}{4}$. In contrast to the previous problem, now one action has an outcome with probability 1 and this is therefore also a particular case of certainty effect.

Table 9: Outcomes of a_1, a_2, a_3 and a_4 for Problem 2.

		θ_{i1}	θ_{i2}	$\mathbb{E}[a]$	$\mathbb{E}[\sqrt{a}]$	$\mathbb{E}[a^2]$	Var[a]	$H(a)$	$NH(a)$
	x_{1j}	3000							
a_1	p_{1j}	1		3000	54.7723	9000000	0	0	0
	x_{2j}	4000	0						
a_2	p_{2j}	0.80	0.20	3200	50.5964	12800000	2560000	0.5004	0.7219
	x_{3j}	3000	0						
a_3	p_{3j}	0.25	0.75	750	13.693	2250000	1687500	0.5623	0.8113
	x_{4j}	4000	0						
a_4	p_{4j}	0.20	0.80	800	12.6491	3200000	2560000	0.5004	0.7219

Experiments revealed that the sure gain a_1 is preferred to a_2, however when offering the choice between a_3 and a_4, the option with the largest gain, in this case a_4, is the preferred one and not the most probable option a_3 (see [13]). This choice pattern is again in conflict with expected utility theory. People choose a sure gain over an uncertain gain, however when both probabilities of winning are less probable (in this case 0.2 and 0.25), then people are attracted by the prospect with largest gain. In [31] it was shown that the NEU-E decision model is able to describe this preference pattern with a risk neutral utility $u(x) = x$, where $R(a_1) < R(a_2)$ for $\lambda \in (0.08, 1]$ and $R(a_4) < R(a_3)$ for $\lambda \in [0, 1]$, and a risk-averse utility $u(x) = \sqrt{x}$, where $R(a_1) < R(a_2)$ for $\lambda \in [0, 1]$ and $R(a_4) < R(a_3)$ for $\lambda \in (0.47, 1]$. Now, this example will be analysed including the effect of variance using the NEU-EV model. Table 10 contains the NEU-EV risk measures and Table 11 the tradeoff factors for the various decisions using a risk-neutral utility $u(x) = x$, a risk averse utility $u(x) = \sqrt{x}$ and a risk-seeking utility $u(x) = x^2$.

Table 10: Risk measures for a_1, a_2, a_3 and a_4 for Problem 2.

Utility function	$R(a_i)$	NEU-EV
$u(x) = x$	$R(a_1)$	$0.9375\lambda - 0.9375$
	$R(a_2)$	$1.8609\lambda - 1$
	$R(a_3)$	$1.6727\lambda - 0.9375$
	$R(a_4)$	$1.8609\lambda - 1$
$u(x) = \sqrt{x}$	$R(a_1)$	$\lambda - 1$
	$R(a_2)$	$1.7848\lambda - 0.9239$
	$R(a_3)$	$1.7352\lambda - 1$
	$R(a_4)$	$1.7850\lambda - 0.9240$
$u(x) = x^2$	$R(a_1)$	$0.7031\lambda - 0.7031$
	$R(a_2)$	$1.8610\lambda - 1$
	$R(a_3)$	$1.4383\lambda - 0.7031$
	$R(a_4)$	$1.8610\lambda - 1$

Table 11: Tradeoff factors for decisions with NEU-EV for Problem 2.

Utility function	Risk decision	NEU-EV
$u(x) = x$	$R(a_1) < R(a_2)$	$0.0677 < \lambda \leq 1$
	$R(a_1) = R(a_2)$	$\lambda = 0.0677$
	$R(a_1) > R(a_2)$	$0 \leq \lambda < 0.0677$
	$R(a_3) < R(a_4)$	$0.3320 < \lambda \leq 1$
	$R(a_3) = R(a_4)$	$\lambda = 0.3320$
	$R(a_3) > R(a_4)$	$0 \leq \lambda < 0.3320$
$u(x) = \sqrt{x}$	$R(a_1) < R(a_2)$	$0 \leq \lambda \leq 1$
	$R(a_1) = R(a_2)$	—
	$R(a_1) > R(a_2)$	—
	$R(a_3) < R(a_4)$	$0 \leq \lambda \leq 1$
	$R(a_3) = R(a_4)$	—
	$R(a_3) > R(a_4)$	—
$u(x) = x^2$	$R(a_1) < R(a_2)$	$0.2564 < \lambda \leq 1$
	$R(a_1) = R(a_2)$	$\lambda = 0.2564$
	$R(a_1) > R(a_2)$	$0 \leq \lambda < 0.2564$
	$R(a_3) < R(a_4)$	$0.7024 < \lambda \leq 1$
	$R(a_3) = R(a_4)$	$\lambda = 0.7024$
	$R(a_3) > R(a_4)$	$0 \leq \lambda < 0.7024$

Analysing the results obtained with the NEU-EV model, the certain gain a_1 is completely preferred over a_2 with the risk-averse utility function $u(x) = \sqrt{x}$. Note that this is the only case where the utility function yields a higher expected utility for a_1 than for a_2, the variance and entropy being obviously always higher for a_2 (see Table 9). Decision makers with a risk neutral utility function choose a_1 for tradeoff factors $\lambda \in (0.0677, 1]$ and decision makers with a risk-seeking utility function choose a_1 for values $\lambda \in (0.2564, 1]$. The high variance of prospect a_2 penalizes its risk and therefore a_2 could only be a possible option for an individual using a risk neutral utility with a value of λ very close to zero

($\lambda \in [0, 0.0677)$), therefore almost ignoring the effects of variance and entropy, or for an individual using a risk-seeking utility with a little larger interval for λ: $[0, 0.2564)$, since a risk-seeking individual is inclined to take possibly a bigger risk than a risk-neutral individual. Regarding the prospects a_3 and a_4, the normalized entropy and variance risk components satisfy

$$NEV(a_3) = 0.7353 < NEV(a_4) = 0.8610,$$

so that considering only uncertainty, option a_3 would be more favourable. Therefore it is clear that the utility function has here a significant role in modelling the observed preference of a_4 over a_3. Since for the risk-averse utility $u(x) = \sqrt{x}$ one has $\mathbb{E}[u(a_3)] > \mathbb{E}[u(a_4)]$, then in this case, one always chooses a_3, which is confirmed by the interval for λ in Table 11. In the risk-neutral case, where the expected utilities coincide with the expected values, the inequality changes to $\mathbb{E}[u(a_3)] < \mathbb{E}[u(a_4)]$ giving rise to a chance for the preference of a_4 over a_3. One can observe in Table 11 that there exists an interval for λ where in fact a_4 is preferred over a_3, this interval being $[0, 0.3320)$, which means that for this preference more weight must be given to the expected utility component. Since the expected utility of a_4 increases with respect to those of a_3 with the convexity of the utility function (cf. Table 9), one expects a higher interval range for λ in the risk-seeking case corresponding to the quadratic utility than in the risk-neutral case. In fact, a closer look reveals that the interval range for λ increases, the risk decision $R(a_4) < R(a_3)$ corresponding to $\lambda \in [0, 0.7024)$. In this case the expected utility for a_4 is considerably high so that more weight can also be given to the entropy and variance component, since there exist values of $\lambda > 0.5$ which lead to the choice of a_4.

Thus, the NEU-EV model with the risk-neutral and with the risk-seeking utility function can be used to model the observed phenomenon of common ratio effect presented in this problem when $\lambda \in (0.2564, 0.3320)$. This means that more weight is given to the expected utility component, since the values of the tradeoff factor satisfy $\lambda < 0.5$. This interval is obtained intersecting the interval corresponding to the risk-neutral utility $(0.0677, 0.3320)$ with the interval for the

risk-seeking utility $(0.2564, 0.7024)$. Using the risk-seeking utility, the problem can also be explained with a higher weight given to the uncertainty component and it contains the equilibrium situation $\lambda = 0.5$. The risk-seeking utility has therefore advantages in modelling the problem due to the larger interval for λ. With the risk-neutral utility the explained decisions are always made with a higher weight given to the expected utility component.

Also in this example, one can observe that the NEU-EV model captures the effect of the higher variance of a_4 in the choice between a_3 and a_4, with the consequence that it is not possible to choose a_4 with a risk-averse utility. On the contrary, with the NEU-E model, where the effect of variance is ignored, it is possible to choose a_4 with the risk-averse utility. The NEU-EV is through the additional variance component more sensitive to uncertainty and this is reflected in the choice of a compatible utility function.

Observe that the results obtained in the last example are consistent with Theorem 2. One can also apply directly Theorem 2 to describe the preference of the certain gain a_1 over a_2 in the following way. Considering the risk-neutral utility $u(x) = x$, since $\mathbb{E}[a_1] < \mathbb{E}[a_2]$, then according to Theorem 2 (b), one has $R(a_1) < R(a_2)$ for $\frac{\frac{\mathbb{E}[a_1]}{\mathbb{E}[a_2]}-1}{\frac{\mathbb{E}[a_1]}{\mathbb{E}[a_2]}-1-NEV(a_2)} < \lambda \leq 1$. Calculating the lower limit for λ one obtains $\frac{\frac{\mathbb{E}[a_1]}{\mathbb{E}[a_2]}-1}{\frac{\mathbb{E}[a_1]}{\mathbb{E}[a_2]}-1-NEV(a_2)} = 0.0677$, which coincides with the corresponding result presented in Table 11. As for the risk-averse utility $u(x) = \sqrt{x}$, since $\mathbb{E}[u(a_1)] > \mathbb{E}[u(a_2)]$, then applying Theorem 2 (a), it is always true that $R(a_1) < R(a_2)$ and therefore $0 \leq \lambda \leq 1$ (cf. Table 11). For the risk-seeking utility $u(x) = x^2$, one has $\mathbb{E}[u(a_1)] < \mathbb{E}[u(a_2)]$ (see Table 9) and therefore Theorem 2 (b) implies that $R(a_1) < R(a_2)$ for $\frac{\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]}-1}{\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]}-1-NEV(a_2)} < \lambda \leq 1$. Calculating the lower limit for λ, one gets $\frac{\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]}-1}{\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]}-1-NEV(a_2)} = 0.2564$, as indicated in Table 11.

5.2. Common consequence effects

The commmon consequence effect is another paradox, which was described by Machina [18] as follows.

Consider two pairs of actions defined by probability mixtures in the form

$$a_1:\ \alpha\delta_x + (1-\alpha)P^{\star\star} \quad \text{versus} \quad a_2:\ \alpha P + (1-\alpha)P^{\star\star}$$
$$a_3:\ \alpha\delta_x + (1-\alpha)P^{\star} \quad \text{versus} \quad a_4:\ \alpha P + (1-\alpha)P^{\star},$$

where $0 \leq \alpha \leq 1$, δ_x is a prospect yielding x with certainty, P involves outcomes that are both greater and less than x and $P^{\star\star}$ stochastically dominates $P^{\star}$. The options a_1 and a_2 in the first pair have the common consequence $P^{\star\star}$ and, in the second pair of choices, a_3 and a_4 share the common consequence $P^{\star}$. The common consequence effect includes the famous Allais paradox [2] and it includes the certainty effect if $P^{\star\star} = x$ with probability 1. Considering the certainty effect, the following version of common consequence effect will be analysed, which generalizes the problem of the Allais paradox and was considered in [16]. The Allais paradox will then be studied separately as a particular case (Problem 3) as well as another particular problem of certainty effect (Problem 4).

Consider the pairs of prospects (a_1, a_2) and (a_3, a_4), where $P^{\star\star} = y = \delta_y$, P has outcomes x with probability $\frac{p}{p+s}$ and 0 with probability $\frac{s}{p+s}$, $P^{\star} = 0$ with probability 1, $\alpha = p + s$, $1 - \alpha = q$, with $p + q + s = 1$ and $x > y > 0$, so that the prospects are given by

$$\begin{aligned} a_1 &= (y, 1) && \text{versus} \quad a_2 = (x, p; y, q; 0, s), \\ a_3 &= (y, p+s; 0, q) && \text{versus} \quad a_4 = (x, p; 0, q+s). \end{aligned} \tag{29}$$

For the problems presented below (Problem 3 and Problem 4), the observed preferences are the choice of a_1 over a_2 and the choice of a_4 over a_3. The next theorem shows that the preference predictions with the NEU-EV model are compatible with these observed choices, with one exception contradicting the preference of a_4 over a_3 in a special situation presented in Theorem 6 $(b)(ii)$.

Theorem 6. *Consider the NEU-EV risk measure defined in (4), with increasing utility function $u(x)$, and the problem of the common consequence effect with pairs of prospects (a_1, a_2) and (a_3, a_4) defined in (29).*

(a) (i) If $\mathbb{E}[u(a_1)] > \mathbb{E}[u(a_2)]$, then $R(a_1) < R(a_2)$.

(ii) If $\mathbb{E}[u(a_1)] < \mathbb{E}[u(a_2)]$*, then* $R(a_1) < R(a_2)$ *for*

$$\frac{\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]} - 1}{\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]} - 1 - NEV(a_2)} < \lambda \leq 1.$$

(b) (i) Let $\mathbb{E}[u(a_4)] > \mathbb{E}[u(a_3)]$*. If* $NEV(a_4) - NEV(a_3) < 0$*, then* $R(a_4) < R(a_3)$*.*

If $NEV(a_4) - NEV(a_3) > 0$*, then* $R(a_4) < R(a_3)$ *for*

$$0 \leq \lambda < \frac{1 - \frac{\mathbb{E}[u(a_3)]}{\mathbb{E}[u(a_4)]}}{1 - \frac{\mathbb{E}[u(a_3)]}{\mathbb{E}[u(a_4)]} + NEV(a_4) - NEV(a_3)}.$$

(ii) Let $\mathbb{E}[u(a_4)] < \mathbb{E}[u(a_3)]$*. If* $NEV(a_4) - NEV(a_3) < 0$*, then* $R(a_4) < R(a_3)$ *for*

$$\frac{\frac{\mathbb{E}[u(a_4)]}{\mathbb{E}[u(a_3)]} - 1}{\frac{\mathbb{E}[u(a_4)]}{\mathbb{E}[u(a_3)]} - 1 + NEV(a_4) - NEV(a_3)} < \lambda \leq 1.$$

If $NEV(a_4) - NEV(a_3) > 0$*, then* $R(a_4) > R(a_3)$*.*

PROOF. (a) (i) Consider the case $\mathbb{E}[u(a_1)] > \mathbb{E}[u(a_2)]$. From the fact that $NEV(a_1) = 0$ the condition $R(a_1) < R(a_2)$ is equivalent to

$$\lambda \left(-NEV(a_2) + 1 - \frac{\mathbb{E}[u(a_2)]}{\mathbb{E}[u(a_1)]} \right) < 1 - \frac{\mathbb{E}[u(a_2)]}{\mathbb{E}[u(a_1)]},$$

where $-NEV(a_2) < 0$. There are now two possibilities: if $-NEV(a_2) + 1 - \frac{\mathbb{E}[u(a_2)]}{\mathbb{E}[u(a_1)]} > 0$, then

$$\lambda < \frac{1 - \frac{\mathbb{E}[u(a_2)]}{\mathbb{E}[u(a_1)]}}{-NEV(a_2) + 1 - \frac{\mathbb{E}[u(a_2)]}{\mathbb{E}[u(a_1)]}},$$

where the term on the right hand side of the last inequality is greater than 1, so that $R(a_1) < R(a_2)$ is satisfied for $\lambda \in [0, 1]$; if $-NEV(a_2) + 1 - \frac{\mathbb{E}[u(a_2)]}{\mathbb{E}[u(a_1)]} < 0$, then

$$\lambda > \frac{1 - \frac{\mathbb{E}[u(a_2)]}{\mathbb{E}[u(a_1)]}}{-NEV(a_2) + 1 - \frac{\mathbb{E}[u(a_2)]}{\mathbb{E}[u(a_1)]}},$$

where the term on the right hand side of the last inequality is negative, so that $R(a_1) < R(a_2)$ is satisfied for $\lambda \in [0, 1]$. One concludes that in the case $\mathbb{E}[u(a_1)] > \mathbb{E}[u(a_2)]$ the inequality $R(a_1) < R(a_2)$ holds for $\lambda \in [0, 1]$.

(ii) Consider the case $\mathbb{E}[u(a_1)] < \mathbb{E}[u(a_2)]$. Then, since $-NEV(a_2) < 0$, one has

$$R(a_1) < R(a_2) \Leftrightarrow \lambda > \frac{\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]} - 1}{-NEV(a_2) + \frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]} - 1},$$

where the term on the right hand side of the last inequality is positive and less than 1. Therefore, one concludes that there exist intervals for λ contained in $[0, 1]$ given by

$$\frac{\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]} - 1}{-NEV(a_2) + \frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]} - 1} < \lambda \leq 1,$$

where $R(a_1) < R(a_2)$ is valid.

(b) (i) Let $\mathbb{E}[u(a_4)] > \mathbb{E}[u(a_3)]$. First, consider that $NEV(a_4) - NEV(a_3) < 0$. Then,

$$R(a_4) < R(a_3) \Leftrightarrow \lambda \left(NEV(a_4) - NEV(a_3) + 1 - \frac{\mathbb{E}[u(a_3)]}{\mathbb{E}[u(a_4)]} \right) < 1 - \frac{\mathbb{E}[u(a_3)]}{\mathbb{E}[u(a_4)]},$$

where $NEV(a_4) - NEV(a_3) + 1 - \frac{\mathbb{E}[u(a_3)]}{\mathbb{E}[u(a_4)]}$ may be positive, leading to

$$\lambda < \frac{1 - \frac{\mathbb{E}[u(a_3)]}{\mathbb{E}[u(a_4)]}}{NEV(a_4) - NEV(a_3) + 1 - \frac{\mathbb{E}[u(a_3)]}{\mathbb{E}[u(a_4)]}},$$

where the term on the right is greater than 1, or negative, yielding

$$\lambda > \frac{1 - \frac{\mathbb{E}[u(a_3)]}{\mathbb{E}[u(a_4)]}}{NEV(a_4) - NEV(a_3) + 1 - \frac{\mathbb{E}[u(a_3)]}{\mathbb{E}[u(a_4)]}},$$

where the term on the right is negative. In both situations, the inequalities hold for the entire range of $\lambda \in [0, 1]$, so that $R(a_4) < R(a_3)$ is always true.

Now, consider that $NEV(a_4) - NEV(a_3) > 0$. Then,

$$R(a_4) < R(a_3) \Leftrightarrow \lambda < \frac{1 - \frac{\mathbb{E}[u(a_3)]}{\mathbb{E}[u(a_4)]}}{1 - \frac{\mathbb{E}[u(a_3)]}{\mathbb{E}[u(a_4)]} + NEV(a_4) - NEV(a_3)},$$

where the term on the right hand side of the last inequality is positive and less than 1. This means that $R(a_4) < R(a_3)$ is valid for

$$0 \leq \lambda < \frac{1 - \frac{\mathbb{E}[u(a_3)]}{\mathbb{E}[u(a_4)]}}{1 - \frac{\mathbb{E}[u(a_3)]}{\mathbb{E}[u(a_4)]} + NEV(a_4) - NEV(a_3)}.$$

(ii) Let $\mathbb{E}[u(a_4)] < \mathbb{E}[u(a_3)]$. If $NEV(a_4) - NEV(a_3) < 0$, then

$$R(a_4) < R(a_3) \Leftrightarrow \lambda > \frac{\frac{\mathbb{E}[u(a_4)]}{\mathbb{E}[u(a_3)]} - 1}{\frac{\mathbb{E}[u(a_4)]}{\mathbb{E}[u(a_3)]} - 1 + NEV(a_4) - NEV(a_3)},$$

where the term on the right hand side of the last inequality is positive and less than 1. Consequently, $R(a_4) < R(a_3)$ holds for

$$\frac{\frac{\mathbb{E}[u(a_4)]}{\mathbb{E}[u(a_3)]} - 1}{\frac{\mathbb{E}[u(a_4)]}{\mathbb{E}[u(a_3)]} - 1 + NEV(a_4) - NEV(a_3)} < \lambda \leq 1.$$

If $NEV(a_4) - NEV(a_3) > 0$, then

$$R(a_4) < R(a_3) \Leftrightarrow \lambda \left(NEV(a_4) - NEV(a_3) + \frac{\mathbb{E}[u(a_4)]}{\mathbb{E}[u(a_3)]} - 1 \right) < \frac{\mathbb{E}[u(a_4)]}{\mathbb{E}[u(a_3)]} - 1,$$

where the term $NEV(a_4) - NEV(a_3) + \frac{\mathbb{E}[u(a_4)]}{\mathbb{E}[u(a_3)]} - 1$ may be positive or negative. However, in both cases the condition turns out to be impossible to hold for $\lambda \in [0, 1]$. Therefore, if $\mathbb{E}[u(a_4)] < \mathbb{E}[u(a_3)]$ and $NEV(a_4) - NEV(a_3) > 0$, the opposite inequality $R(a_4) > R(a_3)$ is satisfied for $\lambda \in [0, 1]$.

□

Problem 3 (Common consequence effect, certainty effect and Allais paradox)

The Allais paradox can be represented as a special case of (29) by

$$\begin{aligned} a_1 &= (1M, 1) && \text{versus} \quad a_2 = (5M, 0.1; 1M, 0.89; 0, 0.01), \\ a_3 &= (1M, 0.11; 0, 0.89) && \text{versus} \quad a_4 = (5M, 0.1; 0, 0.9). \end{aligned}$$

Table 12 contains the expected utilities for the utility functions $u(x) = x$, $u(x) = \sqrt{x}$ and $u(x) = x^2$, the variances and entropies corresponding to the four options.

Table 12: Outcomes of a_1, a_2, a_3 and a_4 for Problem 3.

		θ_{i1}	θ_{i2}	θ_{i3}	$\mathbb{E}[a]$	$\mathbb{E}[\sqrt{a}]$	$\mathbb{E}[a^2]$	Var[a]	$H(a)$	$NH(a)$
	x_{1j}	$1M$								
a_1	p_{1j}	1			$1M$	$1M$	$1M$	0	0	0
	x_{2j}	$5M$	$1M$	0						
a_2	p_{2j}	0.1	0.89	0.01	$1.39M$	$1.1136M$	$3.39M$	$1.4579M^2$	0.3800	0.3459
	x_{3j}	$1M$	0							
a_3	p_{3j}	0.11	0.89		$0.11M$	$0.11M$	$0.11M$	$0.0979M^2$	0.3465	0.4999
	x_{4j}	$5M$	0							
a_4	p_{4j}	0.1	0.9		$0.5M$	$0.2236M$	$2.5M$	$2.25M^2$	0.3251	0.4690

Recall that the observed preferences are a_1 over a_2 and a_4 over a_3. Calculating the risk measures from the outputs in Table 12, one can observe that $R(a_2)$ and $R(a_4)$ remain unaltered for the three utilities (see Table 13). This is due to the fact that the expected utilities and the variances are always higher for a_2 and a_4 than for a_1 and a_3, respectively. The preference of a_1 with the certain outcome over a_2 could be predicted from Theorem 2 (b) for all three utilities. The results in Table 14 agree with those predictions and one can observe that the interval range for λ is larger for the risk-averse utility, $(0.1317, 1]$, than for the risk-seeking utility, $(0.5117, 1]$. It is clear that the intervals must contain the value 1 since with the normalized entropy and variance term alone, one would prefer a_1 over a_2. With the risk-averse utility it is possible to have a higher weighted normalized expected utility term in the NEU-EV risk measure than with the risk-seeking utility. As for a_3 and a_4, the results show that it is also possible to describe the choice of a_4 with all three utility functions, where the risk-seeking utility leads in this case to a larger interval range for λ, given by $[0, 0.6738)$, than the risk-averse utility, the interval being $[0, 0.5233)$ (see Table 14). Here, it is possible to predict the choice of a_4 over a_3 with the expected utility term alone, when $\lambda = 0$. The larger interval for λ with the risk-seeking utility allows chosing a_4 with the NEU-EV risk measure giving a higher weight

to the normalized entropy and variance term.

Table 13: Risk measures for a_1, a_2, a_3 and a_4 for Problem 3.

Utility function	$R(a_i)$	NEU-EV
$u(x) = x$	$R(a_1)$	$0.7194\lambda - 0.7194$
	$R(a_2)$	$1.6729\lambda - 1$
	$R(a_3)$	$0.4917\lambda - 0.22$
	$R(a_4)$	$1.7345\lambda - 1$
$u(x) = \sqrt{x}$	$R(a_1)$	$0.8979\lambda - 0.8979$
	$R(a_2)$	$1.6729\lambda - 1$
	$R(a_3)$	$0.7636\lambda - 0.4919$
	$R(a_4)$	$1.7345\lambda - 1$
$u(x) = x^2$	$R(a_1)$	$0.2949\lambda - 0.2949$
	$R(a_2)$	$1.6729\lambda - 1$
	$R(a_3)$	$0.3157\lambda - 0.044$
	$R(a_4)$	$1.7345\lambda - 1$

Table 14: Tradeoff factors for decisions with NEU-EV for Problem 3.

Utility function	Risk decision	NEU-EV
$u(x) = x$	$R(a_1) < R(a_2)$	$0.2943 < \lambda \leq 1$
	$R(a_1) = R(a_2)$	$\lambda = 0.2943$
	$R(a_1) > R(a_2)$	$0 \leq \lambda < 0.2943$
	$R(a_3) < R(a_4)$	$0.6276 < \lambda \leq 1$
	$R(a_3) = R(a_4)$	$\lambda = 0.6276$
	$R(a_3) > R(a_4)$	$0 \leq \lambda < 0.6276$
$u(x) = \sqrt{x}$	$R(a_1) < R(a_2)$	$0.1317 < \lambda \leq 1$
	$R(a_1) = R(a_2)$	$\lambda = 0.1317$
	$R(a_1) > R(a_2)$	$0 \leq \lambda < 0.1317$
	$R(a_3) < R(a_4)$	$0.5233 < \lambda \leq 1$
	$R(a_3) = R(a_4)$	$\lambda = 0.5233$
	$R(a_3) > R(a_4)$	$0 \leq \lambda < 0.5233$
$u(x) = x^2$	$R(a_1) < R(a_2)$	$0.5117 < \lambda \leq 1$
	$R(a_1) = R(a_2)$	$\lambda = 0.5117$
	$R(a_1) > R(a_2)$	$0 \leq \lambda < 0.5117$
	$R(a_3) < R(a_4)$	$0.6738 < \lambda \leq 1$
	$R(a_3) = R(a_4)$	$\lambda = 0.6738$
	$R(a_3) > R(a_4)$	$0 \leq \lambda < 0.6738$

According to the results of Theorem 6, one can also conclude that $R(a_1) < R(a_2)$ from Theorem 6 $(a)(ii)$ and the interval for λ can also be calculated from the expression indicated there. Considering the choice between a_4 and a_3, since for all three utilities one has $\mathbb{E}[u(a_4)] > \mathbb{E}[u(a_3)]$ and $NEV(a_4) = 0.7345 > NEV(a_3) = 0.2717$, then, according to Theorem 6 $(b)(i)$, $R(a_4) < R(a_3)$ holds for the values of λ belonging to the indicated interval, which yield the corresponding intervals presented in Table 14.

One concludes that the Allais paradox can be well explained with the NEU-EV model. Note that, although a_4 has a higher variance than a_3, even $NEV(a_4)$

being higher than $NEV(a_3)$, the NEU-EV risk measure is able to model the preference of a_4 and this is done better with the risk-seeking utility (since it leads to a higher interval for the tradeoff factor). This prediction of a_4 over a_3 could be achieved with the EU model. The advantage of the NEU-EV model is that it is flexible to model the EU prediction for $\lambda = 0$, but it also models the prediction taking additional information into account regarding uncertainty, given by entropy and variance. Considering the common choice of a_1 over a_2 and of a_4 over a_3, the preferences can be explained with all three utility functions for values of $\lambda \in (0.5117, 0.5233)$, which suggests that, with the NEU-EV model, a decision maker weighting approximately equally the uncertainty component and the expected utility component agrees with those options. This interval is obtained from the intersection of the interval $(0.1317, 0.5233)$, corresponding to $u(x) = \sqrt{x}$, with the interval $(0.2943, 0.6276)$, corresponding to $u(x) = x$, and with $(0.5117, 0.6738)$, corresponding to $u(x) = x^2$. These intervals show that a decision maker favouring more the uncertainty component has a higher chance to prefer the options with the risk-seeking utility and a decision maker favouring more the expected utility component has a higher chance to prefer the options with the risk-averse utility.

Problem 4 (Common consequence effect and certainty effect)

The example presented in Table 15, considered in [13] and analysed in [31] can be written in Machina's notation as

$$\begin{aligned} a_1 &: \ 0.34\delta_{2400} + 0.66P^{\star\star} \quad \text{versus} \quad a_2 : \ 0.34P + 0.66P^{\star\star} \\ a_3 &: \ 0.34\delta_{2400} + 0.66P^{\star} \quad \text{versus} \quad a_4 : \ 0.34P + 0.66P^{\star}, \end{aligned}$$

where δ_{2400} represents the certain outcome (that is $x = 2400$ with probability 1), $P^{\star\star} = 2400$ with probability 1, $P^{\star} = 0$ with probability 1 and $P = 2500$ with probability $\frac{33}{34}$ and $P = 0$ with probability $\frac{1}{34}$ Alternatively, according to the notation used in this paper, this problem can be written as

$$\begin{aligned} a_1 &= (2400, 1) \quad \text{versus} \quad a_2 = (2500, 0.33; 2400, 0.66; 0, 0.01), \\ a_3 &= (2400, 0.34; 0, 0.66) \quad \text{versus} \quad a_4 = (2500, 0.33; 0, 0.67). \end{aligned}$$

This problem is also a particular case of (29), where $y = 2400$, $x = 2500$, $p = 0.33$, $q = 0.66$ and $s = 0.01$.

Research experiments showed that people have the tendency to prefer a_1 in the first pair of choices and to choose a_4 in the second pair [13],[18]. Note that the pair of choices (a_3, a_4) is obtained from the first pair (a_1, a_2) by eliminating the 0.66 chance of $P^{\star\star}$ in a_1 and in a_2.

Table 15: Outcomes of a_1, a_2, a_3 and a_4 for Problem 4.

		θ_{i1}	θ_{i2}	θ_{i3}	$\mathbb{E}[a]$	$\mathbb{E}[\sqrt{a}]$	$\mathbb{E}[a^2]$	Var[a]	$H(a)$	$NH(a)$
	x_{1j}	2400								
a_1	p_{1j}	1			2400	48.9898	5760000	0	0	0
	x_{2j}	2500	2400	0						
a_2	p_{2j}	0.33	0.66	0.01	2409	48.8333	5864100	60819	0.6862	0.6246
	x_{3j}	2400	0							
a_3	p_{3j}	0.34	0.66		816	16.6565	1958400	1292544	0.641	0.9248
	x_{4j}	2500	0							
a_4	p_{4j}	0.33	0.67		825	16.5	2062500	1381875	0.6342	0.9149

From the results in Table 16 and Table 17 it becomes clear that the NEU-EV model with the risk-averse utility $u(x) = \sqrt{x}$ does only predict the choice of a_1 over a_2 and of a_3 over a_4. One can see in Table 15 that only with this utility the preferences seem to be in accordance with the ordering of the corresponding expected utilities: $\mathbb{E}[\sqrt{a_1}] > \mathbb{E}[\sqrt{a_2}]$ and $\mathbb{E}[\sqrt{a_3}] > \mathbb{E}[\sqrt{a_4}]$. Theorem 6 (a) (ii) predicts then the preference of a_1 and, since $NEV(a_4) = 0.9575 > NEV(a_3) = 0.9301$, Theorem 6 (b) (ii), the preference of a_3. With the risk-neutral utility $u(x) = x$ the risk of a_1 continues inferior to the risk of a_2 for a very high interval of the tradeoff factor, where in this case the result of Theorem 6 (a) (ii) can be applied (since $\mathbb{E}[a_2] > \mathbb{E}[a_1]$), although there is now a possibility for the choice of a_2 over a_1, but only for tradeoff factors very close to zero ($\lambda \in [0, 0.0045)$). Now, in the risk-neutral case, a_4 is preferred to a_3 for $\lambda \in [0, 0.2739)$ (cf. Theorem 6 (b) (i), where now $\mathbb{E}[a_4] > \mathbb{E}[a_3]$ and $NEV(a_4) > NEV(a_3)$). With the risk-seeking

utility $u(x) = x^2$ the decision $R(a_1) < R(a_2)$ is obtained for $\lambda \in (0.0214, 1]$ and $R(a_4) < R(a_3)$ for $\lambda \in [0, 0.636)$. These intervals could be calculated from the expressions in Theorem 6 (a) (ii) and Theorem 6 (b) (i) with $NEV(a_4) > NEV(a_3)$, respectively. Thus, the convexity of the utility function increases the possibility for choosing a_4 over a_3. The choice of a_4 instead of a_3 is perhaps more characteristic for aggressive decision makers. In fact, with the risk-seeking utility even a risk-measure where more weight, i.e. $0.5 < \lambda < 0.636$, is given to the uncertainty components normalized entropy and variance will lead to the preference of a_4 over a_3. Also in this example, the NEU-EV model is sensitive to the uncertainty implied by the variance component in a_4, which is crucial for the decision between a_4 and a_3 with the risk-averse utility function, having the consequence that it is not possible to prefer a_4 over a_3 with this utility. One concludes that the NEU-EV model can be used to explain the observed decisions in the experiments with the risk-neutral and with the risk-seeking utility function, taking into account, besides expected utility, the risk factors normalized entropy and variance, with tradeoff factors $\lambda \in (0.0214, 0.2739)$. Considering $u(x) = x$ the interval for λ leading to the choices is $(0.0045, 0.2739)$ and considering $u(x) = x^2$, it is given by $(0.0214, 0.636)$. Here one can see the advantage of modelling the decision problem with the risk-seeking utility. It has associated a larger interval for λ, contains values satisfying $\lambda < 0$, where the expected utility term is favoured, contains $\lambda = 0.5$ corresponding to an equilibrium between the uncertainty and the expected utility, and it permits also that a decision maker favouring more the uncertainty component $(\lambda > 0.5)$ is also able to prefer the options a_1 and a_4.

Table 16: Risk measures for a_1, a_2, a_3 and a_4 for Problem 4.

Utility function	$R(a_i)$	NEU-EV
$u(x) = x$	$R(a_1)$	$0.9963\lambda - 0.9963$
	$R(a_2)$	$1.8123\lambda - 1$
	$R(a_3)$	$1.7773\lambda - 0.9891$
	$R(a_4)$	$1.8171\lambda - 1$
$u(x) = \sqrt{x}$	$R(a_1)$	$\lambda - 1$
	$R(a_2)$	$1.8091\lambda - 0.9968$
	$R(a_3)$	$1.7882\lambda - 1$
	$R(a_4)$	$1.8077\lambda - 0.9906$
$u(x) = x^2$	$R(a_1)$	$0.9822\lambda - 0.9822$
	$R(a_2)$	$1.8123\lambda - 1$
	$R(a_3)$	$1.7377\lambda - 0.9495$
	$R(a_4)$	$1.8171\lambda - 1$

Table 17: Tradeoff factors for decisions with NEU-EV for Problem 4.

Utility function	Risk decision	NEU-EV
$u(x) = x$	$R(a_1) < R(a_2)$	$0.0045 < \lambda \leq 1$
	$R(a_1) = R(a_2)$	$\lambda = 0.0045$
	$R(a_1) > R(a_2)$	$0 \leq \lambda < 0.0045$
	$R(a_3) < R(a_4)$	$0.2739 < \lambda \leq 1$
	$R(a_3) = R(a_4)$	$\lambda = 0.2739$
	$R(a_3) > R(a_4)$	$0 \leq \lambda < 0.2739$
$u(x) = \sqrt{x}$	$R(a_1) < R(a_2)$	$0 \leq \lambda \leq 1$
	$R(a_1) = R(a_2)$	—
	$R(a_1) > R(a_2)$	—
	$R(a_3) < R(a_4)$	$0 \leq \lambda \leq 1$
	$R(a_3) = R(a_4)$	—
	$R(a_3) > R(a_4)$	—
$u(x) = x^2$	$R(a_1) < R(a_2)$	$0.0214 < \lambda \leq 1$
	$R(a_1) = R(a_2)$	$\lambda = 0.0214$
	$R(a_1) > R(a_2)$	$0 \leq \lambda < 0.0214$
	$R(a_3) < R(a_4)$	$0.636 < \lambda \leq 1$
	$R(a_3) = R(a_4)$	$\lambda = 0.636$
	$R(a_3) > R(a_4)$	$0 \leq \lambda < 0.636$

Since this example of common consequence effect also depends on an action with certain outcome a_1 one can also apply Theorem 2 to predict the preference $R(a_1) < R(a_2)$ as explained below.

Considering the risk-neutral utility $u(x) = x$, since $\mathbb{E}[a_1] < \mathbb{E}[a_2]$ (see Table 15), then according to Theorem 2 (b), one has $R(a_1) < R(a_2)$ for $\frac{\frac{\mathbb{E}[a_1]}{\mathbb{E}[a_2]}-1}{\frac{\mathbb{E}[a_1]}{\mathbb{E}[a_2]}-1-NEV(a_2)} < \lambda \leq 1$. Calculating the lower limit for λ gives $\frac{\frac{\mathbb{E}[a_1]}{\mathbb{E}[a_2]}-1}{\frac{\mathbb{E}[a_1]}{\mathbb{E}[a_2]}-1-NEV(a_2)} = 0.0045$, which coincides with the corresponding result in Table 17. For the risk-averse utility $u(x) = \sqrt{x}$, since $\mathbb{E}[u(a_1)] > \mathbb{E}[u(a_2)]$ (see Table 15), then applying Theorem 2 (a), one concludes that $R(a_1) < R(a_2)$ and therefore $0 \leq \lambda \leq 1$, which can be

confirmed in Table 17. Considering the risk-seeking utility $u(x) = x^2$, one has $\mathbb{E}[u(a_1)] < \mathbb{E}[u(a_2)]$ (see Table 15) and therefore Theorem 2 (b) implies that $R(a_1) < R(a_2)$ for $\frac{\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]}-1}{\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]}-1-NEV(a_2)} < \lambda \leq 1$. Calculating the lower limit for λ, one gets $\frac{\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]}-1}{\frac{\mathbb{E}[u(a_1)]}{\mathbb{E}[u(a_2)]}-1-NEV(a_2)} = 0.0214$, as indicated in Table 17.

6. Conclusions

The normalized expected utility-entropy and variance (NEU-EV) model is a new risk decision model which improves the existing EU-EV model by using normalized entropy and generalizes the normalized EU-E model by including an additional variance component.

The NEU-EV model can adequately be applied to actions having states with different numbers of outcomes, whereas the non-normalized models are not appropriate to explain these kind of problems, as shown in an example analysed in this paper. Another advantage of the NEU-EV model is that due to the inclusion of variance as risk factor, in certain situations, where actions have equal expected values and equal entropy, the variance can be different and thus decisive for decision making. An example of this situation is studied in this paper. It is also shown that the model has mathematical properties that are consistent with several properties of risk perception.

The NEU-EV model is applied to examples of certainty effects, common ratio effects and common consequence effects and the results are investigated taking into account different utility functions, where the influence of risk-averse, risk-neutral and risk-seeking utility functions is analysed. It is shown that the NEU-EV model is an adequate descriptive model for explaining the effects, since observed results obtained from empirical studies, which were presented in earlier literature, are well explained with this model. Furthermore, the results suggest that a risk-seeking utility function and in some cases also a risk neutral utility function are more appropriate in modelling these decision paradoxes, when risk is assessed by expected utility together with the uncertainty factors entropy and variance.

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Appendix A. Decision models

The following table contains the acronyms of the models considered in this paper.

Table A.18: Decision models.

acronym	model
EU	expected utility
EU-E	expected utility and entropy
NEU-E	normalized expected utility and entropy
EU-EV	expected utility, entropy and variance
NEU-EV	normalized expected utility, entropy and variance

The EU-EV risk measure is given by (see [4])

$$R(a) = \frac{\lambda}{2}\left[H_a(\theta) + \frac{\mathrm{Var[X(a,\theta)]}}{\max\limits_{a\in A}\{\mathrm{Var[X(a,\theta)]}\}}\right] - (1-\lambda)\frac{\mathbb{E}[u(X(a,\theta))]}{\max\limits_{a\in A}\{\mathbb{E}[u(X(a,\theta))]\}}. \quad \text{(A.1)}$$

The NEU-E risk measure is defined by (see [31])

$$R(a) = \begin{cases} \lambda N H_a(\theta) - (1-\lambda)\dfrac{\mathbb{E}[u(X(a,\theta))]}{\max\limits_{a\in A}\{\mathbb{E}[u(X(a,\theta))]\}}, & \text{if } n > 1, \\ -(1-\lambda)\dfrac{\mathbb{E}[u(X(a,\theta))]}{\max\limits_{a\in A}\{\mathbb{E}[u(X(a,\theta))]\}}, & \text{if } n = 1. \end{cases} \quad \text{(A.2)}$$

The EU-E risk measure uses the simple entropy (2) and is defined by (see [30])

$$R(a) = \lambda H_a(\theta) - (1-\lambda)\frac{\mathbb{E}[u(X(a,\theta))]}{\max\limits_{a\in A}\{\mathbb{E}[u(X(a,\theta))]\}}. \quad \text{(A.3)}$$

Note that if the variance term $\frac{\text{Var[X(a,}\theta)]}{\max_{a \in A}\{\text{Var[X(a,}\theta)]\}}$ in the NEU-EV risk measure (4) is substituted by the normalized entropy $NH_a(\theta)$, then the NEU-E risk measure is recovered. If the variance term in the EU-EV risk measure is substituted by the entropy $H_a(\theta)$, then the EU-E risk measure is recovered.

Figure A.2 illustrates the implications between the models listed in Table A.18. Setting the tradeoff factor λ equal to 0 in the EU-E, NEU-E, EU-EV and NEU-EV models, one recovers the expected utility decision model. If $\lambda = 1$ in the EU-E model or in the NEU-E model, one obtains a decision model based only on entropy (E) or on normalized entropy (NE), respectively. If $\lambda = 1$ in the EU-EV model or in the NEU-EV model, one obtains a decision model based only on entropy and (normalized) variance (EV) or on normalized entropy and (normalized) variance (NEV), respectively. Substituting the normalized entropy by the simple entropy ($NE := E$) reduces the NEU-EV model to the EU-EV model and the NEU-E model to the EU-E model.

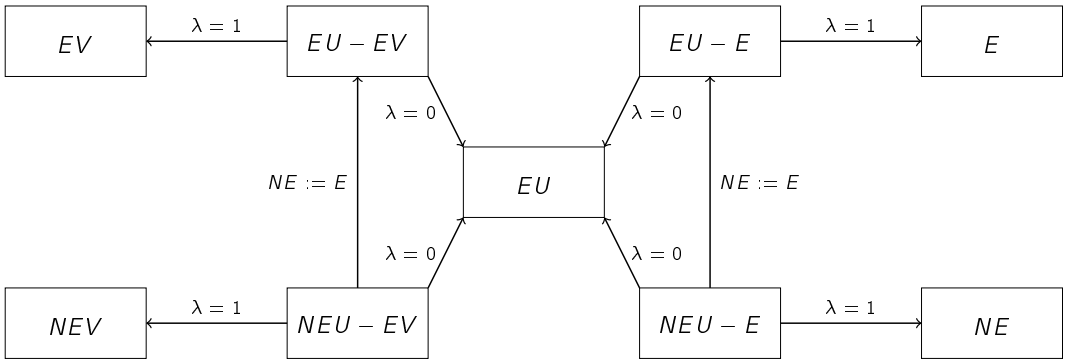


Figure A.2: Implications between the models (E - entropy, NE - normalized entropy, EV - entropy and (normalized) variance, NEV - normalized entropy and (normalized) variance).

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