COMPUTING WITH THE TC-CANONICAL CLASS

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Abstract. The TC-canonical class introduced by Costa and Farber is a very useful tool for studying the maximality of TC, especially in the context of closed manifolds. We survey some calculations where this class has been used to prove the maximality of TC. We also give some non-maximality results which have been established for manifolds with abelian fundamental group.

In many works, the topological complexity of a path-connected CW-complex X was successfully determined by using the cohomological lower bound provided by the zero-divisors cup-length (zcl) together with the dimensional upper bound $\mathsf{TC}(X) \leq 2\dim(X)$ or the better upper bound $\mathsf{TC}(X) \leq 2\mathsf{cat}(X)$ given by the LS-category. However, the classical zero-divisors cup-length with coefficients in a field $\Bbbk$ (or even in an arbitrary commutative ring) sometimes gives a number less than the known upper bound and the best we get with these methods is an interval estimate of $\mathsf{TC}(X)$. Numerous efforts have been made to refine the invariant zcl by potentially better algebraic lower bounds using, for instance, TC-weight techniques or tools from rational homotopy theory (see, for instance, [16], [3]).

When $\pi := \pi_1(X)$ is non-trivial, another useful and powerful tool is the TC-canonical class $\mathfrak{v}_X$ introduced by Costa and Farber in [7]. This class is also a zero-divisor, but with coefficients in a certain $\mathbb{Z}[\pi \times \pi]$-module, and it plays a role for TC similar to that of the Berstein-Schwarz class for the LS-category ([1], [19], [12]). In particular, if $\dim X = n$, Costa and Farber have proved that $\mathsf{TC}(X) = 2n$ (so, TC is maximal) if and only if $\mathfrak{v}_X^{2n} \neq 0$.

In this text we will review algebraic methods developed by Cohen and Vandembroucq in [4], [5] and [6] for studying the maximality of the topological complexity of certain closed manifolds by means of the TC-canonical class. These methods were first used to compute the topological complexity of the Klein bottle and thereby to complete the results of Dranishnikov in [9] on the topological complexity of non-orientable surfaces. In our maximality examples we will here mainly follow the presentation of [5] where it was proved that the topological complexity of the connected sum of at least two copies of $\mathbb{RP}^n$ (with $n \geq 2$) is maximal. Note that this result contrasts drastically with the famous result by Farber, Tabachnikov and Yuzvinsky [15] stating that, for a single copy of $\mathbb{RP}^n$, TC coincides with the smallest integer k for which there exists an immersion of $\mathbb{RP}^n$ in $\mathbb{R}^k$ and is consequently non-maximal. It is worth noting that, as in the example above, the only

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closed surfaces which have non-maximal TC are exactly those with abelian fundamental group (that is, S^2, $S^1 \times S^1$ and $\mathbb{RP}^2$). In [7], Costa and Farber have used the class $\mathfrak{v}_X$ to study the maximality of $\mathsf{TC}(X)$ when X is a CW-complex with fundamental group $\mathbb{Z}_2$ or $\mathbb{Z}_3$. They have, in particular, proved that when $\pi_1(X) = \mathbb{Z}_2$, we always have $\mathsf{TC}(X) < 2\dim(X)$. In Section 4, we present some non-maximality results from [6] where the study of Costa and Farber was extended to manifolds with abelian fundamental group. However, as shown in [7] for general CW-complexes and in [6] for manifolds, it is in general not true that the commutativity of $\pi_1(X)$ implies $\mathsf{TC}(X) < 2\dim(X)$ and we finish with an example illustrating this fact in the context of manifolds.

1. TC-canonical class

In this section we recall some elements about the TC-canonical class. As this class is a zero-divisor in cohomology with local coefficients, we start with some preliminaries on this notion. We here set some notations and conventions which will be used throughout this text.

For a discrete group π (with unit 1), we denote by $\bar{a}$ the inverse of an element $a \in \pi$ and by $\mathbb{Z}[\pi]$ the associated integral group ring. Recall that an element of $\mathbb{Z}[\pi]$ is a finite sum $\sum n_i a_i$ where $n_i \in \mathbb{Z}$ and $a_i \in \pi$. All our $\mathbb{Z}[\pi]$-modules will be left modules. If Γ is a $\mathbb{Z}[\pi]$-module, we denote by Γ_π the coinvariants of Γ,

$$\Gamma_\pi = \Gamma / \mathrm{span}_{\mathbb{Z}}\{z - a \cdot z \,|\, z \in \Gamma, a \in \pi\}.$$

For (left) $\mathbb{Z}[\pi]$-modules Γ and Γ', the tensor product $\Gamma \otimes \Gamma' = \Gamma \otimes_{\mathbb{Z}} \Gamma'$ is a left $\mathbb{Z}[\pi]$-module via the diagonal action of π, and we denote by $\Gamma \otimes_\pi \Gamma'$ the coinvariants of $\Gamma \otimes \Gamma'$. Note that the projections $\Gamma \to \Gamma_\pi$ and $\Gamma' \to \Gamma'_\pi$ induce a projection $\Gamma \otimes_\pi \Gamma' \to \Gamma_\pi \otimes \Gamma'_\pi$.

1.1. **Zero-divisors with local coefficients.** Let X be a path-connected CW-complex with base point x_0 and let $\pi = \pi_1(X, x_0)$ be its fundamental group. We recall that a local system of coefficients over X can be seen as a $\mathbb{Z}[\pi]$-module Γ and that the (co)homology groups of X with coefficients in Γ can be described as follows (see [20, Chap. VI] and [8, Chap. V] as general references). We consider the universal cover $\widetilde{X}$ of X with $\tilde{x}_0$ a lift of x_0 and look at the singular chain complex $C_*(\widetilde{X})$ as a free $\mathbb{Z}[\pi]$-module. The homology $H_*(X;\Gamma)$ is the homology of the $\mathbb{Z}$-chain complex $C_*(\widetilde{X}) \otimes_\pi \Gamma$ given with the differential $\partial \otimes \mathrm{id}$ (where ∂ is the differential of $C_*(\widetilde{X})$). In particular, we have

$$H_0(X;\Gamma) = \Gamma_\pi.$$

For the cohomology, we consider the $\mathbb{Z}$-cochain complex $(\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_*(\widetilde{X}), \Gamma), \delta)$, where Γ is viewed as a $\mathbb{Z}[\pi]$-chain complex concentrated in degree 0 and δ is given by $\delta\varphi = (-1)^{|\varphi|+1}\varphi\partial$. As usual $|\varphi|$ denotes the degree of φ. The cohomology groups of X with coefficients in Γ can then be described as

$$H^*(X;\Gamma) = H(\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_*(\widetilde{X}), \Gamma)).$$

If Γ and Γ' are two $\mathbb{Z}[\pi]$-modules, the cup product of classes $\mathfrak{u} \in H^*(X;\Gamma)$ and $\mathfrak{u}' \in H^*(X;\Gamma')$ is an element of $H^*(X;\Gamma \otimes \Gamma')$ where $\Gamma \otimes \Gamma'$ is given with the

diagonal action of π. We also note that $H^1(X;\Gamma)$ can be identified with the set of crossed homomorphisms $u : \pi \to \Gamma$ (that is, $u(ab) = u(a) + au(b)$ for $a, b \in \pi$) modulo the principal homomorphisms (that is, u is given by $u(a) = az - z$ for some $z \in \Gamma$). Let us quickly describe the class $\mathfrak{u}$ associated to a crossed homomorphism $u : \pi \to \Gamma$. As X is path-connected, we can (as in [20, Theorem VI(3.3)]) consider only the singular simplexes $\Delta^k \to X$ of X which map all the vertices of Δ^k to x_0. Accordingly, we can work with a sub-complex of $C_*(\widetilde{X})$ in which $C_0(\widetilde{X})$ is substituted by $\mathbb{Z}[\pi] \cdot \tilde{x}_0$ and $C_1(\widetilde{X})$ is substituted by the free $\mathbb{Z}[\pi]$-module S_1 with basis $\{\tilde{\sigma}\}$ where, for any 1-simplex $\sigma : \Delta^1 \to X$ of X, $\tilde{\sigma} : \Delta^1 \to \widetilde{X}$ is the lifting of σ satisfying $\tilde{\sigma}(1,0) = \tilde{x}_0$. We can then consider the $\mathbb{Z}[\pi]$-cochain $S_1 \to \Gamma$ defined by $\tilde{\sigma} \to u([\sigma])$. Since u is a crossed homomorphism, this cochain is a cocycle and its class $\mathfrak{u}$ is the cohomology class associated to u.

If Γ is a system of coefficients over X and $f : (X', x'_0) \to (X, x_0)$ is a pointed continuous map, then we denote by $f^*(\Gamma)$ the local system on X' induced by f, that is, the abelian group Γ with the action of $\pi_1(X', x'_0)$ given by $a \cdot z = \pi_1(f)(a) \cdot z$ for $z \in \Gamma$ and $a \in \pi_1(X', x'_0)$. We then have induced morphisms $f_* : H_*(X'; f^*(\Gamma)) \to H_*(X;\Gamma)$ and $f^* : H^*(X;\Gamma) \to H^*(X'; f^*(\Gamma))$. In what follows we will suppress the base points from the notation and, when several spaces are involved, we will often write π_X instead of $\pi_1(X)$.

We now come to the notion of zero-divisor with local coefficients. Let Γ be a local coefficient system on $X \times X$ and let $\Delta^*(\Gamma)$ be the local system on X induced by the diagonal map $\Delta\colon X \to X \times X$. A cohomology class $\mathfrak{u} \in H^*(X \times X;\Gamma)$ is called a *zero-divisor* if

$$\Delta^*(\mathfrak{u}) = 0 \text{ in } H^*(X; \Delta^*(\Gamma)).$$

Proposition 1. [14, Corollary 4.40] *If the cup product of n zero-divisors $\mathfrak{u}_i \in H^*(X \times X;\Gamma_i)$ is non-zero in $H^*(X \times X;\Gamma_1 \otimes \cdots \otimes \Gamma_n)$, then $\mathsf{TC}(X) \geq n$.*

Note that, if we take coefficients in a field $\Bbbk$ with trivial action of $\pi_1(X \times X)$, then the zero-divisors can be identified with the elements of the kernel of the cup product $\cup\colon H^*(X;\Bbbk) \otimes H^*(X;\Bbbk) \to H^*(X;\Bbbk)$, and we recover the lower bound of TC given by the classical zero-divisors cup-length of $H^*(X;\Bbbk)$.

1.2. **TC-canonical class and Costa-Farber's Theorem.** We now describe the TC-*canonical class* $\mathfrak{v}_X$ associated to a path-connected CW-complex X.

Let $\pi = \pi_X$ be the fundamental group of X. Let $I(\pi) = \ker(\varepsilon\colon \mathbb{Z}[\pi] \to \mathbb{Z})$ be the augmentation ideal. The elements of $I(\pi)$ are finite sums $\sum n_i a_i$ where $n_i \in \mathbb{Z}$, $a_i \in \pi$ and $\sum n_i = 0$. Note that $I(\pi)$ and $\mathbb{Z}[\pi]$ are (left) $\mathbb{Z}[\pi]$- and $\mathbb{Z}[\pi \times \pi]$-module through the actions given by:

$$a \cdot \sum n_i a_i = \sum n_i a a_i \quad (a,b) \cdot \sum n_i a_i = \sum n_i (a a_i \bar{b}).$$

Here $n_i \in \mathbb{Z}$, $a, b, a_i \in \pi$ and recall that $\bar{a}$ denotes the inverse of a. The ideal $I(\pi)$ can then be regarded as a system of local coefficients over both X and $X \times X$ and we will use the notations $\mathrm{I}[\pi]$ and $\mathbb{I}[\pi]$ to distinguish the two structures.

In [7], Costa and Farber defined the class $\mathfrak{v} = \mathfrak{v}_X \in H^1(X \times X; \mathbb{I}[\pi])$ as the cohomology class induced by the crossed homomorphism

$$v = v_X : \pi \times \pi \to \mathbb{I}[\pi],\ (a, b) \mapsto a\bar{b} - 1.$$

Alternatively, by [7, Lemma 5], $\mathfrak{v}_X$ can be seen as the image of $1 \in H_0(X \times X; \mathbb{Z})$ by the connecting homomorphism

$$H_0(X \times X; \mathbb{Z}) \to H^1(X \times X; \mathbb{I}[\pi])$$

in the long exact sequence associated to the exact sequence of coefficients over $X \times X$

$$0 \to \mathbb{I}[\pi] \to \mathbb{Z}[\pi] \xrightarrow{\varepsilon} \mathbb{Z} \to 0.$$

Here the action of $\pi \times \pi$ on $\mathbb{Z}[\pi]$ is as described before and the action on $\mathbb{Z}$ is trivial.

A fundamental property of the class $\mathfrak{v}$ is to be a zero-divisor (with coefficients in $\mathbb{I}[\pi]$). Indeed, $\Delta^*(\mathfrak{v})$ is the class in $H^1(X; \Delta^*(\mathbb{I}[\pi]))$ corresponding to the crossed homomorphism

$$\pi \xrightarrow{\Delta} \pi \times \pi \xrightarrow{v_X} I(\pi),\ a \mapsto a\bar{a} - 1 = 0.$$

For $k \geq 1$, we denote by $\mathbb{I}^k[\pi]$ the tensor product of k copies of $I(\pi)$ with the diagonal action of $\pi \times \pi$. It follows from Proposition 1 that, if $\mathfrak{v}_X^k \in H^k(X \times X; \mathbb{I}^k[\pi])$ does not vanish, then $\mathsf{TC}(X) \geq k$. The following theorem of Costa and Farber [7] shows the crucial role played by $\mathfrak{v}_X$ in the study of the maximality of TC.

Theorem 2. *(*[7, Theorem 7]*) Suppose that X is a CW-complex of dimension $n \geq 2$. Then $\mathsf{TC}(X) = 2n$ if and only if the $2n$-th power of $\mathfrak{v}_X$ does not vanish:*

$$\mathsf{TC}(X) = 2n \iff \mathfrak{v}_X^{2n} \neq 0 \text{ in } H^{2n}(X \times X; \mathbb{I}^{2n}[\pi]).$$

Recall that $\mathsf{TC}(X)$ can be seen as the sectional category of the fibration $ev_{0,1} : X^{[0,1]} \to X \times X$, $\lambda \mapsto (\lambda(0), \lambda(1))$. The proof of Theorem 2 relies essentially on the fact that the class $\mathfrak{v}_X$ can be identified with the homological obstruction (in the sense of [19]) for the fibration $ev_{0,1}$ to have a section over the 1-skeleton of $X \times X$.

Costa and Farber used Theorem 2 to study the topological complexity of CW-complexes with fundamental group equal to $\mathbb{Z}_2$ or $\mathbb{Z}_3$. In particular, they proved:

Theorem 3. *(*[7, Theorem 1]*) Suppose that X is a path-connected CW-complex of dimension $n \geq 2$ such that $\pi_1(X) = \mathbb{Z}_2$. Then $\mathsf{TC}(X) < 2n$.*

The proof is based on an explicit understanding in this particular case of the class $\mathfrak{v}_X^2$ leading to the equality $\mathfrak{v}_X^{2n} = 0$. However, such a direct approach to the class $\mathfrak{v}_X$ and its powers seems to be very difficult in general. In the next section, we will see an alternative approach consisting in evaluating the class $\mathfrak{v}^{2n}$ on suitable homology classes. We will actually mainly restrict ourselves to closed manifolds and consider evaluation on the fundamental class.

As mentioned before, the TC-canonical class is an analogue for TC of the Berstein-Schwarz class $\mathfrak{b} = \mathfrak{b}_X \in H^1(X; \mathrm{I}[\pi])$, which can be seen as the class associated to the crossed homomorphism

$$b_X : \pi \to \mathrm{I}[\pi],\ a \mapsto a - 1.$$

In this analogy, Theorem 2 can be seen as a version for TC of the following result due to Berstein [1] and Schwarz [19] for $n \geq 3$ and to Dranisknikov and Rudyak [12] for $n \geq 2$.

Theorem 4 ([1], [19], [12])**.** *Suppose that* $\dim(X) = n \geq 2$. *Then* $\mathsf{cat}(X) = n$ *if and only if* $\mathfrak{b}_X^n \in H^n(X; \mathrm{I}^n[\pi])$ *does not vanish.*

Let $B\pi$ be the classifying space of the group π. The classes $\mathfrak{b}_X$ and $\mathfrak{v}_X$ can be expressed through a classifying map of the universal cover of X, that is, a map $\gamma : X \to B\pi$ inducing an isomorphism of fundamental groups. Indeed, through isomorphisms $\mathrm{I}[\pi] \cong \gamma^*\mathrm{I}[\pi]$ and $\mathbb{I}[\pi] \cong (\gamma \times \gamma)^*\mathbb{I}[\pi]$, we have the following identifications:

$$\mathfrak{b}_X = \gamma^*\mathfrak{b}_\pi \qquad \mathfrak{v}_X = (\gamma \times \gamma)^*\mathfrak{v}_\pi$$

where $\mathfrak{b}_\pi := \mathfrak{b}_{B\pi}$ and $\mathfrak{v}_\pi := \mathfrak{v}_{B\pi}$. By using the usual identification between the cohomology groups of the classifying space of a discrete group and the cohomology groups of the group itself we have

$$\mathfrak{b}_\pi \in H^1(\pi; \mathrm{I}[\pi]) \qquad \mathfrak{v}_\pi \in H^1(\pi \times \pi; \mathbb{I}[\pi]).$$

An important breakup in the parallel between the Berstein-Schwarz class and the TC-canonical class is that, unlike the class $\mathfrak{b}_\pi$, the class $\mathfrak{v}_\pi$ is not universal. The universality of the class $\mathfrak{b}_\pi$ means that, for any class $\mathfrak{c} \in H^k(\pi; \Gamma)$ there exists a morphism of $\mathbb{Z}[\pi]$-module $\varphi : \mathrm{I}^k[\pi] \to \Gamma$ such that $\mathfrak{c} = \varphi_*(\mathfrak{b}_\pi^k)$ (see [12]). As discussed in [17], the analogous property for $\mathfrak{v}_\pi$ is not true. The study in [17] of the classes which can be expressed from powers of $\mathfrak{v}_\pi$ has led to important progress in the understanding of $\mathsf{TC}(B\pi)$.

1.3. **TC-canonical class and fundamental class.** In this section, we reformulate Theorem 2 for closed manifolds.

Let us first recall that if X be a path-connected CW-complex with fundamental group π_X and if Γ and J are $\mathbb{Z}[\pi_X]$-modules then the evaluation of $\mathfrak{u} \in H^k(X; J)$ on $\mathbf{c} \in H_k(X; \Gamma)$ is given by

$$\mathfrak{u}(\mathbf{c}) := \mathbf{c} \cap \mathfrak{u} \in H_0(X; J \otimes \Gamma) = J \otimes_{\pi_X} \Gamma$$

where $\cap$ is the cap product

$$H_k(X; \Gamma) \otimes H^k(X; J) \xrightarrow{\cap} I \otimes_{\pi_X} \Gamma.$$

Given a map $f : X' \to X$ and classes $\mathbf{c} \in H_k(X'; f^*(\Gamma))$, $\mathfrak{u} \in H^k(X; J)$, we will frequently use the formula

$$f_*(\mathbf{c} \cap f^*(\mathfrak{u})) = f_*(\mathbf{c}) \cap \mathfrak{u}$$

coming from the naturality of the cap product and represented by the following cap product diagram

$$\begin{array}{ccc} H_k(X'; f^*(\Gamma)) \otimes H^k(X'; f^*(J)) & \xrightarrow{\cap} & H_0(X'; f^*(J \otimes \Gamma)) = f^*(J) \otimes_{\pi_Y} f^*(\Gamma) \\ \downarrow{\scriptstyle f_*} \quad \uparrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f_*} \\ H_k(X; \Gamma) \otimes H^k(X; J) & \xrightarrow{\cap} & H_0(X; J \otimes \Gamma) = J \otimes_{\pi_X} \Gamma. \end{array}$$

If f induces an epimorphism of fundamental groups then the morphism in the rightmost column is an isomorphism. In such a case, we have

$$f^*(\mathfrak{u})(\mathbf{c}) = 0 \Leftrightarrow \mathfrak{u}(f_*(\mathbf{c})) = 0.$$

The evaluation of a cohomology class on a suitable homology class may permit us to see that the cohomology class does not vanish. For instance, if $\dim(X) = n$ and if, for some $\mathbb{Z}[\pi \times \pi]$-module Γ, there exists a class $\mathbf{c} \in H_{2n}(X \times X; \Gamma)$ such that $\mathfrak{v}^{2n}(\mathbf{c}) \in \mathbb{I}^{2n}[\pi] \otimes_{\pi\times\pi} \Gamma$ does not vanish, then we can assert that $\mathsf{TC}(X) = 2n$. When $X = M$ is a closed manifold, we can use Poincaré duality to say more.

Let M be a connected closed manifold of dimension $n \geq 2$ and let $\pi = \pi_1(M)$. We denote by $\widetilde{\mathbb{Z}} = \widetilde{\mathbb{Z}}_M$ the $\mathbb{Z}[\pi]$ orientation module of M, that is, $\widetilde{\mathbb{Z}}$ represents the integers with an action of π given by $a \cdot t = \omega(a)t$ where $a \in \pi$, t is an integer, and $\omega = \omega_M : \pi \to \{\pm 1\}$ is the homomorphism determined by the first Stiefel-Whitney class of M. If the manifold is orientable, the action is trivial and coefficients in $\widetilde{\mathbb{Z}}$ correspond to the usual coefficients in $\mathbb{Z}$. When the manifold is non-orientable, the action is necessarily non-trivial. For instance, the action of $\pi_1(\mathbb{RP}^2)$, written multiplicatively as $\langle a \mid a^2 = 1\rangle$, on $\widetilde{\mathbb{Z}}_{\mathbb{RP}^2}$ is given by $a \cdot t = -t$. We will also use that the orientation module of a product $M \times N$ is given by the action $(a, b) \cdot t = \omega_M(a)\omega_N(b)t$ for $(a, b) \in \pi_1(M) \times \pi_1(N)$.

Coefficients in $\widetilde{\mathbb{Z}}$ ensure the existence of an isomorphism $H_n(M; \widetilde{\mathbb{Z}}) \cong \mathbb{Z}$ and consequently of a generator $[M] \in H_n(M; \widetilde{\mathbb{Z}}) \cong \mathbb{Z}$ referred as fundamental class of M. When M is non-orientable, we will sometimes refer to $[M]$ as the twisted fundamental class of M and use the notation $[M]_{\widetilde{\mathbb{Z}}}$.

Using Poincaré duality, Theorem 2 restricted to manifolds can be reformulated as follows:

Corollary 5. *Let M be a connected closed manifold of dimension $n \geq 2$ and let $\pi = \pi_1(M)$. We have $\mathsf{TC}(M) = 2n$ if and only if the element*

$$\mathfrak{v}_M^{2n}([M \times M]) = [M \times M] \cap \mathfrak{v}_M^{2n} \in H_0(M \times M; \mathbb{I}^{2n}[\pi] \otimes \widetilde{\mathbb{Z}}) = \mathbb{I}^{2n}[\pi] \otimes_{\pi\times\pi} \widetilde{\mathbb{Z}}$$

does not vanish.

Note that using the Künneth formula, the fundamental class of $M\times M$, $[M\times M] \in H_{2n}(M \times M; \widetilde{\mathbb{Z}}) = \mathbb{Z}$ can be identified with the cross product $[M] \times [M]$.

We will now give another reformulation of Corollary 5 by using a classifying map $\gamma : M \to B\pi$. The orientation module $\widetilde{\mathbb{Z}}$ of M can be considered as a local system of coefficients over $B\pi$ through the π action induced by the isomorphism $\pi_1(\gamma)$. In this way, γ is compatible with the action and

$$\gamma_* : H_*(M; \widetilde{\mathbb{Z}}) \to H_*(B\pi; \widetilde{\mathbb{Z}}) = H_*(\pi; \widetilde{\mathbb{Z}})$$

is well-defined. The same can be done for the products $M \times M$ and $\gamma \times \gamma$. Then, through the identification $\mathfrak{v}_X = (\gamma \times \gamma)^* \mathfrak{v}_\pi$ and the naturality of the cap product

given by the diagram

$$\begin{array}{ccc} H_{2n}(M\times M;\widetilde{\mathbb{Z}})\otimes H^{2n}(M\times M;\mathbb{I}^{2n}[\pi]) & \xrightarrow{\cap} & \mathbb{I}^{2n}[\pi]\otimes_{\pi\times\pi}\widetilde{\mathbb{Z}} \\ \downarrow{\scriptstyle (\gamma\times\gamma)_*} \quad \uparrow{\scriptstyle (\gamma\times\gamma)^*} & & \downarrow{\scriptstyle =} \\ H_{2n}(B\pi\times B\pi;\widetilde{\mathbb{Z}})\otimes H^{2n}(B\pi\times B\pi;\mathbb{I}^{2n}[\pi]) & \xrightarrow{\cap} & \mathbb{I}^{2n}[\pi]\otimes_{\pi\times\pi}\widetilde{\mathbb{Z}} \end{array}$$

we see that $\mathsf{TC}(M) = 2n$ if and only if $(\gamma\times\gamma)_*[M\times M]\cap \mathfrak{v}_\pi^{2n}$ does not vanish. In what follows, we will use the notation

$$\mathbf{m} = \gamma_*([M]) \in H_n(\pi;\widetilde{\mathbb{Z}}).$$

We then have

$$\mathbf{m}\times\mathbf{m} = \gamma_*([M])\times\gamma_*([M]) = (\gamma\times\gamma)_*([M\times M]) \in H_{2n}(\pi\times\pi;\widetilde{\mathbb{Z}}).$$

and obtain the following reformulation of Corollary 5:

Corollary 6. *Let M be a connected closed manifold of dimension $n \geq 2$ with $\pi = \pi_1(M)$. Let $\gamma : M \to B\pi$ be a classifying map and let $\mathbf{m} = \gamma_*([M]) \in H_n(\pi;\widetilde{\mathbb{Z}})$. We have*

$$\mathsf{TC}(M) = 2n \Leftrightarrow \mathfrak{v}_\pi^{2n}(\mathbf{m}\times\mathbf{m}) \neq 0.$$

Note that a similar reformulation of Theorem 4 can be done. Namely, with the notation above, we have

$$\mathsf{cat}(M) = n \Leftrightarrow \mathfrak{b}_\pi^n(\mathbf{m}) \neq 0. \tag{1}$$

In particular, the class $\mathbf{m}$ is necessarily non-trivial if $\mathsf{cat}(M) = \dim(M)$.

2. Computing with the bar resolution – a first example

In Corollary 6 above, we have seen that the study of the maximality of TC for a closed manifold of dimension $n \geq 2$ can be reduced to the study of the element

$$\mathfrak{v}_\pi^{2n}(\mathbf{m}\times\mathbf{m}) \in \mathbb{I}^{2n}[\pi]\otimes_{\pi\times\pi}\widetilde{\mathbb{Z}} \tag{2}$$

where $\pi = \pi_1(M)$, $\mathfrak{v}_\pi := \mathfrak{v}_{B\pi}$, $\gamma : M \to B\pi$ is a classifying map and $\mathbf{m} = \gamma_*([M]) \in H_n(B\pi;\widetilde{\mathbb{Z}}) = H_n(\pi;\widetilde{\mathbb{Z}})$ is the image through γ_* of the (possibly twisted) fundamental class of M.

The (co)homology groups of a discrete group π can be expressed through a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}[\pi]$-module. In this section we will see how we can use the normalized bar resolution to calculate the element (2). We will illustrate the method to recover the well-known fact that $\mathsf{TC}(\mathbb{RP}^2) \leq 3$ (see [15] or Theorem 2 above). We refer to Brown [2] as a standard reference for cohomology of groups.

2.1. **Bar resolution.** Let π be a discrete group and let $\bar{\pi} = \{a \in \pi \mid a \neq 1\}$. Recall that the normalized bar resolution $\bar{B}_*(\pi)$ of $\mathbb{Z}$ as a trivial $\mathbb{Z}[pi]$-module is given by the following augmented $\mathbb{Z}[\pi]$-chain complex

$$\cdots \longrightarrow \bar{B}_n(\pi) \xrightarrow{\partial_n} \cdots \longrightarrow \bar{B}_1(\pi) \xrightarrow{\partial_1} \bar{B}_0(\pi) = \mathbb{Z}[\pi] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

where $\bar{B}_n(\pi)$ is the free $\mathbb{Z}[\pi]$-module with basis $\{[a_1|\cdots|a_n], (a_1, \ldots, a_n) \in \bar{\pi}^n\}$ and ∂_n is the $\mathbb{Z}[\pi]$ morphism given by

$$\partial_n([a_1|\cdots|a_n]) = a_1 \cdot [a_2|\cdots|a_n] + \sum_{i=1}^{n-1}(-1)^i[a_1|\cdots|a_{i-1}|a_ia_{i+1}|a_{i+2}|\cdots|a_n] + (-1)^n[a_1|\cdots|a_{n-1}]$$

(with $[g_1|\cdots|g_k] = 0$ if $g_i = 1$ for some i). In particular, $\partial_1[a] = (a-1)[\,]$ where $[\,]$ is the generator of $B_0(\pi)$.

As a $\mathbb{Z}[\pi]$-chain complex, $\bar{B}_*(\pi)$ is equivalent to the singular chain complex of the universal cover of $B\pi$. The homology of π (or of the space $B\pi$) with coefficients in a $\mathbb{Z}[\pi]$-module Γ is then the homology of the chain complex $\bar{B}_*(\pi;\Gamma) := \bar{B}_*(\pi)\otimes_\pi \Gamma$ (with differential $\partial \otimes$ id). In what follows, we will consider homology with coefficients in either a system $\Gamma = \widetilde{\mathbb{Z}}$ corresponding to the orientation module of a manifold or one of the systems $\mathbb{Z}$ or $\mathbb{Z}_2$ with trivial action of π.

Example 7. *Consider the cyclic group $\pi = \langle a \mid a^2 = 1\rangle$ and the system of coefficients $\Gamma = \widetilde{\mathbb{Z}}$ over $B\pi = \mathbb{RP}^\infty$ given by $a \cdot t = -t$ for $t \in \mathbb{Z}$. The normalized bar resolution gives the $\mathbb{Z}$-chain complex $\bar{B}_*(\pi;\widetilde{\mathbb{Z}}) := \bar{B}_*(\pi) \otimes_\pi \widetilde{\mathbb{Z}}$ with a single generator of degree n given by $\alpha_n \otimes_\pi 1$ where $\alpha_n = [a|\cdots a|a] \in \bar{B}_n(\pi)$. The differential is given by*

$$\begin{aligned}\partial\alpha_n \otimes_\pi 1 &= a\alpha_{n-1} \otimes_\pi 1 + (-1)^n\alpha_{n-1} \otimes_\pi 1\\ &= \alpha_{n-1} \otimes_\pi a \cdot 1 + (-1)^n\alpha_{n-1} \otimes_\pi 1\\ &= (-1 + (-1)^n)\alpha_{n-1} \otimes_\pi 1.\end{aligned}$$

This gives the homology

$$H_n(\pi;\widetilde{\mathbb{Z}}) = H_n(\mathbb{RP}^\infty;\widetilde{\mathbb{Z}}) = \begin{cases}\mathbb{Z}_2 & \text{if } n \text{ is even,}\\ 0 & \text{if } n \text{ is odd.}\end{cases}$$

In particular the image of the (twisted) fundamental class $[\mathbb{RP}_2]_{\widetilde{\mathbb{Z}}} \in H_2(\mathbb{RP}_2;\widetilde{\mathbb{Z}})$ through the homomorphism induced by the inclusion $\gamma : \mathbb{RP}_2 \hookrightarrow \mathbb{RP}^\infty = B\pi$, which is a classifying map, is given by

$$\mathbf{m} = \gamma_*([\mathbb{RP}_2]_{\widetilde{\mathbb{Z}}}) = [a|a] \otimes_\pi 1 \in H_2(\pi;\widetilde{\mathbb{Z}}) = \mathbb{Z}_2.$$

Note that we do not distinguish between the cycle $[a|a]\otimes_\pi 1$ and its homology class. Note also that in this particular case ($\pi = \langle a|a^2 = 1\rangle$) the reduced bar resolution $\bar{B}_*(\pi)$ is isomorphic to the classical resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}[\pi]$-module given by

$$\cdots \longrightarrow \mathbb{Z}[\pi] \xrightarrow{a+1} \mathbb{Z}[\pi] \xrightarrow{a-1} \mathbb{Z}[\pi] \xrightarrow{a+1} \mathbb{Z}[\pi] \xrightarrow{a-1} \mathbb{Z}[\pi] \xrightarrow{\varepsilon} \mathbb{Z}$$

2.2. **Cross product.** In order to calculate the cross product of two homology classes, we use the Eilenberg-Zilber map

$$EZ\colon \bar{B}_*(\pi)\otimes\bar{B}_*(\pi)\longrightarrow\bar{B}_*(\pi\times\pi),$$

which is the $\mathbb{Z}[\pi\times\pi]\cong\mathbb{Z}[\pi]\otimes\mathbb{Z}[\pi]$ morphism given by

$$\begin{array}{rcl} EZ_n\colon \bigoplus_{i=0}^{n}\bar{B}_i(\pi)\otimes\bar{B}_{n-i}(\pi) & \to & \bar{B}_n(\pi\times\pi)\\ [a_1|\cdots|a_i]\otimes[b_{i+1}|\cdots|b_n] & \mapsto & \sum_{\sigma\in\mathcal{S}_{i,n-i}}\operatorname{sgn}(\sigma)[q_{\sigma^{-1}(1)}|\cdots|q_{\sigma^{-1}(n)}]\end{array}$$

where $\mathcal{S}_{i,n-i}$ denotes the set of $(i,n-i)$ shuffles, $\operatorname{sgn}(\sigma)$ is the signature of the shuffle σ, and

$$q_k=\begin{cases}(a_k,1) & \text{if } 1\le k\le i,\\ (1,b_k) & \text{if } i+1\le k\le n.\end{cases}$$

For instance, for $a,b\in\pi$, we have

$$EZ([a|a]\otimes[b|b])=\begin{cases}[a_1|a_1|b_2|b_2]-[a_1|b_2|a_1|b_2]+[a_1|b_2|b_2|a_1]\\ +[b_2|a_1|a_1|b_2]-[b_2|a_1|b_2|a_1]+[b_2|b_2|a_1|a_1]\end{cases}\tag{3}$$

where $a_1=(a,1)$, $a_2=(1,a)$, $b_1=(b,1)$, $b_2=(1,b)$.

Considering $\Gamma=\widetilde{\mathbb{Z}},\mathbb{Z}$ or $\mathbb{Z}_2$, the ring multiplication induces an isomorphism of $\mathbb{Z}[\pi\times\pi]$- module $\Gamma\otimes\Gamma\cong\Gamma$ and the map EZ induces a $\mathbb{Z}$-chain map

$$EZ\colon \bar{B}_*(\pi;\Gamma)\otimes\bar{B}_*(\pi;\Gamma)\longrightarrow\bar{B}_*(\pi\times\pi;\Gamma),\tag{4}$$

yielding the cross product in homology with coefficients in Γ.

Example 8. *As in Example 7 we consider the cyclic group $\pi=\langle a|a^2=1\rangle$ and the class $\mathbf{m}=\gamma_*([\mathbb{RP}_2]_{\widetilde{\mathbb{Z}}})=[a|a]\otimes_\pi 1$. The element*

$$EZ(([a|a]\otimes_\pi 1)\otimes([a|a]\otimes_\pi 1))=EZ([a|a]\otimes[a|a])\otimes_\pi 1$$

is a cycle representing the class $\mathbf{m}\times\mathbf{m}$ in $H_4(\pi\times\pi;\widetilde{\mathbb{Z}})$.

2.3. **Canonical cocycle.** We now represent the TC-canonical class $\mathfrak{v}_\pi=\mathfrak{v}_{B\pi}\in H^1(\pi\times\pi;\mathbb{I}[\pi])=H^1(B\pi\times B\pi;\mathbb{I}[\pi])$ by using the bar resolution. Consider the universal cover $E\pi$ of $B\pi$. As $\bar{B}_*(\pi\times\pi)$ is a $\mathbb{Z}[\pi\times\pi]$-chain complex equivalent to the singular chain complex $C_*(E\pi\times E\pi)$ we can see from the description given in Section 1 that $\mathfrak{v}_\pi$ can be described as the class of the following $\mathbb{Z}[\pi\times\pi]$-cocycle of degree 1:

$$\begin{aligned}\nu_\pi\colon \bar{B}_1(\pi\times\pi)&\to\mathbb{I}[\pi]\\ [(a,b)]&\mapsto a\bar{b}-1\end{aligned}$$

The fact that it is cocycle can here be checked by the following calculation:

$$\begin{aligned}\nu_\pi\partial_2([(a_1,b_1)|(a_2,b_2)])&=\nu_\pi((a_1,b_1)\,[(a_2,b_2)]-[(a_1a_2,b_1b_2)]+[(a_1,b_1)])\\ &=(a_1,b_1)\cdot(a_2\bar{b}_2-1)-(a_1a_2\bar{b}_2\bar{b}_1-1)+a_1\bar{b}_1-1\\ &=a_1(a_2\bar{b}_2-1)\bar{b}_1-a_1a_2\bar{b}_2\bar{b}_1+a_1\bar{b}_1\\ &=0.\end{aligned}$$

As in [4], we have the following explicit expression of the n-th power of $\mathfrak{v}_\pi$:

Lemma 9. *The n-th power of the* TC*-canonical class* $\mathfrak{v}$ *is the class of the cocyle* ν^n *of degree n given by*

$$\begin{aligned} \nu_\pi^n \colon \bar{B}_n(\pi\times\pi) &\to \mathbb{I}^n[\pi] \\ [(a_1,b_1)|\cdots|(a_n,b_n)] &\mapsto (-1)^{n(n-1)/2}\cdot z_1\otimes z_2\otimes\cdots\otimes z_n \end{aligned}$$

where $z_1 = a_1\bar{b}_1 - 1$ *and* $z_i = (a_1\cdots a_{i-1})(a_i\bar{b}_i - 1)(\bar{b}_{i-1}\cdots\bar{b}_1)$ *for* $2\le i\le n$.

The calculation of this expression can be made by using (for $G = \pi\times\pi$) the following Alexander-Whitney diagonal, which is, for a general group G, the $\mathbb{Z}[G]$-chain map given by

$$\begin{aligned} \Delta\colon \bar{B}_*(G) &\to \bar{B}_*(G)\otimes\bar{B}_*(G) \\ [g_1|\cdots|g_n] &\mapsto \sum_{i=0}^{n}[g_1|\cdots|g_i]\otimes g_1\cdots g_i[g_{i+1}|\cdots|g_n] \end{aligned}$$

Here the $\mathbb{Z}[G]$-structure on $\bar{B}_*(G)\otimes\bar{B}_*(G)$ is induced by the diagonal action of G.

If $\varphi\colon \bar{B}_i(G)\to I$ and $\psi\colon \bar{B}_{n-i}(G)\to J$ are $\mathbb{Z}[G]$-cochains of degrees i and $n-i$ respectively, their cup product $\varphi\cup\psi$ is the $\mathbb{Z}[G]$-cochain of degree n given by

$$\varphi\cup\psi\colon \bar{B}_n(G)\xrightarrow{\Delta}(\bar{B}_*(G)\otimes\bar{B}_*(G))_n\xrightarrow{(-1)^{i(n-i)}\varphi\otimes\psi} I\otimes J.$$

Example 10. *Again we consider the cyclic group* $\pi = \langle a \mid a^2 = 1\rangle$ *and the class* $\mathbf{m} = \gamma_*([\mathbb{RP}_2]_{\widetilde{\mathbb{Z}}}) = [a|a]\otimes_\pi 1$. *The element* $\mathfrak{v}_\pi^4(\mathbf{m}\times\mathbf{m})$ *is given by*

$$\mathfrak{v}_\pi^4(\mathbf{m}\times\mathbf{m}) = \nu^4(EZ([a|a]\otimes[a|a]))\otimes_{\pi\times\pi}1\in\mathbb{I}^4[\pi]\otimes_{\pi\times\pi}\widetilde{\mathbb{Z}}$$

Taking into account the formula (2.2), Lemma 9 and the fact that $a^2 = 1$, *we have*

$$\nu^4(EZ([a|a]\otimes[a|a])) = 2(a-1)\otimes(a-1)\otimes(a-1)\otimes(a-1)\ \text{in}\ \mathbb{I}^4[\pi]$$

Calculating in $\mathbb{I}^4[\pi]\otimes_{\pi\times\pi}\widetilde{\mathbb{Z}}$ *we have:*

$$\begin{aligned} &(a-1)\otimes(a-1)\otimes(a-1)\otimes(a-1)\otimes_{\pi\times\pi}1 \\ &= (a,1)\cdot((a-1)\otimes(a-1)\otimes(a-1)\otimes(a-1)\otimes_{\pi\times\pi}1) \\ &= (a-1)\otimes(a-1)\otimes(a-1)\otimes(a-1)\otimes_{\pi\times\pi}(a,1)\cdot 1 \\ &= (a-1)\otimes(a-1)\otimes(a-1)\otimes(a-1)\otimes_{\pi\times\pi}(-1) \\ &= -(a-1)\otimes(a-1)\otimes(a-1)\otimes(a-1)\otimes_{\pi\times\pi}1 \end{aligned}$$

Consequently $\mathfrak{v}_\pi^4(\mathbf{m}\times\mathbf{m}) = 2(a-1)\otimes(a-1)\otimes(a-1)\otimes(a-1)\otimes_\pi 1 = 0$. *As mentioned before, this recovers the well-known fact that* $\mathsf{TC}(\mathbb{RP}^2)\le 3$.

3. Klein bottle and related manifolds

Recall that the Klein bottle K can be seen as the connected sum of two copies of $\mathbb{RP}^2$. The classical estimates of $\mathsf{TC}(K)$ by means of the dimension and of the zero-divisors cup-length over $\mathbb{Z}_2$ gives $3\le\mathsf{TC}(K)\le 4$. In this section we will see that

Theorem 11 ([4], [5]). $\mathsf{TC}(K) = 4$ *and, more generally, for any* $n \geq 2$,

$$\mathsf{TC}(\mathbb{RP}^n \# \mathbb{RP}^n) = 2n.$$

For $g \geq 2$, the non-orientable surface of genus g is the connected sum of g copies of $\mathbb{RP}^2$. Let $\mathcal{P}^n_g = \mathbb{RP}^n \# \cdots \# \mathbb{RP}^n$ be the connected sum of g copies of the real projective space $\mathbb{RP}^n$. As a consequence of Theorem 11 (or, more exactly, of Propositions 13 and 15 below), we will obtain:

Theorem 12 ([4], [5], [9]). *For any* $g \geq 2$, $\mathsf{TC}(N_g) = 4$ *and, more generally, for any* $n \geq 2$, $\mathsf{TC}(\mathcal{P}^n_g) = 2n$.

For $g \geq 4$, $\mathsf{TC}(N_g)$ has been first determined by Dranishnikov in [9] and [10]. His method relies on appropriate decompositions of N_g as a connected sum of a non-orientable surface with an orientable surface of genus > 1. The value $\mathsf{TC}(K) = 4$ has been also recovered by Iwase, Sakai and Tsutaya [18] by using a fibrewise version of categorical weight.

We will here prove Theorem 11 by using the algebraic approach explained in the previous section to see that $\mathfrak{v}^{2n}$ does not vanish. We will mainly use the unified presentation developed in [5] as well as the fact (already used in [4] when $n = 2$) that it is sufficient to evaluate $\mathfrak{v}^{2n}$ on the $\mathbb{Z}_2$-top class of $\mathbb{RP}^n \# \mathbb{RP}^n$, that is on the generator $[(\mathbb{RP}^n \# \mathbb{RP}^n) \times (\mathbb{RP}^n \# \mathbb{RP}^n)]_{\mathbb{Z}_2}$ of $H_{2n}(\mathbb{RP}^n \# \mathbb{RP}^n) \times (\mathbb{RP}^n \# \mathbb{RP}^n); \mathbb{Z}_2)$. Theorem 11 will indeed follow immediately from the following result:

Proposition 13. *For* $n \geq 2$, $\mathfrak{v}^{2n}([(\mathbb{RP}^n \# \mathbb{RP}^n) \times (\mathbb{RP}^n \# \mathbb{RP}^n)]_{\mathbb{Z}_2}) \neq 0$.

We will use the map

$$f : \mathbb{RP}^n \# \mathbb{RP}^n \to \mathbb{RP}^\infty \vee \mathbb{RP}^\infty$$

obtained by composing the pinch map $\mathbb{RP}^n \# \mathbb{RP}^n \to \mathbb{RP}^n \vee \mathbb{RP}^n$ with the inclusion $\mathbb{RP}^n \vee \mathbb{RP}^n \hookrightarrow \mathbb{RP}^\infty \vee \mathbb{RP}^\infty$. Observe that $\mathbb{RP}^\infty \vee \mathbb{RP}^\infty$ can be identified with $B\pi_2$ where

$$\pi_2 := \langle a, b \,|\, a^2 = 1, b^2 = 1 \rangle.$$

For $n \geq 3$, π_2 is exactly the fundamental group of $\mathbb{RP}^n \# \mathbb{RP}^n$ and we look at f as a classifying map such that $\pi_1(f) = \mathrm{Id}$.

For $n = 2$, we use the following presentation of the fundamental group of the Klein bottle $K = \mathbb{RP}^2 \# \mathbb{RP}^2$:

$$\pi_K = \langle a, b \,|\, a^2 b^2 = 1 \rangle.$$

The morphism $\pi_1(f) : \pi_K \to \pi_2$ can then be identified with the epimorphism $\rho : \pi_K \to \pi_2$, $a \mapsto a$, $b \mapsto b$. Since ρ is an epimorphism, the $\pi_K \times \pi_K$ module $(f \times f)^*(\mathbb{I}(\pi_2))$ is isomorphic to the $\pi_K \times \pi_K$ module $\rho_*(\mathbb{I}[\pi_K]) = \{\sum n_i \rho(a_i) \mid a_i \in \pi_K, \sum n_i = 0\}$ where the action is given by $(g, h) \cdot \sum n_i \rho(a_i) = \sum n_i \rho(g) \rho(a_i) \rho(\bar{h})$. We also have the identification

$$\rho_* \mathfrak{v}_K = (f \times f)^*(\mathfrak{v}_{\pi_2}) \in H^1(K \times K; \rho_*(\mathbb{I}[\pi_K])) = H^1(K \times K; (f \times f)^*(\mathbb{I}[\pi_2]))$$

and similarly for the powers of $\mathfrak{v}_K$. In order to establish that $\mathfrak{v}^4_K$ does not vanish it is therefore sufficient to see that

$$\rho_* \mathfrak{v}^4_K = (f \times f)^*(\mathfrak{v}^4_{\pi_2}) \in H^4(K \times K; \rho_*(\mathbb{I}^4[\pi_K]))$$

does not vanish. Since ρ is surjective, we also have an isomorphism

$$\rho_*(\mathbb{I}^4(\pi_K)) \otimes_{\pi_K\times\pi_K} \mathbb{Z}_2 \cong \mathbb{I}^4(\pi_2) \otimes_{\pi_2\times\pi_2} \mathbb{Z}_2.$$

In conclusion of this preliminary discussion, it appears that, using the notations $M = \mathbb{RP}^n\#\mathbb{RP}^n$, $\pi = \pi_1(\mathbb{RP}^n\#\mathbb{RP}^n)$ and $\rho = \pi_1(f)$ for all cases ($n = 2$ or $n > 2$), we can identify the classes

$$[M \times M]_{\mathbb{Z}_2} \cap \rho_* \mathfrak{v}_M^{2n} \quad \text{and} \quad (f \times f)_*([M \times M]_{\mathbb{Z}_2}) \cap \mathfrak{v}_{\pi_2}^{2n}$$

and it is sufficient, for establishing Proposition 13, to prove that

$$\mathfrak{v}_{\pi_2}^{2n}((f \times f)_*([M \times M]_{\mathbb{Z}_2})) \in \mathbb{I}^{2n}(\pi_2) \otimes_{\pi_2\times\pi_2} \mathbb{Z}_2 \tag{5}$$

does not vanish.

We now describe a cycle representing the image of the $\mathbb{Z}_2$ top class of $\mathbb{RP}^n\#\mathbb{RP}^n$ under the map induced by $f : \mathbb{RP}^n\#\mathbb{RP}^n \to B\pi_2 = \mathbb{RP}^\infty \vee \mathbb{RP}^\infty$. Let $\mathbf{a}_i, \mathbf{b}_i$ be the $\mathbb{Z}_2$-homology classes (with $\mathbf{a}_0 = \mathbf{b}_0$) corresponding to the two branches of $\mathbb{RP}^\infty \vee \mathbb{RP}^\infty$. As the two projections $\mathbb{RP}^n\#\mathbb{RP}^n \to \mathbb{RP}^n$ each induce an isomorphism $H_n(\mathbb{RP}^n\#\mathbb{RP}^n; \mathbb{Z}_2) \to H_n(\mathbb{RP}^n; \mathbb{Z}_2)$, the element $f([\mathbb{RP}^n\#\mathbb{RP}^n]_{\mathbb{Z}_2})$ can be identified with $\mathbf{c}_n = \mathbf{a}_n + \mathbf{b}_n$ of $H_n(B\pi_2; \mathbb{Z}_2)$.

Considering the normalized bar resolution, the classes $\mathbf{a}_i, \mathbf{b}_i$ are represented by the following cycles of $B_i(\pi_2; \mathbb{Z}_2)$:

$$\alpha_i = [a|a|\cdots|a], \qquad \beta_i = [b|b|\cdots|b].$$

Note that, since the action of $\pi_2\times\pi_2$ on the coefficients $\mathbb{Z}_2$ is trivial, we have omitted the part "$\otimes_{\pi_2\times\pi_2} 1$" from the notation keeping in mind that we are working over $\mathbb{Z}_2$. We will also use the notation $\mathbb{I}[\pi_2; \mathbb{Z}_2] := \mathbb{I}[\pi_2] \otimes \mathbb{Z}_2$ and identify $\mathbb{I}^4[\pi_2] \otimes_{\pi_2\times\pi_2} \mathbb{Z}_2$ with the coinvariants of $\mathbb{I}^4[\pi_2; \mathbb{Z}_2]$ with respect to the diagonal action of $\pi_2 \times \pi_2$.

With the notation above, the class (5) is given by $\mathfrak{v}^{2n}(\mathbf{c}_n \times \mathbf{c}_n)$ and we will use the following decomposition:

$$\mathbf{c}_n \times \mathbf{c}_n = \mathbf{a}_n \times \mathbf{a}_n + \mathbf{a}_n \times \mathbf{b}_n + \mathbf{b}_n \times \mathbf{a}_n + \mathbf{b}_n \times \mathbf{b}_n \in H_{2n}(\pi_2 \times \pi_2; \mathbb{Z}_2)). \tag{6}$$

For $n = 2$, we will see that $\mathfrak{v}^4(\mathbf{c}_2 \times \mathbf{c}_2) \neq 0$ in $\mathbb{I}^4[\pi_2] \otimes_{\pi_2\times\pi_2} \mathbb{Z}_2$ by showing that:

Lemma 14. *We have:*

(i) $\mathfrak{v}^4(\mathbf{a}_2 \times \mathbf{a}_2) = \mathfrak{v}^4(\mathbf{b}_2 \times \mathbf{b}_2) = 0$
(ii) $\mathfrak{v}^4(\mathbf{a}_2 \times \mathbf{b}_2) + \mathfrak{v}^4(\mathbf{b}_2 \times \mathbf{a}_2) \neq 0.$

Proof. As seen in Example 10, we have:

$$\nu^4(EZ([a|a] \otimes [a|a])) = 2(a-1) \otimes (a-1) \otimes (a-1) \otimes (a-1) \text{ in } \mathbb{I}^4[\pi_2].$$

The element $\mathfrak{v}^4(\mathbf{a}_2\times\mathbf{a}_2)$ is obtained by considering $\nu^4(EZ([a|a]\otimes[a|a]))$ in $\mathbb{I}^4[\pi_2; \mathbb{Z}_2]$ and then passing to the coinvariants with respect to the diagonal action of $\pi_2 \times \pi_2$. Tensoring by $\mathbb{Z}_2$ we directly see that $\mathfrak{v}^4(\mathbf{a}_2 \times \mathbf{a}_2) = 0$. The calculation of $\mathfrak{v}^4(\mathbf{b}_2 \times \mathbf{b}_2)$ is similar and the point (i) follows.

In order to calculate $\mathfrak{v}^4(\mathbf{a}_2 \times \mathbf{b}_2)$, we use the expression (3) and compute the element $\nu^4(EZ([a|a] \otimes [b|b]))$ in $\mathbb{I}^4(\pi_2; \mathbb{Z}_2)$, which permits us in particular to omit the signs of the shuffles. We get

$$\nu^4(EZ([a|a] \otimes [b|b])) = \begin{cases} (a-1) \otimes (1-a) \otimes (b-1) \otimes (1-b) \\ +(a-1) \otimes a(b-1) \otimes (1-a)b \otimes (1-b) \\ +(a-1) \otimes a(b-1) \otimes a(1-b) \otimes (1-a) \\ +(b-1) \otimes (a-1)b \otimes (1-a)b \otimes (1-b) \\ +(b-1) \otimes (a-1)b \otimes a(1-b) \otimes (1-a) \\ +(b-1) \otimes (1-b) \otimes (a-1) \otimes (1-a) \end{cases} \tag{7}$$

Let $A = \langle a \,|\, a^2 = 1 \rangle$. We have $\mathbb{I}[A; \mathbb{Z}_2] \cong \mathbb{Z}_2(a-1)$. Consider the projection $\mathbb{I}[\pi_2; \mathbb{Z}_2] \to \mathbb{I}[A; \mathbb{Z}_2]$ sending b to 1. As in [4], we consider the third exterior power $\wedge^3 \mathbb{I}[\pi_2; \mathbb{Z}_2]$ of $\mathbb{I}[\pi_2; \mathbb{Z}_2]$ given with the diagonal action of $\pi_2 \times \pi_2$. Through the projection

$$\mathbb{I}^4[\pi_2; \mathbb{Z}_2] \to \mathbb{I}[A; \mathbb{Z}_2] \otimes \wedge^3 \mathbb{I}[\pi_2; \mathbb{Z}_2]$$

the element (7) yields

$$(a-1) \otimes (a-1) \wedge (ab-1) \wedge (b-1).$$

Similarly, the projection onto $\mathbb{I}[A; \mathbb{Z}_2] \otimes \wedge^3 \mathbb{I}[\pi_2; \mathbb{Z}_2]$ of $\nu^4(EZ([b|b] \otimes [a|a]))$ gives

$$(a-1) \otimes (a-1) \wedge (ba-1) \wedge (b-1).$$

We are thus reduced to see that the element

$$(a-1) \otimes (a-1) \wedge (ab-ba) \wedge (b-1)$$

does not vanish when passing to the coinvariants with respect to the action of $\pi_2 \times \pi_2$. Since this action is trivial on the first component $\mathbb{I}[A; \mathbb{Z}_2]$, it is sufficient to see that

$$s = (a-1) \wedge (ab-ba) \wedge (b-1)$$

does not vanish in the coinvariants $(\wedge^3 \mathbb{I}[\pi_2; \mathbb{Z}_2])_{\pi_2 \times \pi_2}$.

The element s described here corresponds exactly to the element s considered in [4] and the following is a translation of [4][Section 3.3.2] to the notation adopted here. The strategy is to project $\wedge^3 \mathbb{I}[\pi_2; \mathbb{Z}_2]$ to a small $\mathbb{Z}[\pi_2 \times \pi_2]$-module in which it will be possible to check by inspection that the element s survives when passing to the coinvariants. We first reduce $\mathbb{I}[\pi_2; \mathbb{Z}_2]$ to a smaller $\mathbb{Z}[\pi_2 \times \pi_2]$-module in which it is fundamental to keep the elements $a-1$, $b-1$ and $ab-ba$. Note that the presence of $ab-ba$ in s gives an evidence of the role of the non-commutativity of π_2 in this calculation.

Let $v = a-1$, $w = ba-1$ and let (w^3) be the sub-$\mathbb{Z}[\pi_2 \times \pi_2]$-module of $\mathbb{I}[\pi_2; \mathbb{Z}_2]$ generated by w^3. The quotient $J = \mathbb{I}[\pi_2; \mathbb{Z}_2]/(w^3)$ is a $\mathbb{Z}[\pi_2 \times \pi_2]$-module which can be described as the $\mathbb{Z}_2$-vector space with basis

$$v, w, wa, w^2, w^2a$$

and $\pi_2 \times \pi_2$ action given by the following table:

	v	w	wa	w^2	w^2a
$(a,1)$	v	$wa + w^2a$	$w + w^2$	w^2a	w^2
$(1,a)$	v	wa	w	w^2a	w^2
$(b,1)$	$v + w + wa$	wa	w	w^2a	w^2
$(1,b)$	$v + w + w^2 + wa$	$wa + w^2a$	$w^2 + w$	w^2a	w^2

We note that in J we have

$$wa - v = b - 1 \quad \text{and} \quad w^2 = baba - 1 = ba - ab = 1 - abab.$$

Using the projection $\bigwedge^3 \mathbb{I}[\pi_2; \mathbb{Z}_2] \to \bigwedge^3 J$, we look at our element s in $\bigwedge^3 J$. We obtain

$$s = v \wedge wa \wedge w^2.$$

Now considering the table above we see that $L = \text{span}_{\mathbb{Z}_2}\{w, wa, w^2, w^2a\}$ is a sub-$\mathbb{Z}[D \times D]$-module of J. We can therefore consider the projection

$$\bigwedge^3 J \longrightarrow \bigwedge^3 J / \bigwedge^3 L.$$

The quotient $\bigwedge^3 J / \bigwedge^3 L$ is the $\mathbb{Z}_2$ vector space with basis $\{t_1, t_2, t_3, t_4, t_5, t_6\}$, where

$$t_1 = v \wedge w \wedge wa, \qquad t_2 = v \wedge w \wedge w^2, \qquad t_3 = v \wedge w \wedge w^2a,$$
$$t_4 = v \wedge wa \wedge w^2, \qquad t_5 = v \wedge wa \wedge w^2a, \qquad t_6 = v \wedge w^2 \wedge w^2a.$$

Note that t_4 is exactly our element s. We still have to see that s does not vanish when passing to the coinvariants. The computation of the $\pi_2 \times \pi_2$ action on $\bigwedge^3 J / \bigwedge^3 L$ gives:

	t_1	t_2	t_3	t_4	t_5	t_6
$(a,1)$	$t_1 + t_3 + t_4 + t_6$	t_5	$t_4 + t_6$	$t_3 + t_6$	t_2	t_6
$(1,a)$	t_1	t_5	t_4	t_3	t_2	t_6
$(b,1)$	t_1	t_5	t_4	t_3	t_2	t_6
$(1,b)$	$t_1 + t_3 + t_4 + t_6$	t_5	$t_4 + t_6$	$t_3 + t_6$	t_2	t_6

We can thus see that, modulo the $\pi_2 \times \pi_2$ action, we have

$$t_2 = t_5, \qquad t_3 = t_4, \qquad t_6 = 0,$$

and that there are no more relations. Therefore our element s, which corresponds to $t_3 = t_4$, is not 0 modulo the $\pi_2 \times \pi_2$ action. This achieves the proof of (ii). □

We now complete the proof of Proposition 13 (and hence of Theorem 11.)

Proof of Proposition 13. Recall that we have to see that $\mathfrak{v}^{2n}(\mathbf{c}_n \times \mathbf{c}_n) \neq 0$. For $n = 2$, this follows from Lemma 14. For $n > 2$, we decompose $\mathfrak{v}^{2n}$ as $\mathfrak{v}^4 \cup \mathfrak{v}^{2n-4}$ and use

the following diagram (see [5][(Diagram 4.3)])

$$\begin{array}{ccc} H_{2n}(\pi_2\times\pi_2,\mathbb{Z}_2) & \xrightarrow{\quad\mathfrak{v}^{2n}\quad} & \mathbb{I}^{2n}[\pi_2;\mathbb{Z}_2]_{\pi_2\times\pi_2} \\ \downarrow{\scriptstyle \Delta_{4,2n-4}} & & \downarrow{\scriptstyle q_{4,2n-4}} \\ H_4(\pi_2\times\pi_2;\mathbb{Z}_2)\otimes H_{2n-4}(\pi_2\times\pi_2;\mathbb{Z}_2) & \xrightarrow{\mathfrak{v}^4\otimes\mathfrak{v}^{2n-4}} & \mathbb{I}^4[\pi_2;\mathbb{Z}_2]_{\pi_2\times\pi_2}\otimes\mathbb{I}^{2n-4}[\pi_2;\mathbb{Z}_2]_{\pi_2\times\pi_2} \end{array}$$

where $\Delta_{4,2n-4}$ is the projection on $H_4(\pi_2\times\pi_2;\mathbb{Z}_2)\otimes H_{2n-4}(\pi_2\times\pi_2;\mathbb{Z}_2)$ of the homology diagonal $H_*(\pi_2\times\pi_2;\mathbb{Z}_2)\to H_*(\pi_2\times\pi_2;\mathbb{Z}_2)\otimes H_*(\pi_2\times\pi_2;\mathbb{Z}_2)$. Note also that the horizontal arrows correspond to the cap product with the respective powers of $\mathfrak{v}=\mathfrak{v}_{\pi_2}$ and that $q_{4,2n-4}$ is the obvious projection.

Taking into account the decomposition (6), we obtain:

$$\begin{aligned} q_{4,2n-4}\mathfrak{v}^{2n}(\mathbf{c}_n\times\mathbf{c}_n) &= (\mathfrak{v}^4\ \otimes\ \mathfrak{v}^{2n-4})\Delta_{4,2n-4}(\mathbf{c}_n\times\mathbf{c}_n) \\ &= \sum_{i=0}^{4}\mathfrak{v}^4(\mathbf{a}_i\times\mathbf{a}_{4-i})\otimes\mathfrak{v}^{2n-4}(\mathbf{a}_{n-i}\times\mathbf{a}_{n-4+i}) \\ &+ \sum_{i=0}^{4}\mathfrak{v}^4(\mathbf{a}_i\times\mathbf{b}_{4-i})\otimes\mathfrak{v}^{2n-4}(\mathbf{a}_{n-i}\times\mathbf{b}_{n-4+i}) \\ &+ \sum_{i=0}^{4}\mathfrak{v}^4(\mathbf{b}_i\times\mathbf{a}_{4-i})\otimes\mathfrak{v}^{2n-4}(\mathbf{b}_{n-i}\times\mathbf{a}_{n-4+i}) \\ &+ \sum_{i=0}^{4}\mathfrak{v}^4(\mathbf{b}_i\times\mathbf{b}_{4-i})\otimes\mathfrak{v}^{2n-4}(\mathbf{b}_{n-i}\times\mathbf{b}_{n-4+i}). \end{aligned}$$

Consider, as before, $A=\langle a\,|\,a^2=1\rangle$ and set $B=\langle a\,|\,b^2=1\rangle$. Since $\pi_2\times\pi_2$ acts trivially on both $\mathbb{I}[A;\mathbb{Z}_2]\cong\mathbb{Z}_2(a-1)$ and $\mathbb{I}[B;\mathbb{Z}_2]\cong\mathbb{Z}_2(b-1)$, we have

$$\mathbb{I}^{n-2}[A;\mathbb{Z}_2]_{\pi_2\times\pi_2}\otimes\mathbb{I}^{n-2}[B;\mathbb{Z}_2]_{\pi_2\times\pi_2}\cong\mathbb{Z}_2(a-1)^{\otimes n-2}\otimes\mathbb{Z}_2(b-1)^{\otimes n-2}\cong\mathbb{Z}_2.$$

Let $q_{A,B}$ be the following projection:

$$\mathbb{I}^{2n-4}[\pi_2;\mathbb{Z}_2]_{\pi_2\times\pi_2}\to\mathbb{I}^{n-2}[A;\mathbb{Z}_2]_{\pi_2\times\pi_2}\otimes\mathbb{I}^{n-2}[B;\mathbb{Z}_2]_{\pi_2\times\pi_2}\cong\mathbb{Z}_2(a-1)^{\otimes n-2}\otimes(b-1)^{\otimes n-2}.$$

We can check that, by applying $q_{A,B}$ to the $\mathfrak{v}^{2n-4}$ terms in the sum above, the only non-vanishing terms are:

$$q_{A,B}\mathfrak{v}^{2n-4}(\mathbf{a}_{n-2}\times\mathbf{b}_{n-2})=q_{A,B}\mathfrak{v}^{2n-4}(\mathbf{b}_{n-2}\times\mathbf{a}_{n-2})=(a-1)^{\otimes n-2}\otimes(b-1)^{\otimes n-2}.$$

Consequently, in $\mathbb{I}^4[\pi_2;\mathbb{Z}_2]_{\pi_2\times\pi_2}\otimes\mathbb{Z}_2(a-1)^{\otimes n-2}\otimes(b-1)^{\otimes n-2}$ we have:

$$\begin{aligned} (\mathrm{Id}\otimes q_{A,B})q_{4,2n-4}\mathfrak{v}^{2n}(\mathbf{c}_n\times\mathbf{c}_n) &= \mathfrak{v}^4(\mathbf{a}_2\times\mathbf{b}_2)\otimes q_2\mathfrak{v}^{2n-4}(\mathbf{a}_{n-2}\times\mathbf{b}_{n-2}) \\ &+ \mathfrak{v}^4(\mathbf{b}_2\times\mathbf{a}_2)\otimes q_2\mathfrak{v}^{2n-4}(\mathbf{b}_{n-2}\times\mathbf{a}_{n-2}) \\ &= \mathfrak{v}^4(\mathbf{a}_2\times\mathbf{b}_2+\mathbf{b}_2\times\mathbf{a}_2)\otimes(a-1)^{\otimes n-2}\otimes(b-1)^{\otimes n-2} \end{aligned}$$

We can then conclude that $\mathfrak{v}^{2n}(\mathbf{c}_n\times\mathbf{c}_n)\neq 0$ since $\mathfrak{v}^4(\mathbf{a}_2\times\mathbf{b}_2+\mathbf{b}_2\times\mathbf{a}_2)$ does not vanish in $\mathbb{I}^4[\pi_2;\mathbb{Z}_2]_{\pi_2\times\pi_2}$ by Lemma 14. □

Finally, Theorem 12 can be obtained from Proposition 13 through the following Proposition 15, which we explain now.

Let M and N two n-dimensional connected closed manifolds. The collapsing map $M\#N \to M$ induces an homomorphism $\pi_{M\#N} \to \pi_M$ and an isomorphism $H_n(M\#N;\mathbb{Z}_2) \to H_n(M;\mathbb{Z}_2)$. We have a commutative diagram

$$\begin{array}{ccc} H_{2n}(M\#N \times M\#N;\mathbb{Z}_2) & \xrightarrow{\cap \mathfrak{v}^{2n}_{M\#N}} & \mathbb{I}^{2n}[\pi_{M\#N};\mathbb{Z}_2]_{\pi_{M\#N}\times\pi_{M\#N}} \\ \cong \downarrow & & \downarrow \\ H_{2n}(M\times M;\mathbb{Z}_2) & \xrightarrow{\cap \mathfrak{v}^{2n}_{M}} & \mathbb{I}^{2n}[\pi_M;\mathbb{Z}_2]_{\pi_M\times\pi_M}. \end{array}$$

in which the left-hand vertical map is an isomorphism. It is therefore clear that the non-triviality of the bottom line implies the non-triviality of the top line. We can conclude:

Proposition 15. $\mathsf{TC}(M\#N) = \mathsf{TC}(M) = 2n$ *as soon as* $\mathfrak{v}^{2n}([M\times M]_{\mathbb{Z}_2})$ *is non- zero.*

We note that a similar argument with coefficients in $\widetilde{\mathbb{Z}}$ (corresponding to the argument given in [10][Prop. 2.1] for surfaces) shows that for connected, closed n dimensional manifolds M and N, $\mathsf{TC}(M\#N)$ is maximal as soon as $\mathsf{TC}(M)$ is maximal and N is orientable. The orientability of N guarantees that the map $H_n(M\#N,\widetilde{\mathbb{Z}}) \to H_n(M,\widetilde{\mathbb{Z}})$ is well-defined. Note also that, assuming that M e N are simply-connected, Dranishnikov and Sadykov established the stronger general result that $\mathsf{TC}(M\#N) \geq \mathsf{TC}(M)$ [11].

4. Manifolds with fundamental abelian group

As mentioned in the proof of Lemma 14, the non-commutativity of the fundamental group was an important ingredient for the maximality of the topological complexity of the Klein bottle. It is of course not sufficient. For instance, $K \times S^1$ has a non-abelian fundamental group but, by the product formula, $\mathsf{TC}(K \times S^1) \leq 5 < 2\dim(K \times S^1)$. In the other direction, given a path-connected CW-complex X, we can ask if the maximality of TC, that is, $\mathsf{TC}(X) = 2\dim(X)$, implies the non-comutativity of $\pi_1(X)$. As an extreme case, we know that, if X is simply-connected, then $\mathsf{TC}(X) \leq \dim(X)$ and is therefore non-maximal. We have also seen in Theorem 3 that if $\pi_1(X) = \mathbb{Z}_2$, then $\mathsf{TC}(X)$ is non-maximal. In [7], Costa and Farber have also studied the (non)maximality of TC when $\pi_1(X) = \mathbb{Z}_3$. In this case, they have established that $\mathsf{TC}(X) < 2\dim(X)$ when either $\dim X$ is odd or $\dim X$ is even such that the 3-adic expansion of $\dim(X)/2$ contains at least one digit 2 (see [7, Theorem 2]). In addition, they have exhibited examples of even dimensional CW-complexes which do not satisfy the latter condition and have TC maximal. So, in general the maximality of TC does not imply the non-commutativity of $\pi_1(X)$. However, it is worth noting that these examples are not closed manifolds and we can try to understand in which extent the commutativity of the fundamental group implies the non-maximality of TC in the context of manifolds. In this section we will see some non-maximality results which have been established in [6] for closed

manifolds with fundamental abelian group. Nevertheless, as also proved in [6], we will see that, even in this restricted setting, maximality can occur.

Let M be an n-dimensional connected closed manifold with $n \geq 2$ and $\pi := \pi_1(M)$ abelian. As π is abelian, the inversion $j : \pi \to \pi$, $j(a) = \bar{a}$ and the map

$$\chi : \pi \times \pi \to \pi, \quad (a, b) \mapsto a\bar{b}$$

are both homomorphism. The homomorphism χ makes compatible the π and $\pi \times \pi$ action on $I(\pi)$ in the sense that (with the notations of Section 1.2) the $\mathbb{Z}[\pi \times \pi]$-module $\chi^*(\mathrm{I}[\pi])$ is exactly the $\mathbb{Z}[\pi \times \pi]$-module $\mathbb{I}[\pi]$. It is also compatible with any fixed $\mathbb{Z}[\pi]$ orientation module $\widetilde{\mathbb{Z}}$ since, for any homomorphism $\omega : \pi \to \{\pm 1\}$ and any $(a, b) \in \pi \times \pi$, one has $\omega(a\bar{b}) = \omega(a)\omega(b)$. Consequently χ induces a map $\chi_* : H_*(\pi \times \pi; \widetilde{\mathbb{Z}}) \to H_*(\pi; \widetilde{\mathbb{Z}})$.

Considering the crossed homomorphisms associated to the classes $\mathfrak{v}_\pi$ and $\mathfrak{b}_\pi$, we have, for any $(a, b) \in \pi \times \pi$,

$$v_\pi(a, b) = b_\pi(a\bar{b}) = b_\pi\chi(a, b).$$

We then have $\mathfrak{v}_\pi = \chi^*\mathfrak{b}_\pi$ in $H^1(\pi \times \pi; \mathbb{I}[\pi])$ and similarly for the powers.

Considering a classifying map $\gamma : M \to B\pi$ we then have for any $k \geq 1$

$$\mathfrak{v}_M^k = (\gamma \times \gamma)^*\mathfrak{v}_\pi^k = (\gamma \times \gamma)^*\chi^*\mathfrak{b}_\pi^k \text{ in } H^k(M \times M; \mathbb{I}^k[\pi]).$$

This link between the TC-canonical class and the Berstein-Schwarz class of an abelian group, that was also observed in [17], yields to the following result, which was the starting point of the study in [6] of TC for manifolds with abelian fundamental group.

Proposition 16. *Let M be an n-dimensional connected closed manifold with $n \geq 2$ and $\pi := \pi_1(M)$ abelian. Let $\gamma : M \to B\pi$ be a classifying map, let $[M] \in H_n(M; \widetilde{\mathbb{Z}})$ be the fundamental class of M and let $\mathbf{m} := \gamma_*([M]) \in H_n(\pi; \widetilde{\mathbb{Z}})$. Then*

$$\chi_*(\mathbf{m} \times \mathbf{m}) = 0 \Rightarrow \mathsf{TC}(M) < 2n.$$

Note that Proposition 2.6 of [6] gives the converse of this result:

Proposition 17 ([6])**.** *In the same conditions as in Proposition 16, we have:*

$$\mathsf{TC}(M) < 2n \Rightarrow \chi_*(\mathbf{m} \times \mathbf{m}) = 0.$$

We here only give the proof of Proposition 16.

Proof of Proposition 16. Recall from Theorem 6 that $\mathsf{TC}(M) = 2n$ if and only if $(\mathbf{m} \times \mathbf{m}) \cap \mathfrak{v}_\pi^{2n} \neq 0$. Since $\mathfrak{v}_\pi^{2n} = \chi^*\mathfrak{b}_\pi^{2n}$ we then have

$$\mathsf{TC}(M) = 2n \Leftrightarrow (\mathbf{m} \times \mathbf{m}) \cap \chi^*\mathfrak{b}_\pi^{2n} \neq 0.$$

By using the naturality of the cap product together with the fact that the morphism $\mathbb{I}^{2n}[\pi] \otimes_{\pi\times\pi} \widetilde{\mathbb{Z}} \to \mathrm{I}^{2n}[\pi] \otimes_\pi \widetilde{\mathbb{Z}}$ induced by the epimorphism χ is an isomorphism, we obtain

$$\mathsf{TC}(M) = 2n \Leftrightarrow \chi_*(\mathbf{m} \times \mathbf{m}) \cap \mathfrak{b}_\pi^{2n} \neq 0$$

and the conclusion follows. □

We will use Proposition 16 to prove the following theorem which generalizes, in the setting of orientable manifolds, the results of Costa and Farber mentioned in the introduction of this section. Note, in particular, that an orientable manifold with cyclic fundamental group has always non-maximal topological complexity.

Theorem 18 ([6])**.** *Let M be a connected closed orientable manifold with $\pi_1(M)$ abelian. We have $\mathsf{TC}(M) < 2\dim(M)$ when:*

(1) $\pi_1(M) = \mathbb{Z}^r$ and $\dim(M)$ odd;

(2) $\pi_1(M) = \mathbb{Z}^r$, $\dim(M)$ even and $r < 2\dim(M)$;

(3) $\pi_1(M) = \mathbb{Z}_q$ with $q \geq 2$;

(4) $\pi_1(M) = \mathbb{Z}^r \times \mathbb{Z}_q$ with $\dim M$ odd and $r < \dim(M)$;

(5) $\pi_1(M) = \mathbb{Z}^r \times \mathbb{Z}_q$ with $\dim M$ even and $r \leq \dim(M)$.

Since M is orientable, the orientation module $\widetilde{\mathbb{Z}}$ is just $\mathbb{Z}$ with trivial action of π and we suppress the coefficients from the notation. Since $\chi = \mu \times (\mathrm{Id} \times \mathrm{j})$, we have

$$\chi_*(\mathbf{m} \times \mathbf{m}) = \mathbf{m} \wedge j_*\mathbf{m}$$

where $\wedge$ denotes the Pontryagin product $H_*(\pi) \otimes H_*(\pi) \to H_*(\pi)$ induced by μ, see [2]. Recall that $\wedge$ is a strictly anti-commutative product, that is, for classes $\mathbf{c}$, $\mathbf{d}$ we have

$$\mathbf{c} \wedge \mathbf{d} = (-1)^{|\mathbf{c}||\mathbf{d}|}\mathbf{d} \wedge \mathbf{c} \quad \text{and } \mathbf{c} \wedge \mathbf{c} = 0 \text{ when } |\mathbf{c}| \text{ is odd.}$$

In the following calculations, we will use that:

(i) The Pontryagin algebra $H_*(\mathbb{Z}^r)$ is an exterior algebra $\bigwedge(x_1, \ldots, x_r)$ where $|x_i| = 1$ for any $1 \leq i \leq r$ and that for any $\mathbf{c} \in H_*(\mathbb{Z}^r)$ we have $j_*(\mathbf{c}) = (-1)^{|\mathbf{c}|}\mathbf{c}$.

(ii) For $q \geq 2$, the homology $H_+(\mathbb{Z}_q)$ is concentrated in odd degrees with trivial Pontryagin product and, for any $\mathbf{c} \in H_+(\mathbb{Z}_q)$, $j_*(\mathbf{c}) = (-1)^{\frac{|\mathbf{c}|+1}{2}}\mathbf{c}$.

Proof of Theorem 18. For (1) we just observe that

$$\chi_*(\mathbf{m} \times \mathbf{m}) = \mathbf{m} \wedge j_*\mathbf{m} = (-1)^{|\mathbf{m}|}\mathbf{m} \wedge \mathbf{m} = 0$$

since the degree of $\mathbf{m}$ is odd. Item (2) is immediate since $\chi_*(\mathbf{m} \times \mathbf{m})$ has degree $2\dim(M)$ and $H_*(\mathbb{Z}^r)$ vanishes in degrees greater than r. The proof of (3) is immediate when $\dim M$ is even (since, in this case, $\mathbf{m} = 0$) and similar to the proof of (1) when $\dim M$ is odd. Finally, suppose that $\pi_1(M) = \mathbb{Z}^r \times \mathbb{Z}_q$. By the Künneth formula, we see that $H_*(\mathbb{Z}^r \times \mathbb{Z}_q) = H_*(\mathbb{Z}^r) \otimes H_*(\mathbb{Z}_q)$. For $\mathbf{c} \otimes \mathbf{d}$ and $\mathbf{c}' \otimes \mathbf{d}' \in H_*(\mathbb{Z}^r) \otimes H_*(\mathbb{Z}_q)$, the Pontryagin product is given by $(-1)^{|\mathbf{c}'||\mathbf{d}|}(\mathbf{c} \wedge \mathbf{c}') \otimes (\mathbf{d} \wedge \mathbf{d}')$. Suppose first that $\dim(M) > r$. We can write $\mathbf{m} = \sum \mathbf{c}_i \otimes \mathbf{d}_i$ where $\mathbf{c}_i \in H_*(\mathbb{Z}^r)$ and $\mathbf{d}_i \in H_+(\mathbb{Z}_q)$. From the expression of j_* given above, we have $j_*(\mathbf{m}) = \sum \lambda_i \mathbf{c}_i \otimes \mathbf{d}_i$ where $\lambda_i = \pm 1$. Since $H_+(\mathbb{Z}_q)$ is concentrated in odd degrees and $\mathbf{d}_i \wedge \mathbf{d}_j = 0$ for any i, j we can conclude that $\chi_*(\mathbf{m} \times \mathbf{m}) = \mathbf{m} \wedge j_*(\mathbf{m}) = 0$. This proves (4) and (5) when $\dim(M) > r$. Suppose now that $\dim M$ is even and equal to r. We then have $\mathbf{m} = \mathbf{m}' \otimes 1 + \sum \mathbf{c}_i \otimes \mathbf{d}_i$ where $\mathbf{m}' \in H_r(\mathbb{Z}^r) = \mathbb{Z}$ is of maximal length in the exterior algebra $H_*(\mathbb{Z}^r)$ and $\sum \mathbf{c}_i \otimes \mathbf{d}_i$ is as before. Since r is even, we obtain

$j_*(\mathbf{m}) = \mathbf{m}' \otimes 1 + \sum \lambda_i \mathbf{c}_i \otimes \mathbf{d}_i$ and $\mathbf{m} \wedge j_*(\mathbf{m}) = (\mathbf{m}' \wedge \mathbf{m}') \otimes 1 + 2 \sum \lambda_i (\mathbf{m}' \wedge \mathbf{c}_i) \otimes \mathbf{d}_i$, which is 0 because $|\mathbf{c}_i|$ must be positive and $\mathbf{m}'$ is of maximal length. □

In [6], more sufficient conditions ensuring that $\mathsf{TC}(M) < 2\dim(M)$ have been established including in particular the following result

Theorem 19. [6] *Let M be a connected closed orientable manifold with $\pi_1(M) = \mathbb{Z}_{p^a} \times \mathbb{Z}_{p^b}$ where p is a prime and $a, b \geq 1$ are integers. We have $\mathsf{TC}(M) < 2\dim(M)$.*

as well as the following general statement in the non-orientable case:

Theorem 20. [6] *Let M be a connected closed non-orientable manifold and $\pi_1(M)$ abelian. Then $\mathsf{TC}(M) < 2\dim(M)$.*

In these cases, the relevant homology ($H_*(\pi)$ or $H_*(\pi; \widetilde{\mathbb{Z}})$ where $\pi = \pi_1(M)$) does not necessarily decompose as a tensor product of the homology groups of the factor of π. In other words, the Tor part in the Künneth Formula is not necessarily trivial. The calculations developed in [6] for establishing the vanishing of the class $\chi_*(\mathbf{m} \times \mathbf{m})$ have been partially carried out at the chain level in order to deal with the possible class coming from the Tor part. We also note that an important ingredient in the non-orientable case is that, for any non-trivial $\mathbb{Z}[\pi]$-module $\widetilde{\mathbb{Z}}$, the homology groups $H_*(\pi; \widetilde{\mathbb{Z}})$ are either 0 or $\mathbb{Z}_2$ and that, additionally, a Pontryagin square is always divisible by 2.

We finish by exhibiting a manifold N with $\pi_1(N) = \mathbb{Z}^r$, $\dim(N)$ even and $r = 2\dim(N)$ satisfying $\mathsf{TC}(N) = 2\dim N$. This shows that the condition $r < 2\dim(N)$ in Theorem 18 (2) cannot be relaxed and produces an example of a manifold with abelian fundamental group and maximal topological complexity. We refer to [6, Examples 4.2 and 4.3] for limiting examples associated to the conditions given in Theorem 18 (4) and (5).

Example 21. *(*[6, Example 4.1]*) We construct a manifold N with $\pi_1(N) = \mathbb{Z}^8$, $\dim(N) = 4$ and $\mathsf{TC}(N) = 2\dim(N)$. We first consider the manifold $M = T^4 \# T^4$, where $T^4 = (S^1)^4$, as well as the map f*

$$M = T^4 \# T^4 \xrightarrow{pinch} T^4 \vee T^4 \hookrightarrow T^4 \times T^4 = B(\mathbb{Z}^4 \times \mathbb{Z}^4).$$

This map induces the abelianization homomorphism $\pi_1(f) : \pi_1(M) = \mathbb{Z}^4 * \mathbb{Z}^4 \to G = \mathbb{Z}^4 \times \mathbb{Z}^4$. The kernel of $\pi_1(f)$ is, as a normal subgroup, generated by a finite number of commutators. By using surgery to kill these commutators we obtain an orientable manifold N of dimension 4 and a map $g : N \to BG$ which induces an isomorphism of fundamental groups and such that $g_*([N]) = f_*([M])$ in $H_4(\mathbb{Z}^4 \times \mathbb{Z}^4)$. In particular, N is a manifold with $\pi_1(N) = \mathbb{Z}^8$ and $g : N \to BG$ is a classifying map. Writing $H_*(\mathbb{Z}^4 \times \mathbb{Z}^4)$ as the exterior algebra $\bigwedge(x_1, \ldots, x_4, x'_1, \cdots, x'_4)$, the element $g_*([N]) = f_*([M])$ can be identified with the class $\mathbf{c} = x_1x_2x_3x_4 + x'_1x'_2x'_3x'_4$. We then calculate that $\mathbf{c} \wedge j_*(\mathbf{c}) = 2x_1x_2x_3x_4x'_1x'_2x'_3x'_4 \neq 0$ in $H_8(\mathbb{Z}^4 \times \mathbb{Z}^4) = \mathbb{Z}$. We can finally use Proposition 17 to conclude that $\mathsf{TC}(N) = 8$.

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