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Research paper

Kinematic and dynamic analysis of spatial multibody systems based on a formulation with fully Cartesian coordinates and a generic rigid body

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A B S T R A C T

This work introduces the Fully Cartesian Coordinates Formulation with a Generic Rigid Body (FCC-GRB), a novel global multibody formulation for three-dimensional mechanical system analysis. The formulation's intrinsic characteristics are thoroughly detailed and compared with other widely-used global formulations, enabling its application in both kinematic and dynamic analysis of complex mechanical systems and as a teaching tool in advanced multibody dynamics courses.

FCC-GRB formulation is founded on two main premises: multibody systems are described using only Cartesian coordinates, and the rigid bodies are modeled with a fixed and predetermined structure. Consequently, the kinematic constraints are described by lower-degree equations and the system mass matrix is highly sparse. Additionally, the introduction of the generic rigid body simplifies the modeling process by making the definition of the bodies independent of system topology. To reduce the number of generalized coordinates, a reduced modeling approach using less coordinates for describing the generic rigid body is also introduced and compared with the fully-defined alternative.

The formulation's accuracy was validated through forward dynamic analysis of benchmark problems. Simulations demonstrated excellent agreement with reference data, with both modeling approaches yielding comparable kinematic results. The reduced approach offered faster computational performance, particularly in more complex models.

Abbreviations	Description
2D	Two-dimensional
3D	Three-dimensional

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AR	Angular Relation
CAR	Constant Angular Relation
CP	Coincident Point
CV	Coincident Vector
DAE	Differential Algebraic Equations
DC	Driver Constraint
DoF	Degree-of-Freedom
EoM	Equations-of-Motion
FCC	Fully Cartesian Coordinates
FCC-GRB	Fully Cartesian Coordinates with a Generic Rigid Body
FD	Forward Dynamics
GRB	Generic Rigid Body
GSJ	Grounded Spherical Joint
ICC	Intraclass Correlation
KC	Kinematic Constraint
KJ	Kinematic Joint
ODE	Ordinary Differential Equations
OV	Orientation of a Generic Vector
PJ	Prismatic Joint
PRJ	Pinned Revolute Joint
RB	Rigid Body
RJ	Revolute Joint
RoM	Range-of-Motion
RGRB	Reduced Generic Rigid Body
SJ	Spherical Joint
TP	Translation of a Generic Point
UJ	Universal Joint
Symbol (Latin)	**Description**
$\mathbf{0}_3$, $\mathbf{I}_3$	Null and identity matrices
$\mathbf{b}_{\boldsymbol{\tau}}$, $\mathbf{b}_{-\boldsymbol{\tau}}$	Moment arms of the equivalent force pair $\mathbf{f}_{\boldsymbol{\tau}}$ and $\mathbf{f}_{-\boldsymbol{\tau}}$
c_1^P, c_2^P, c_3^P	Scaling factors representing the local coordinates of point P in the local reference frame of the rigid body
$\mathbf{C}_i^P$	Constant transformation matrix that describes the kinematics of a generic point P with respect to body i
$\mathbf{C}_i^{\mathbf{s}}$	Constant transformation matrix that describes the kinematics of a generic vector $\mathbf{s}$ with respect to body i
$\mathscr{C}^{\mathbf{s}_1\mathbf{s}_2}$	Constant transformation matrix that relates the generic vectors $\mathbf{s}_1$ and $\mathbf{s}_2$
$\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$	Basis vectors that define the global reference frame of the multibody system
$\mathbf{f}$	Generic external force $\mathbf{f}$
$\mathbf{f}_R^k$	Internal reaction forces associated to the kinematic constraint of type k
$\mathbf{f}_{\boldsymbol{\tau}}$, $\mathbf{f}_{-\boldsymbol{\tau}}$	Equivalent force pair of the external moment of force $\boldsymbol{\tau}$
$\mathbf{g}$	Generalized forces of the system
$\mathbf{g}_{2v_i}$	Velocity-dependent inertial force of the rigid body i defined in its reduced form
$\mathbf{g}_i^{\mathbf{f}}$	Generalized equivalent force of a concentrated external force $\mathbf{f}$ applied in body i
$\mathbf{g}_i^{\boldsymbol{\tau}}$	Generalized equivalent force of an external moment of force $\boldsymbol{\tau}$ applied in body i
$\mathbf{g}^{\boldsymbol{\Phi}}$	Generalized internal forces
$\mathbf{g}_i^{\boldsymbol{\Phi}^k}$	Generalized internal forces for the kinematic constraint of type k with respect to body i
I_{ii}	Components of the inertia tensor of the rigid body i
m_i	Mass of the rigid body i
$\mathbf{M}$	Mass matrix of the system
$\mathbf{M}_i$	Mass matrix of the rigid body i
$\mathbf{M}_{2v_i}$	Equivalent mass matrix of the rigid body i defined in its reduced form
nb, nc, nq, nv	Number of bodies, kinematic constraints, generalized coordinates and vectors of the system
P	Generic point P
P^*	Reference point P^*
O_i	Origin of the local reference frame of body i
$\mathbf{q}$, $\dot{\mathbf{q}}$, $\ddot{\mathbf{q}}$	Generalized positions, velocities and accelerations of the system
$\mathbf{q}_{3v_i}$, $\dot{\mathbf{q}}_{3v_i}$, $\ddot{\mathbf{q}}_{3v_i}$	Position, velocity and acceleration of the fully-defined equivalent vector of the reduced rigid body i
$\dot{\mathbf{q}}_i^*$	Generalized virtual velocities of the rigid body i
$\dot{\mathbf{q}}_{3v_i}^*$	Generalized virtual velocities of the rigid body i described in terms of the fully-defined equivalent form
$\mathbf{r}_P$, $\dot{\mathbf{r}}_P$, $\ddot{\mathbf{r}}_P$	Position, velocity and acceleration vectors of the generic point P in the global reference frame
$\mathbf{r}_{P^*}$, $\dot{\mathbf{r}}_{P^*}$, $\ddot{\mathbf{r}}_{P^*}$	Position, velocity and acceleration vectors of a reference point P^* in the global reference frame
$\dot{\mathbf{r}}_P^*$	Virtual velocities of the generic point P
$\mathbf{s}$, $\dot{\mathbf{s}}$, $\ddot{\mathbf{s}}$	Position, velocity and acceleration vectors of a generic unit vector $\mathbf{s}$
$\mathbf{s}^*$, $\dot{\mathbf{s}}^*$, $\ddot{\mathbf{s}}^*$	Position, velocity and acceleration vectors of a reference unit vector $\mathbf{s}^*$
$\mathbf{S}$	Constant transformation matrix that converts the reduced rigid body i in its fully-defined equivalent form
t	Time
T	Kinetic energy
$\mathbf{u}_i$, $\dot{\mathbf{u}}_i$, $\ddot{\mathbf{u}}_i$ $\mathbf{v}_i$, $\dot{\mathbf{v}}_i$, $\ddot{\mathbf{v}}_i$ $\mathbf{w}_i$, $\dot{\mathbf{w}}_i$, $\ddot{\mathbf{w}}_i$	Position, velocity and acceleration of the generic rigid body vectors **u, v** and **w** of body i in the global reference frame
$\mathbf{u}_i'$, $\mathbf{v}_i'$, $\mathbf{w}_i'$	Position of the generic rigid body vectors **u, v** and **w** in the local reference frame of body i
$\widetilde{\mathbf{u}}$	Skew-symmetric matrix of vector $\mathbf{u}$ used to compute the cross product

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V	Potential energy
$\overline{\mathbf{V}}_i$, $\mathbf{V}_i$, $\dot{\mathbf{V}}_i$	Transformation matrices that convert the reduced rigid body i in its fully-defined equivalent form
W_i^*	Virtual Power generated by the internal forces of body i
$\dot{\mathbf{y}}$	State vector including the generalized velocities and accelerations utilized in the direct integration method
Symbol (Greek)	**Description**
α, β	Baumgarte stabilization coefficients
$\boldsymbol{\gamma}$	Right-hand-side vector of the acceleration equations of the system
γ^k	Contributions to the right-hand-side vector of the acceleration equations for the kinematic constraint of type k
$\theta_{\mathbf{s}_1\mathbf{s}_2}$	Angle between two generic unit vectors $\mathbf{s}_1$ and $\mathbf{s}_2$
$\theta^*_{\mathbf{s}_1\mathbf{s}_2}$, $\dot{\theta}^*_{\mathbf{s}_1\mathbf{s}_2}$, $\ddot{\theta}^*_{\mathbf{s}_1\mathbf{s}_2}$	Angular position, velocity and acceleration of a reference angle θ^* between two generic unit vectors $\mathbf{s}_1$ and $\mathbf{s}_2$
$\boldsymbol{\lambda}$	Lagrange multipliers of the system
$\boldsymbol{\lambda}^k$	Lagrange multipliers for the kinematic constraint of type k
$\boldsymbol{\nu}$	Right-hand-side vector of the velocity equations of the system
$\boldsymbol{\nu}_t$	Vector of the partial derivatives of $\boldsymbol{\nu}$ with respect to t
ν^k	Contributions to the right-hand-side vector of the velocity equations for the kinematic constraint of type k
$\widehat{\boldsymbol{\xi}}$, $\widehat{\boldsymbol{\eta}}$, $\widehat{\boldsymbol{\zeta}}$	Basis vectors that define the local reference frame of a generic rigid body
ξ_i^P, η_i^P, ζ_i^P	Coordinate of the point P in the $\widehat{\boldsymbol{\xi}}$, $\widehat{\boldsymbol{\eta}}$ and $\widehat{\boldsymbol{\zeta}}$ axes of the local reference frame of body i
ρ_i	Mass density of rigid body i
$\boldsymbol{\tau}$	Generic external moment of force $\boldsymbol{\tau}$
$\boldsymbol{\tau}_k^{int}$	Internal joint moment of force associated to joint k
$\boldsymbol{\tau}_{k_m}^{int}$	Internal moment of force of joint k associated to the *m-th* angular driving constraint
$\boldsymbol{\Phi}$, $\dot{\boldsymbol{\Phi}}$, $\ddot{\boldsymbol{\Phi}}$	Vector of the kinematic constraints and respective first derivative and second derivative with respect to time
$\boldsymbol{\Phi}_{\mathbf{q}}$	Jacobian matrix of the system
$\boldsymbol{\Phi}_t$	Vector of the partial derivatives of $\boldsymbol{\Phi}$ with respect to time
Φ^k	Kinematic constraint equation of type k
$\boldsymbol{\Phi}_{\mathbf{q}}^k$	Contributions to the Jacobian matrix from the kinematic constraint of type k
Ω_i	Geometric domain of the rigid body i

1. Introduction

Multibody dynamics methodologies have been successfully applied in the kinematic and dynamic analysis of complex mechanical systems, as they allow for efficient modeling and simulation of numerous problems, providing results with a high physical meaning [1]. In fact, multibody methodologies are frequently applied in many different areas, namely vehicle dynamics [2–4], mechatronics and robotics [5–7], mechanisms and machines [8–12], biomechanics and medical devices [13–18], co-simulation with finite elements method [19–21], media and videogames [22], and in real-time problems [23–27].

Over the last decades, several multibody systems formulations have been proposed, varying in the way the bodies are defined, the nature of the coordinates utilized, and the type of constraint and governing equations needed to describe the system kinematics and dynamics [28–31]. In general, multibody systems formulations can be categorized into two main types: recursive and global. Recursive formulations, also referred to as topological since the model definition depends on the system topology, can be further split into fully- and semi-recursive approaches. It should be noticed that each type of formulation comes with its own set of advantages and drawbacks, which significantly impact the simplicity of the modeling procedure, the systematization of the equations of motion (EoM), as well as computational implementation and efficiency [32]. For a more in-depth discussion on the differences, benefits, and practical applications of each type of multibody system formulation, the interested reader is referred to the works of Jalón et al. [33–35], Cuadrado et al. [36], Bae et al. [37], Roupa et al. [30] or Yu et al. [38].

In global formulations, the position and orientation of each body is usually described in relation to the global reference frame [28–30]. This type of formulation tends to generate dependent generalized coordinates, requiring the incorporation of constraint equations to describe such dependencies and to model the mechanical system. Despite increasing the size of the problem to solve, this approach greatly reduces the dependency of the EoM from the system topology, allowing for an easy systematization of the modeling procedure and definition of the governing equations. By and large, two main families of global formulations are available in literature to model multibody systems, chiefly the reference point coordinates [28,39,40], also referred to as Cartesian coordinates, and the natural coordinates [29,41]. Essentially, these two formulations vary on the type of coordinates adopted to model the bodies, which therefore affect the structure and complexity of the kinematic equations these generate.

In Cartesian coordinates formulation, rigid bodies are typically described by a fixed kinematic structure composed of one point and a set of angular coordinates to describe respectively their position and orientation [28,42]. This modeling approach generates less generalized coordinates per rigid body than the other global formulations and makes the modeling of the rigid bodies independent of the system topology. Moreover, this formulation produces diagonal and constant mass matrices, and the joint reaction forces and moments of force can be obtained directly from the EoM. However, the use of angular variables implies that the kinematic constraint equations are described by nonlinear terms, which eventually penalize the computational efficiency [29].

In natural coordinates, the rigid bodies are defined resorting only on the use of rectangular coordinates of points and vectors, and, therefore, the constraints equations for the most common kinematic pairs present a linear or quadratic dependency on the generalized coordinates [29,43]. Due to the type of the elements that constitute the rigid body, the formulation with natural coordinates utilizes more coordinates per body than the Cartesian coordinates. However, a benefit associated with the natural coordinates approach is the possibility of sharing points and vectors between bodies, defining directly kinematic pairs. This modeling approach allows for the reduction of the number of generalized coordinates and kinematic constraint equations needed to completely describe the configuration of multibody mechanical systems. Therefore, and depending on the topology of the system, the natural coordinates formulation with an implicit definition of the joints tends to produce a smaller number of coordinates than the Cartesian coordinates [29,43]. The description of the mechanical system by means of points located in relevant positions of each model segment implies that the mathematical definition of the rigid bodies and their mass matrices is dependent on the system topology, making the modeling procedure less systematic. Moreover, the use of shared elements also implies that the bodies are not completely independent, meaning that the system mass matrix is coupled, and the joint reaction forces cannot be directly obtained when the EoM are solved. To obtain the reaction forces directly, an explicit definition of the joints can be adopted, increasing the number of generalized coordinates and the computational burden [43].

Based on the natural coordinates, Gameiro et al. [44] proposed a novel global formulation that preserves the main characteristics of the natural and Cartesian formulations, that is, the bodies are modeled with a predetermined kinematic structure and are algebraically defined using only Cartesian coordinates. Similar approaches have been proposed by Uhlar et al. [45], and Pappalardo et al. [46–48]. More recently, Roupa et al. [30] presented a detailed description of the theoretical basis of the formulation for a planar case, formulating the constraint equations and contributions to the Jacobian matrix and right-hand side vectors of velocity and acceleration for the most common kinematic constraints. The formulation was subsequently applied to the inverse and forward dynamic analysis of different classical and biomechanical models, showing an excellent agreement with the benchmark data available in the literature [30, 49–51]. Since the formulation was built upon the two aforementioned premises, it was named as fully Cartesian coordinates with a generic rigid body (FCC-GRB). It is important to note that the term fully Cartesian coordinates were initially proposed by Jalón and his co-authors to describe the natural coordinates formulation, as this formulation considers only the use of Cartesian coordinates to describe the kinematics of the system. However, as the kinematic pairs arise naturally from the sharing of points and vectors, this formulation later became known as natural coordinates formulation [52].

According to Roupa et al. [30], the use of a fixed structure to describe the rigid bodies allows for an easier systematization of the modeling procedure, as their definition becomes independent of the topology of the system. It must be highlighted that this approach differs from the traditional modeling with natural coordinates, where the definition of the system varies with the topology and structure of the system, approximating the formulation to the one with Cartesian coordinates. In turn, the exclusive use of rectangular coordinates implies that, as in the natural coordinates approach, the constraint equations present, at most, a quadratic dependency on the generalized coordinates for the most common kinematic constraints [30]. Moreover, due to the type of the elements that compose the rigid body, it is possible to determine the kinematics of a generic point or vector directly from the generalized coordinates of the system using a set of constant transformation matrices. This procedure, similar to the one utilized in the natural coordinates formulation [29,53], systematizes the evaluation of the coordinates, velocities and acceleration of any point or vector of the system, simplifying the kinematic definition of the constraint equations, the application of external forces and moments or the mathematical derivation of the system mass matrix. Since these transformation matrices are constant in time, the contributions to the Jacobian matrix are mostly constant or linear on the generalized coordinates, meaning that the contributions to the right-hand side vectors of velocity and acceleration can be null, linear, or quadratic [30].

Thus, the present work extends authors' previous developments [30] to formulate spatial multibody mechanical systems using fully Cartesian coordinates (FCC) together with a generic rigid body (GRB) formulation. In the sequel of this process, the main ingredients related to kinematics and dynamics of spatial multibody systems are presented in a detailed manner. Moreover, the main features of the proposed formulation are compared with natural and Cartesian coordinates, which allows for a discussion about the main benefits and limitations of the proposed approach. Finally, the kinematic and dynamic differences between modeling the generic rigid body with a fully-defined and reduced approach are also analyzed and their computational performance examined.

The validity and accuracy of the formulation using both the fully-defined and reduced modeling approaches are assessed by performing a forward dynamic (FD) analysis of five classical benchmark problems. The trajectory of relevant points of the model, the violation of the kinematic constraints, the variation of the mechanical energy and the computational performance are compared with the equivalent models available in the Library of Computational Benchmark Problems available at International Federation for the Promotion of Mechanism and Machine Science website (IFToMM) [54,55]. In order to evaluate the differences in the computational time between a fully-defined or a reduced modeling approach, the FCC-GRB formulation is applied in the simulation of a n-four-bar mechanism with a variable number of segments.

One of the applications previously identified for the planar FCC-GRB formulation is its use in teaching multibody dynamics in higher education and advanced courses [30]. Since the theoretical principles are maintained in the spatial version, this formulation could have an even greater impact in the educational field. Therefore, the main steps required for the complete implementation of the formulation are presented in detail throughout this work, so that, along with the planar approach [30], they can be directly used as a learning tool.

2. Fully Cartesian coordinates formulation in spatial multibody systems

2.1. Fundamental equations

One of the main differences among global, semi-recursive, and recursive formulations lies in the type of governing equations that describe the system kinematics and dynamics. This distinction arises from the type of coordinates used to describe the system configuration and the strategy employed to characterize the degrees-of-freedom (DoF) of the model under analysis. The formulation of the kinematic constraints and EoM for the FCC-GRB formulation is similar to other global approaches. Therefore, only the fundamental equations needed to describe the system kinematics and dynamics are presented in this section. The interested reader is referred to the planar version of the formulation [30] or the works of Nikravesh [28], Jalón and Bayo [29], Haug [40], Shabana [39] and Amirouche [56] for a detailed deduction of the equations.

2.1.1. Kinematic analysis

The configuration of a multibody model is fully defined by a set of independent or dependent coordinates, usually referred to as generalized coordinates ($\mathbf{q}$). Considering a system composed of *nb* bodies, the corresponding set of coordinates can mathematically be expressed as

$$\mathbf{q} = \left\{ \mathbf{q}_1^{\mathrm{T}} \quad \cdots \quad \mathbf{q}_i^{\mathrm{T}} \quad \ldots \quad \mathbf{q}_{nb}^{\mathrm{T}} \right\}^{\mathrm{T}} \tag{1}$$

where $\mathbf{q}_i$ represents the vector of the generalized coordinates of the *i-th* body. Depending on the type of coordinates utilized, the vector $\mathbf{q}$ can describe the position of a set of points and vectors of the system, the global orientation of the bodies, or the relative angular displacements between adjacent elements.

In the presence of dependent coordinates, such as in the global formulations, the number of generalized coordinates (*nq*) surpasses the number of DoFs of the system, requiring the definition of a set of algebraic equations to describe their dependencies. These are usually introduced in the form of kinematic constraints (KC), and gathered in the vector of the kinematic constraints of the system as

$$\mathbf{\Phi}(\mathbf{q}, t) = \begin{Bmatrix} \mathbf{\Phi}_1 \\ \vdots \\ \mathbf{\Phi}_i \\ \vdots \\ \mathbf{\Phi}_{nc} \end{Bmatrix} = 0 \tag{2}$$

in which $\mathbf{\Phi}_i$ is the *i-th* set of kinematic constraint equations, and *nc* the number of kinematic constraints. Besides describing the dependencies between the generalized coordinates, the vector of kinematic constraints also defines the structure and topology of the system, and the kinematics of the motion under analysis. This desideratum is achieved by mathematically describing the geometric relations existing between the generalized coordinates, thus defining the structure of the bodies (e.g., KC of rigid body type), the kinematic pairs and their corresponding DoFs (e.g., KC of joint type), among other topological features. Vector $\mathbf{\Phi}$ also includes equations to express the geometric relations between the generalized coordinates and other external variables, allowing for the definition of topologic relations with external elements (e.g., KC with respect to global reference frame), the implementation of passive and active actuators (e.g., KC of actuator type), or the prescription of the DoFs of the system (e.g., KC of driver type).

Mathematically, vector $\mathbf{\Phi}$ is composed by a set of algebraic equations in their holonomic form that, when properly solved, allows for the computation of the kinematic consistent positions of the system. As some kinematic constraints exhibit a nonlinear behavior, it should be noted that Eq. (2) represents a system of nonlinear equations. Different methods have been proposed to solve the kinematic constraint equations, varying in their complexity and rate of convergence [57,58]. A common approach to compute Eq. (2) deals with the use of the iterative Newton-Raphson method or, when in the presence of redundant constraints, the iterative Newton-Raphson method with the least square approach [29,30].

The full characterization of the system kinematics requires the calculation of the consistent velocities and accelerations for the motion under analysis. For that purpose, the approach available in [29] can be applied, meaning that the generalized velocities of a multibody system ($\dot{\mathbf{q}}$) defined with an FCC-GRB formulation can be directly obtained by solving the velocity constraint equations ($\dot{\mathbf{\Phi}}$) in order to $\dot{\mathbf{q}}$ as

$$\dot{\mathbf{\Phi}}(\mathbf{q}, \dot{\mathbf{q}}, t) = \mathbf{0} \Leftrightarrow \mathbf{\Phi}_{\mathbf{q}} \dot{\mathbf{q}} = \boldsymbol{\nu} \tag{3}$$

with

$$\boldsymbol{\nu}(t) = -\mathbf{\Phi}_t \tag{4}$$

where $\mathbf{\Phi}_{\mathbf{q}}$ represents the Jacobian matrix of the system, containing the derivatives of the kinematic constraints with respect to the generalized coordinates, $\boldsymbol{\nu}$ is the right-hand side vector of velocities, and $\mathbf{\Phi}_t$ denotes the vector containing the partial derivatives of the kinematic constraint equations with respect to time. Similarly, the generalized accelerations of the system ($\ddot{\mathbf{q}}$) can be obtained solving the acceleration constraint equations as

$$\ddot{\mathbf{\Phi}}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, t) = \mathbf{0} \Leftrightarrow \mathbf{\Phi}_{\mathbf{q}} \ddot{\mathbf{q}} = \boldsymbol{\gamma} \tag{5}$$

with

$$\boldsymbol{\gamma}(\mathbf{q}, \dot{\mathbf{q}}, t) = \boldsymbol{\nu}_t - \left(\mathbf{\Phi}_{\mathbf{q}} \dot{\mathbf{q}}\right)_{\mathbf{q}} \dot{\mathbf{q}} \tag{6}$$

in which $\boldsymbol{\gamma}$ and $\boldsymbol{\nu}_t$ represent respectively the vector of the right-hand side vector of accelerations and the vector containing the partial derivatives of vector $\boldsymbol{\nu}$ with respect to time.

2.1.2. Dynamic analysis

The dynamic analysis of a multibody system requires the proper establishment of the EoM, as these equations express the relations between the internal, external, inertial, and velocity-dependent forces and moments that act on the system and its kinematics [32]. In the case of global formulations, the EoM are generic and their derivation independent of the system topology. In fact, this approach simplifies the dynamic analysis of complex multibody systems, since it does not require the deduction of the EoM for each system under analysis, as in the case of recursive formulations [30]. Thus, the EoM for the FCC-GRB formulation with a generic rigid body follows the same approach applied in the natural or Cartesian coordinates and can be expressed as [29,40,56]

$$\mathbf{M}\ddot{\mathbf{q}} + \boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\lambda} = \mathbf{g} \tag{7}$$

where $\mathbf{M}$ represents the mass matrix of the system, $\boldsymbol{\lambda}$ is the vector of Lagrange multipliers, and $\mathbf{g}$ denotes the vector of the generalized forces. Equation (7) describes the relation between the inertial ($\mathbf{M}\ddot{\mathbf{q}}$), internal ($\boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\lambda}$) and external forces ($\mathbf{g}$) that act on the system. When the objective of the analysis is to determine the dynamic response of the system to a set of know external forces (forward dynamics), Eq. (7) represents a second order ordinary differential equation (ODE) that needs to be solved for accelerations, and subsequently integrated over time to obtain the generalized velocities and positions. However, since both the generalized accelerations and Lagrange multipliers are unknown variables, the system becomes underdetermined, and a new set of equations needs to be introduced to allow for its solution. For this purpose, the position constraint equations, given by Eq. (2), can be considered, yielding the follow system

$$\begin{cases} \mathbf{M}\ddot{\mathbf{q}} + \boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\lambda} = \mathbf{g} \\ \boldsymbol{\Phi}(\mathbf{q}, t) = \mathbf{0} \end{cases} \tag{8}$$

Equation (8) represents a set of index-3 differential algebraic equations (DAEs), where the first expression describes the dynamics of the system, and the second one ensures the kinematic consistency of the multibody system. To overcome the numerical problems of solving higher index DAEs, Eq. (8) is usually replaced by a DAE of lower order [59,60]. A common method to reduce the order of the equations is to force the kinematic consistency of the system using the acceleration constraint equations, expressed by Eq. (5), instead of the position constraint equations. Thus, the EoM for a constrained multibody mechanical system can be expressed as a set of linear algebraic equations, which, when solved from a forward dynamics perspective, represent a set of index-1 DAEs as

$$\begin{bmatrix} \mathbf{M} & \boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}} \\ \boldsymbol{\Phi}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{g} \\ \boldsymbol{\gamma} \end{Bmatrix} \tag{9}$$

It must be noticed that the reduction of the index of the DAEs can result in the violation of the kinematic constrains during the integration procedure, leading to the appearance of numerical instabilities and drift problems [35,60,61]. Over the decades, different methods have been proposed to deal with these numerical difficulties, such as stabilization procedures [61–64], augmented Lagrangian formulations [59,65,66], penalty formulations [25,67–69], velocity projections methods [70–73] or other partition methodologies [74–81].

2.2. Fully Cartesian coordinates with a generic rigid body

One of the key characteristics that distinguishes the different global multibody formulations is the approach utilized to represent the bodies that compose the system under study. The different methodologies available influence the number and type of the coordinates and kinematic constraints needed to properly define the system, the methods required to apply the external forces and moments, the definition of the system mass matrix, among other features.

When using Cartesian coordinates formulations, the position and orientation of a given rigid body are described by the Cartesian coordinates of one point and a set of angular-related variables. If applied in the study of spatial systems, six coordinates are required to properly define the six DoFs associated with a free rigid body, namely three Cartesian coordinates of one reference point, and three Euler angles or Rodrigues parameters to describe respectively the translation of the body and its orientation. These representations are characterized by the existence of singular configurations, which can lead to the appearance of numerical issues during the dynamic analysis of the system [39]. A common approach to handle this issue is to replace this type of representation with another (e.g. Euler parameters or the Rodrigues formula), increasing the total number of generalized coordinates per body to seven. However, the incorporation of an extra coordinate leads to the appearance of dependent coordinates, requiring the introduction of one rigid body kinematic constraint [28,39,82].

An alternative solution to describe the kinematics of a multibody system is to utilize the natural coordinates formulation, originally proposed by Jalón and his co-authors [29,41,83]. With this formulation, the position and orientation of each rigid body is defined resorting solely to the use of Cartesian coordinates of points of interest and unit vectors, reason why they are referred to as fully cartesian coordinates [29]. An important feature related to the natural coordinates formulation is the possibility of sharing points and vectors between different bodies, resulting in an implicit definition of several kinematic joints (e.g., spherical joint and revolute joint). As a result, the number of generalized coordinates and joint kinematic constraints needed to describe the configuration of a system is significantly reduced when compared with a procedure based on explicit definition of the joints. This particular characteristic is, in fact, the reason for the name "natural" associated with this formulation, as the kinematic joints arise naturally from the sharing of points and vectors. This modeling procedure is achieved by adopting a variable configuration for the definition of a given rigid body. Depending on the topology of the system to be discretized, different types of bodies with a variable number of points and vectors can be employed, such as two points and two vectors, three non-collinear points and one vector, four or more non-collinear points. As an example, a typical modeling approach to describe a link as a rigid body in a spatial mechanism involves using two points, usually located in its extremities, and two unit vectors, generating a total of 12 generalized coordinates. Consequently, a set of kinematic constraints equations needs to be introduced, at least six, to express the relations between the dependent generalized coordinates. These kinematic restrictions are typically introduced in the form of rigid body kinematic constraints, which include algebraic relations to ensure: (*i*) the constant distance between the points that compose the body, that is constant distance condition; (*ii*) the unitary nature of the unit vectors, that is unitary module condition; (*iii*) the constant angle between unit vectors, that is constant angle condition; (*iv*) the geometric relation between multiple points and vectors with respect to a reference frame composed by three selected vectors, that is linear combination conditions [29,43]. An alternative modeling approach, which uses a smaller number of vectors per rigid body, can be utilized to reduce the dimension of the system to be solved. Taking as example the case of the link considered above, this can be described by the Cartesian coordinates of two points and one unit vector, resulting in a total of nine generalized coordinates. Despite requiring less generalized coordinates and rigid body constraints, this procedure leads to the existence of variable mass matrices, which depend on the generalized velocities of the system. A common methodology to handle this particular aspect consists of decomposing the rigid body mass matrix in two terms, one dependent of the generalized coordinates, which is assembled in system mass matrix, and a second variable term that is treated as a velocity-dependent inertial force [29,53].

2.2.1. Fully-defined rigid body

In this work, a new approach to model spatial multibody systems with fully Cartesian coordinates is proposed. Contrarily to the natural coordinates formulation, in which the definition of each rigid body is dependent on the topology of the system, the presented formulation considers the concept of a generic rigid body. Thus, each body can be described with a fixed and predetermined kinematic structure composed of one vector that defines the reference point ($\mathbf{r}_{O_i}$) and three unit vectors ($\mathbf{u}_i$, $\mathbf{v}_i$, $\mathbf{w}_i$) to represent, respectively, its position and orientation in space (see Fig. 1a). Thus, the generalized coordinates vector for a given rigid body i ($\mathbf{q}_i$) defined according to the proposed FCC-GRB formulation can be written as

$$\mathbf{q}_i = \left\{ \mathbf{r}_{O_i}^{\mathrm{T}} \quad \mathbf{u}_i^{\mathrm{T}} \quad \mathbf{v}_i^{\mathrm{T}} \quad \mathbf{w}_i^{\mathrm{T}} \right\}^{\mathrm{T}} \tag{10}$$

With the purpose to keep the analysis simple, in what follows, the rigid body reference point O_i is located at the center of mass (CoM) of each body, and the rigid body vectors are aligned with the principal axes of inertia of the corresponding rigid bodies. An important feature associated with the FCC-GRB formulation is the definition of the rigid bodies resorting solely to Cartesian coordinates of points and vectors. This modeling approach has the advantage of not requiring the use of explicit angular-related variables, which eventually produce kinematic constraints that have a linear or quadratic dependency on the system generalized coordinates.

The rigid body definition described in Eq. (10) generates 12 dependent generalized coordinates per body, requiring, at least, six constraint equations to describe the topologic relations between them. As in the case of formulation with natural coordinates, the topologic relations are introduced in the form of rigid body constraint equations, in particular the unitary module condition and the

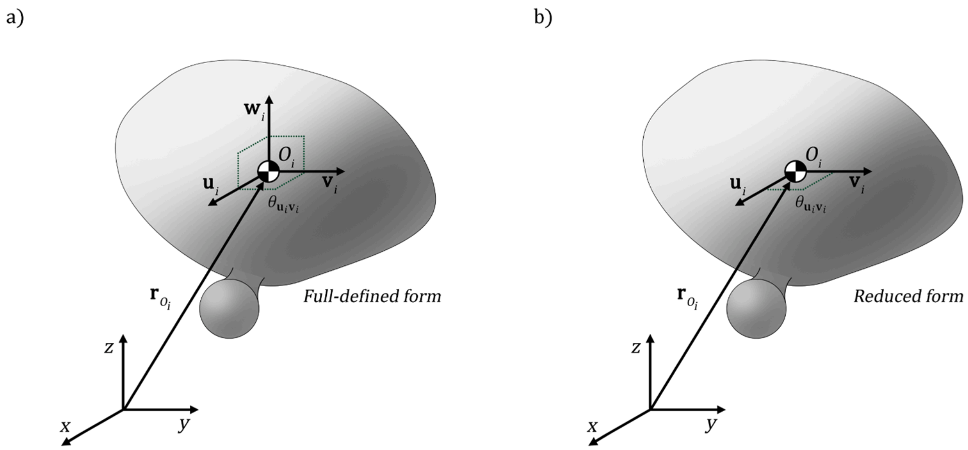

Fig. 1. Representation of the kinematic structure of a generic rigid body defined with FCC: a) Fully-defined variation (12 generalized coordinates); b) Reduced variation (9 generalized coordinates).

constant angle condition. However, in the FCC-GRB formulation, the kinematic constraints required to properly define the bodies are independent of the topology of the segments being modeled. For the case of a rigid body defined with one point and three vectors, three unit module conditions need to be added to the vector of the kinematic constraints ($\mathbf{\Phi}$) to guarantee the unitary nature of the rigid body vectors and three constant angle conditions to ensure that their relative orientation remains constant along the period of analysis, as it will be discussed in Sections 2.3.1 and 2.3.2.

When compared to other common global formulations, defining a multibody system with FCC-GRB tends to generate a larger number of generalized coordinates. In fact, the FCC formulation generates the same number of coordinates per rigid body as the natural coordinates formulation. However, because the natural coordinates formulation allows for the sharing of points and vectors between bodies, it ultimately produces fewer total coordinates. To address this, the next section explores an alternative modeling approach where the base generic rigid body is substituted with a reduced generic rigid body, which comprises fewer generalized coordinates.

2.2.2. Reduced rigid body

Knowing that from two non-collinear vectors it is possible to obtain a vector basis in 3D using orthogonalization methods [84], the kinematic structure of the generic rigid body can be further simplified to reduce the total number of generalized coordinates that each body generates. This procedure is performed by considering one vector describing the position of the reference point ($\mathbf{r}_{O_i}$) and only two directional unit vectors ($\mathbf{u}_i$, $\mathbf{v}_i$) to describe the reduced generic rigid body (RGRB) (see Fig. 1b), resulting in a total of nine generalized coordinates as

$$\mathbf{q}_i = \left\{ \mathbf{r}_{O_i}^{\mathrm{T}} \quad \mathbf{u}_i^{\mathrm{T}} \quad \mathbf{v}_i^{\mathrm{T}} \right\}^{\mathrm{T}} \tag{11}$$

It is important to notice that although the reduced modeling approach decreases the total number of generalized coordinates and kinematic constraints of rigid body type (eliminating two constant angle conditions and one unit module condition), it leads to the appearance of variable mass matrices. Similar to the natural coordinates formulation, this issue can be addressed by dividing the mass matrix into two components: one that is incorporated into the system mass matrix and another that is treated as an external force (see Section 4.2).

2.2.3. Relation between a reduced and fully-defined rigid body

It should be noted that, depending on the modeling strategy adopted to describe a generic rigid body (fully-defined or reduced), the vector containing its generalized coordinates ($\mathbf{q}_i$) differs. Consequently, the kinematic constraint equations that define both the topology and the guiding of the system are not identical. To address this issue, this section introduces the kinematic relations required to convert the reduced system into its fully-defined equivalent. These relations provide a consistent framework for defining the multibody system and the necessary mathematical tools to derive expressions specific to the reduced approach.

Despite allowing for the decrease of the number of generalized coordinates, the use of a reduced rigid body implies that the vector basis, composed of vectors $\mathbf{u}_i$, $\mathbf{v}_i$ and $\mathbf{w}_i$, and which defines the inertial frame of the body ($\widehat{\boldsymbol{\xi}}$, $\widehat{\boldsymbol{\eta}}$, $\widehat{\boldsymbol{\zeta}}$), is not fully determined. Therefore, a procedure to transform the set of generalized coordinates of the reduced rigid body i ($\mathbf{q}_i$) into the equivalent fully-defined vector ($\mathbf{q}_{3v_i}$) is first required. This desideratum can be achieved by determining a vector basis using the rigid body vectors ($\mathbf{u}_i$ and $\mathbf{v}_i$) and orthogonalization methods. Accordingly, let one consider an auxiliary vector $\mathbf{w}_i$ perpendicular to the plane defined by the two rigid body vectors of body i, such that

$$\mathbf{w}_i = \mathbf{u}_i \times \mathbf{v}_i \tag{12}$$

Thus, it is possible to define a set of algebraic equations that relate the set of reduced generalized coordinates of the rigid body i ($\mathbf{q}_i$) and the fully-defined equivalent vector ($\mathbf{q}_{3v_i}$) considering a transformation matrix $\overline{\mathbf{V}}_i$, such that

$$\mathbf{q}_{3v_i} = \left\{ \mathbf{r}_{O_i}^{\mathrm{T}} \quad \mathbf{u}_i^{\mathrm{T}} \quad \mathbf{v}_i^{\mathrm{T}} \quad \mathbf{w}_i^{\mathrm{T}} \right\}^{\mathrm{T}} = \overline{\mathbf{V}}_i \mathbf{q}_i \tag{13}$$

with

$$\overline{\mathbf{V}}_i = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \widetilde{\mathbf{u}}_i \end{bmatrix}_{(12\times 9)} \tag{14}$$

where $\mathbf{I}_3$ and $\mathbf{0}_3$ are a 3×3 identity and null matrices, respectively, and $\widetilde{\mathbf{u}}_i$ represents a skew-symmetric matrix that allows for the computation of the cross product presented on Eq. (12), as

$$\mathbf{w}_i = \widetilde{\mathbf{u}}_i \mathbf{v}_i = \begin{bmatrix} 0 & -u_{i_z} & u_{i_y} \\ u_{i_z} & 0 & -u_{i_x} \\ -u_{i_y} & u_{i_x} & 0 \end{bmatrix} \begin{bmatrix} v_{i_x} \\ v_{i_y} \\ v_{i_z} \end{bmatrix} \tag{15}$$

By algebraically manipulating Eq. (14), it is possible to rewrite it in terms of a constant matrix $\mathbf{S}$ and the transformation matrix $\mathbf{V}_i$, such that

$$\mathbf{q}_{3v_i} = \mathbf{S}\mathbf{V}_i\mathbf{q}_i \tag{16}$$

with

$$\mathbf{S} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \frac{1}{2}\mathbf{I}_3 \end{bmatrix}_{(12\times12)} \tag{17}$$

and

$$\mathbf{V}_i = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \\ \mathbf{0}_3 & -\widetilde{\mathbf{v}}_i & \widetilde{\mathbf{u}}_i \end{bmatrix}_{(12\times9)} \tag{18}$$

The expression for the velocity ($\dot{\mathbf{q}}_{3v_i}$) of the fully-defined equivalent vector can be obtained by differentiating Eq. (16) with respect to time, as

$$\dot{\mathbf{q}}_{3v_i} = \left\{ \dot{\mathbf{r}}_{O_i}^{\mathrm{T}} \quad \dot{\mathbf{u}}_i^{\mathrm{T}} \quad \dot{\mathbf{v}}_i^{\mathrm{T}} \quad \dot{\mathbf{w}}_i^{\mathrm{T}} \right\} = \mathbf{S}\dot{\mathbf{V}}_i\mathbf{q}_i + \mathbf{S}\mathbf{V}_i\dot{\mathbf{q}}_i \tag{19}$$

By employing the following cross product relation

$$\widetilde{\mathbf{v}}_i\mathbf{u}_i = \widetilde{\mathbf{u}}_i^{\mathrm{T}}\mathbf{v}_i = -\widetilde{\mathbf{u}}_i\mathbf{v}_i \tag{20}$$

the expression presented in Eq. (19) can be reduced to

$$\dot{\mathbf{q}}_{3v_i} = \begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & -\frac{1}{2}\widetilde{\mathbf{v}}_i & \frac{1}{2}\widetilde{\mathbf{u}}_i \end{bmatrix} \begin{Bmatrix} \mathbf{r}_{O_i} \\ \mathbf{u}_i \\ \mathbf{v}_i \end{Bmatrix} + \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \\ \mathbf{0}_3 & -\frac{1}{2}\widetilde{\mathbf{v}}_i & \frac{1}{2}\widetilde{\mathbf{u}}_i \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{r}}_{O_i} \\ \dot{\mathbf{u}}_i \\ \dot{\mathbf{v}}_i \end{Bmatrix} = \begin{bmatrix} \dot{\mathbf{r}}_{O_i} \\ \dot{\mathbf{u}}_i \\ \dot{\mathbf{v}}_i \\ \widetilde{\mathbf{u}}_i\dot{\mathbf{v}}_i - \widetilde{\mathbf{v}}_i\dot{\mathbf{u}}_i \end{bmatrix} = \mathbf{V}_i\dot{\mathbf{q}}_i \tag{21}$$

In turn, the acceleration ($\ddot{\mathbf{q}}_{3v_i}$) for the fully-defined equivalent vector can be obtained by differentiating the previous expression with respect to time, yielding

$$\ddot{\mathbf{q}}_{3v_i} = \left\{ \ddot{\mathbf{r}}_{O_i}^{\mathrm{T}} \quad \ddot{\mathbf{u}}_i^{\mathrm{T}} \quad \ddot{\mathbf{v}}_i^{\mathrm{T}} \quad \ddot{\mathbf{w}}_i^{\mathrm{T}} \right\}^{\mathrm{T}} = \dot{\mathbf{V}}_i\dot{\mathbf{q}}_i + \mathbf{V}_i\ddot{\mathbf{q}}_i \tag{22}$$

with

$$\dot{\mathbf{V}}_i = \begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & -\dot{\widetilde{\mathbf{v}}}_i & \dot{\widetilde{\mathbf{u}}}_i \end{bmatrix}_{(12\times9)} \tag{23}$$

It should be noted that by utilizing the two transformation matrices $\mathbf{V}$ and $\dot{\mathbf{V}}$ along with Eqs. (16) to (23), the kinematics of the reduced rigid bodies can be fully determined, providing the necessary mathematical relations for conducting the kinematic and dynamic analysis of the multibody system. For evaluating the kinematic constraints, Eq. (13) and matrix $\overline{\mathbf{V}}$ can be used directly instead of Eq. (16), thereby reducing the computational effort of this step.

2.3. Kinematic constraints of rigid body

The establishment of the kinematic constraints of rigid body in the vector of kinematic constraints is required to ensure the rigid nature of the body. These constraints are defined in the form of algebraic equations that relate the dependent generalized coordinates that describe the rigid body. It is worth noting that aligning the rigid body vectors with the principal axes of inertia of the segment implies the existence of a direct relation between the generalized coordinates of the rigid body and its rotation matrix. However, since the formulation does not include angular variables, the geometric properties intrinsic to the definition of the body topology, namely the orthogonality of the basis vectors defining the local reference frame and their unitary nature, must be explicitly defined as kinematic constraint equations.

As described previously, with the definition of a generic rigid body, the modeling procedure is significantly simplified, requiring the introduction of the same number and type of kinematic constraints regardless of the topology of the body being modeled. In the formulation proposed in this work, two types of kinematic constraints are needed to properly define a given rigid body, namely the unitary module condition of the rigid body vectors and the constant angle condition between two unit vectors of the same rigid body. For the sake of the computational implementation, the kinematic constraint equations and the contributions to the Jacobian matrix and right-hand side vectors of the velocity and acceleration are detailed in the next sections.

2.3.1. Unitary module condition

The unitary module condition ensures that the module of each rigid body vector remains constant and equal to one during the kinematic and dynamic analysis of the multibody system. Mathematically, this condition can be represented by one algebraic equation, which states that the dot product of the rigid body vector with itself is equal to one. Thus, considering the case of the vector **u** of the rigid body i, the unitary module condition can be defined as

$$\Phi^{\mathbf{u}_i} = \mathbf{u}_i{}^{\mathrm{T}}\mathbf{u}_i - 1 = 0 \tag{24}$$

Equation (24) expresses a quadratic relation between the generalized coordinates of the vector being constrained. Consequently, the non-zero contributions to the Jacobian matrix are linear in the generalized coordinates and their computational evaluation straightforward and efficient, that is

$$\mathbf{\Phi}_{\mathbf{q}}^{\mathbf{u}_i} = \left\{ \overbrace{\mathbf{0}^{\mathrm{T}}}^{\mathbf{\Phi}_{\mathbf{r}_{O_i}}^{\mathbf{u}_i}} \quad \overbrace{2\mathbf{u}_i^{\mathrm{T}}}^{\mathbf{\Phi}_{\mathbf{u}_i}^{\mathbf{u}_i}} \quad \overbrace{\mathbf{0}^{\mathrm{T}}}^{\mathbf{\Phi}_{\mathbf{v}_i}^{\mathbf{u}_i}} \quad \overbrace{\mathbf{0}^{\mathrm{T}}}^{\mathbf{\Phi}_{\mathbf{w}_i}^{\mathbf{u}_i}} \right\}_{(1\times 12)} \tag{25}$$

As Eq. (24) presents a scleronomic nature, meaning that its definition is independent of the time, the contributions to the right-hand side vector of the velocities ($\boldsymbol{\nu}$) are null, resulting

$$\nu^{\mathbf{u}_i} = 0 \tag{26}$$

In turn, the contributions to the right-hand side vector of the accelerations ($\boldsymbol{\gamma}$) exhibit a quadratic relation between the generalized velocities of the vector being constrained, and, consequently, their computational evaluation is efficient, yielding

$$\gamma^{\mathbf{u}_i} = -2\dot{\mathbf{u}}_i^{\mathrm{T}}\dot{\mathbf{u}}_i \tag{27}$$

2.3.2. Constant angle between rigid body vectors

The constant angle condition ensures that the different vectors that compose the rigid body maintain their relative orientation during the period of analysis. From a mathematical point of view, this relation can be described by one algebraic equation that forces the dot product between two vectors to be constant. Thus, considering the vectors **u** and **v** from rigid body i, the constant angle condition can be stated as

$$\Phi^{\mathbf{u}_i\mathbf{v}_i} = \mathbf{u}_i^{\mathrm{T}}\mathbf{v}_i - \|\mathbf{u}_i\|\|\mathbf{v}_i\| \cos\left(\theta_{\mathbf{u}_i\mathbf{v}_i}\right) = 0 \tag{28}$$

where $\theta_{\mathbf{u}_i\mathbf{v}_i}$ is the constant angle between vectors $\mathbf{u}_i$ and $\mathbf{v}_i$. Equation (28) expresses a quadratic relation between the generalized coordinates of the two vectors being constrained, presenting also a scleronomic structure. Hence, the contributions to the Jacobian matrix and right-hand-side vector of velocities are respectively linear and null on the generalized coordinates and the contributions to the right-hand-side vector of accelerations present a quadratic relation on the generalized velocities

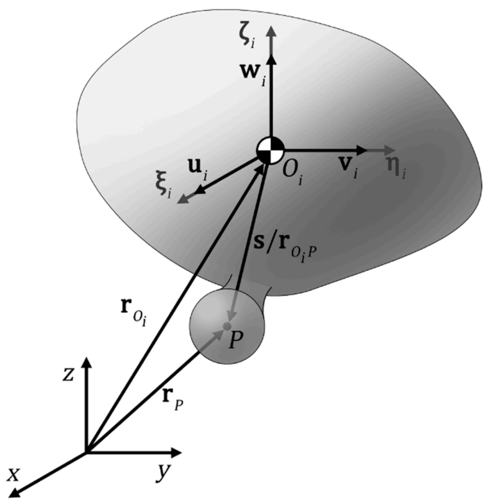

Fig. 2. Schematic representation of a generic point P and vector **s** belonging to rigid body i in the global reference frame.

$$\mathbf{\Phi}_{\mathbf{q}}^{\mathbf{u}_i\mathbf{v}_i} = \left\{ \overbrace{\mathbf{0}^{\mathrm{T}}}^{\mathbf{\Phi}_{\mathbf{r}_{O_i}}^{\mathbf{u}_i\mathbf{v}_i}} \quad \overbrace{\mathbf{v}_i^{\mathrm{T}}}^{\mathbf{\Phi}_{\mathbf{u}_i}^{\mathbf{u}_i\mathbf{v}_i}} \quad \overbrace{\mathbf{u}_i^{\mathrm{T}}}^{\mathbf{\Phi}_{\mathbf{v}_i}^{\mathbf{u}_i\mathbf{v}_i}} \quad \overbrace{\mathbf{0}^{\mathrm{T}}}^{\mathbf{\Phi}_{\mathbf{w}_i}^{\mathbf{u}_i\mathbf{v}_i}} \right\}_{(1\times 12)} \tag{29}$$

$$\nu^{\mathbf{u}_i\mathbf{v}_i} = 0 \tag{30}$$

$$\gamma^{\mathbf{u}_i\mathbf{v}_i} = -2\dot{\mathbf{u}}_{\mathrm{i}}^{\mathrm{T}}\dot{\mathbf{v}}_i \tag{31}$$

3. Kinematics of 3D multibody systems with FCC-GRB formulation

3.1. Kinematics of generic points and vectors

3.1.1. Fully-Defined rigid body

One of the major findings observed in the planar version of the formulation [30], which is similar to the modeling approach utilized with natural coordinates [29], is that the determination of the position of a generic point belonging to a given rigid body can be fully determined by means of a constant matrix **C** and the generalized coordinates of the associated rigid body. Moreover, the coefficients of the matrix **C** are directly the coordinates of the generic point P in the local reference frame of the rigid body to which it is attached, simplifying its determination [30]. A similar approach can be followed to model spatial multibody systems, while preserving the same theoretical basis. Thus, considering a generic point P belonging to the rigid body i, as schematically represented in Fig. 2, its position in the global reference frame ($\mathbf{r}_P$) can be expressed as

$$\mathbf{r}_P = \mathbf{r}_{O_i} + \mathbf{r}_{O_iP} \tag{32}$$

where $\mathbf{r}_{O_iP}$ is the position vector that connects the origin of the local reference frame of rigid body i ($\mathbf{r}_{O_i}$) and the point P in the global reference frame. Let one consider a reference basis defined by the three rigid body unit vectors of body i, the vector $\mathbf{r}_{O_iP}$ can be written as a linear combination of this basis as

$$\mathbf{r}_{O_iP} = c_1^P\mathbf{u}_i + c_2^P\mathbf{v}_i + c_3^P\mathbf{w}_i \tag{33}$$

where c_1^P, c_2^P and c_3^P are the linear combination factors that express this relation. By rewriting vector $\mathbf{r}_{O_iP}$ in the local reference frame of body i ($\mathbf{r}'_{O_iP}$), the following expression unfolds

$$\mathbf{r}'_{O_iP} = \left\{ \begin{matrix} \xi_i^P \\ \eta_i^P \\ \zeta_i^P \end{matrix} \right\} = c_1^P\mathbf{u}'_i + c_2^P\mathbf{v}'_i + c_3^P\mathbf{w}'_i \tag{34}$$

where ξ_i^P, η_i^P and ζ_i^P are the Cartesian coordinates of vector $\mathbf{r}'_{O_iP}$ and $\mathbf{u}'_i$, $\mathbf{v}'_i$ and $\mathbf{w}'_i$ represent the vectors $\mathbf{u}_i$, $\mathbf{v}_i$ and $\mathbf{w}_i$ in the local reference frame of body i. As addressed before, vectors $\mathbf{u}_i$, $\mathbf{v}_i$ and $\mathbf{w}_i$ can also describe a vector basis that defines the inertial reference frame of body i. Therefore, vectors $\mathbf{u}'_i$, $\mathbf{v}'_i$ and $\mathbf{w}'_i$ represent directly the three basis vectors of this reference frame ($\widehat{\boldsymbol{\xi}}$, $\widehat{\boldsymbol{\eta}}$, $\widehat{\boldsymbol{\zeta}}$). Thus, taking this geometric relation into consideration, Eq. (34) can be rewritten as

$$\mathbf{r}'_{O_iP} = c_1^P\widehat{\boldsymbol{\xi}} + c_2^P\widehat{\boldsymbol{\eta}} + c_3^P\widehat{\boldsymbol{\zeta}} = c_1^P\left\{ \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \right\} + c_2^P\left\{ \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} \right\} + c_3^P\left\{ \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \right\} = \left\{ \begin{matrix} c_1^P \\ c_2^P \\ c_3^P \end{matrix} \right\} = \left\{ \begin{matrix} \xi_i^P \\ \eta_i^P \\ \zeta_i^P \end{matrix} \right\} \tag{35}$$

It should be highlighted that Eq. (35) presents an important finding that the linear combination factors c_1^P, c_2^P, and c_3^P are directly the local coordinates of the generic point P in the body's local reference frame. Conversely to the natural coordinates formulation, in which the linear combination factors need to be computed according to the body topology, in the FCC-GRB formulation, the determination of the matrix **C** is a direct assignment of values, not requiring additional calculations [29]. Moreover, if the local coordinates of point P are fixed in the local reference frame of the rigid body, the linear combination factors c_1^P, c_2^P, and c_3^P are constant, and, consequently, the matrix **C** is constant during the analysis. This particular feature implies that this matrix needs to be determined only once at the beginning of the analysis, which is a quite important feature in terms of computational efficiency.

Considering the relation given in Eq. (35) and rearranging Eq. (32) with respect to the generalized coordinates of the system, the following expression unfolds

$$\mathbf{r}_P = \mathbf{C}_i^P\mathbf{q}_i \tag{36}$$

with

$$\mathbf{C}_i^P = \begin{bmatrix} \mathbf{I}_3 & \xi_i^P\mathbf{I}_3 & \eta_i^P\mathbf{I}_3 & \zeta_i^P\mathbf{I}_3 \end{bmatrix} \tag{37}$$

where $\mathbf{C}_i^P$ is a 3×12 constant transformation matrix that relates the global coordinates of a generic point P with the generalized coordinates of rigid body i to which point P is attached to.

Equation (36) can be utilized to fully determine the coordinates of any generic point P belonging to a body in the multibody system, as long as its local coordinates are known. This methodology has the main advantage of simplifying the determination of any point of interest of the model, the definition of several types of kinematic joints, and the application of external forces, among other procedures. Moreover, this modeling approach can be utilized to directly determine the velocities and accelerations of any generic point from the generalized velocities ($\dot{\mathbf{q}}$) and accelerations ($\ddot{\mathbf{q}}$) of the system. Since matrix $\mathbf{C}_i^P$ is time-independent, its time-derivative is null. Thus, considering the first- and second-time derivatives of Eq. (36), the velocity ($\dot{\mathbf{r}}_P$) and acceleration ($\ddot{\mathbf{r}}_P$) of the generic point P can be expressed as

$$\dot{\mathbf{r}}_P = \mathbf{C}_i^P \dot{\mathbf{q}}_i \tag{38}$$

$$\ddot{\mathbf{r}}_P = \mathbf{C}_i^P \ddot{\mathbf{q}}_i \tag{39}$$

A similar approach can be utilized to fully define the kinematics of a generic vector in space. Thus, considering that a generic vector $\mathbf{s}$, belonging to rigid body i, can be defined as the difference of two points, one located in the origin of the rigid body (O_i) and a point P located in the tip of the vector, its kinematics can be described utilizing Eqs. (36) to (39), in the form

$$\mathbf{s} = \mathbf{r}_{O_iP} = \mathbf{r}_P - \mathbf{r}_{O_i} = \mathbf{C}_i^P \mathbf{q}_i - \mathbf{C}_i^{O_i} \mathbf{q}_i = \left(\mathbf{C}_i^P - \mathbf{C}_i^{O_i}\right)\mathbf{q}_i \tag{40}$$

where $\mathbf{C}_i^{O_i}$ denotes the transformation matrix of the rigid body reference point. Since this point also defines the origin of the local reference frame of the generic rigid body, the transformation matrix $\mathbf{C}$ simplifies to

$$\mathbf{C}_i^{O_i} = [\mathbf{I}_3 \quad \mathbf{0}_3 \quad \mathbf{0}_3 \quad \mathbf{0}_3] \tag{41}$$

By rearranging equation Eq. (40), it is possible to obtain an algebraic expression that relates the generalized coordinates of system and a constant matrix $\mathbf{C}_i^{\mathbf{s}}$, namely

$$\mathbf{s} = \mathbf{C}_i^{\mathbf{s}} \mathbf{q}_i \tag{42}$$

with

$$\begin{aligned} \mathbf{C}_i^{\mathbf{s}} = \mathbf{C}_i^P - \mathbf{C}_i^{O_i} &= \left[\mathbf{I}_3 \quad \xi_i^P \mathbf{I}_3 \quad \eta_i^P \mathbf{I}_3 \quad \zeta_i^P \mathbf{I}_3\right] - [\mathbf{I}_3 \quad \mathbf{0}_3 \quad \mathbf{0}_3 \quad \mathbf{0}_3] \\ \mathbf{C}_i^{\mathbf{s}} &= \left[\mathbf{0}_3 \quad \xi_i^{\mathbf{s}} \mathbf{I}_3 \quad \eta_i^{\mathbf{s}} \mathbf{I}_3 \quad \zeta_i^{\mathbf{s}} \mathbf{I}_3\right]_{(3\times 12)} \end{aligned} \tag{43}$$

where $\xi_i^{\mathbf{s}}$, $\eta_i^{\mathbf{s}}$, and $\zeta_i^{\mathbf{s}}$ are the components of vector $\mathbf{s}$ in the local reference frame of body i.

Similarly to the case of the generic point P, Eq. (43) shows that the constant matrix $\mathbf{C}$ for the generic vector $\mathbf{s}$ does not require additional calculations, as its entries are directly the coordinates of the vector $\mathbf{s}$ in the local reference frame of the rigid body. Moreover, the velocity ($\dot{\mathbf{s}}$) and acceleration ($\ddot{\mathbf{s}}$) of the generic vector $\mathbf{s}$ can be expressed as a linear function of the constant matrix $\mathbf{C}_i^{\mathbf{s}}$ and the generalized velocities and accelerations of the rigid body i as follows

$$\dot{\mathbf{s}} = \mathbf{C}_i^{\mathbf{s}} \dot{\mathbf{q}}_i \tag{44}$$

$$\ddot{\mathbf{s}} = \mathbf{C}_i^{\mathbf{s}} \ddot{\mathbf{q}}_i \tag{45}$$

3.1.2. Reduced rigid body

The methodology required to define the kinematics of a generic point for a reduced rigid body, i.e., a rigid body defined with only two vectors, is similar to the one described for the fully-defined case. Since the local reference frame is not explicitly defined in this modeling approach, the methodology described in Section 2.2.3 needs to be implemented to obtain the fully-defined equivalent vectors of the generalized positions ($\mathbf{q}_{3v_i}$), velocities ($\dot{\mathbf{q}}_{3v_i}$) and accelerations ($\ddot{\mathbf{q}}_{3v_i}$) of the reduced rigid body. Hence, the position, velocity and acceleration of a generic point P attached to a reduced rigid body i can be expressed as

$$\mathbf{r}_P = \mathbf{C}_i^P \mathbf{S} \mathbf{V}_i \mathbf{q}_i \tag{46}$$

$$\dot{\mathbf{r}}_P = \mathbf{C}_i^P \mathbf{V}_i \dot{\mathbf{q}}_i \tag{47}$$

$$\ddot{\mathbf{r}}_P = \mathbf{C}_i^P \left(\dot{\mathbf{V}}_i \dot{\mathbf{q}}_i + \mathbf{V}_i \ddot{\mathbf{q}}_i\right) \tag{48}$$

Similarly, the coordinates ($\mathbf{s}$), velocities ($\dot{\mathbf{s}}$) and accelerations ($\ddot{\mathbf{s}}$) for a generic vector $\mathbf{s}$ can be computed using expressions (46) to (48), considering matrix $\mathbf{C}_i^{\mathbf{s}}$ instead of matrix $\mathbf{C}_i^P$.

3.2. Kinematic constraints of joint

Kinematic joints are mechanical elements that restrain the relative movement of the different bodies that compose a constrained multibody system. For this reason, this type of element is also referred to as kinematic pairs in literature [28,29,39,85]. From a mathematical point of view, kinematic joints can be modeled by a set of algebraic equations, which enforce different geometric relations between the generalized coordinates of the model. These equations are usually defined in their homogenous form, being included in the vector of the kinematic constraints of the system. By fulfilling these conditions, the method defines explicitly the structure and topology of the system in analysis, as well as the available DoFs and range-of motion for each joint.

Different kinematic constraints of joint type can be used to model the several types of mechanical joints. These are typically divided into three major groups based on the type of contact between elements: (*i*) Lower pair joints – that are ideal joints characterized by having full contact between the two segments being constrained. Consequently, these tend to present higher contact areas and lower contact stresses (e.g. spherical, universal, revolute, translational, cylindrical, or screw joints); (*ii*) Higher pair joints – that are characterized by having a well-defined contact point, line or partial area between the two bodies. As a result, this type of joints tends to present smaller contact areas and, consequently, these are more likely to experience higher contact stresses and more wear (e.g., cam-type, chain drives, gears or rolling contact joints); (*iii*) Compound pair joints – that are kinematic pairs characterized by combining lower and/or higher pair joints to achieve the desired DoFs (e.g., universal joints defined using two revolute joints or roller/ball bearings joints) [28,86,87].

For the sake of brevity, only the kinematic constraints required to model common lower pair joints will be presented in the following sections. The differences in constraint equations, contributions to the Jacobian matrix, and right-hand side vectors of velocity and acceleration will also be discussed in detail, specifically for the two types of rigid body definitions introduced in Section 2.2.

It is important to note that both the full and reduced rigid body definitions can be utilized simultaneously to describe the constrained multibody system under analysis. For this particular case, the contributions are not detailed in this work, as these will be equal to the respective contribution of the two generic cases.

3.2.1. Coincident points between fully-defined rigid bodies

The coincident points (CP) condition ensures that two points from two distinct bodies share the same position in space. This type of kinematic constraint restrains the relative translation between the two interconnected bodies, allowing only for three rotational DoFs around this point. Consequently, the CP condition defines by itself the spherical joint, which is particularly useful to formulate the joints characterized by having rotational movements (e.g. revolute and universal joints).

Mathematically, the CP condition is guaranteed by stating that the distance between point *P*, which represents the fulcrum of the joint (see Fig. 3), calculated from rigid body *i* position ($\mathbf{r}_{P_i}$) and from rigid body *j* position ($\mathbf{r}_{P_j}$) is null along the period of analysis. This condition can easily be defined recurring to the methodology presented in Section 3.1 to compute the kinematics of a generic point from the generalized coordinates of the system. Hence, the CP constraint ($\mathbf{\Phi}^{CP}$) in its homogenous form can be expressed as

$$\mathbf{\Phi}^{CP}\left(\mathbf{q}_i,\mathbf{q}_j\right) = \mathbf{r}_{P_i} - \mathbf{r}_{P_j} = \begin{Bmatrix} \mathbf{r}_{P_{xi}} - \mathbf{r}_{P_{xj}} \\ \mathbf{r}_{P_{y_i}} - \mathbf{r}_{P_{y_j}} \\ \mathbf{r}_{P_{zi}} - \mathbf{r}_{P_{zj}} \end{Bmatrix} = \mathbf{C}_i^P\mathbf{q}_i - \mathbf{C}_j^P\mathbf{q}_j = \mathbf{0} \tag{49}$$

Equation (49) represents a set of three constraint equations that restrain the relative translations between bodies *i* and *j* in the three canonical directions ($\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$). Algebraically, Eq. (49) is a linear relation between the generalized coordinates of the bodies being constrained. Besides the computational advantages in determining the constraints violation, the linearity of the equations associated with the CP condition implies that the contributions to the Jacobian matrix are constant and equal to the corresponding matrix **C**, and null for the right-hand side vectors of velocity (ν^{CP}) and acceleration (γ^{SP}), namely

$$\mathbf{\Phi}_{\mathbf{q}}^{SP} = \begin{bmatrix} \overbrace{\mathbf{C}_i^P}^{\mathbf{\Phi}_{\mathbf{q}_i}^{CP}} & \overbrace{-\mathbf{C}_j^P}^{\mathbf{\Phi}_{\mathbf{q}_j}^{CP}} \end{bmatrix}_{(3\times 24)} \tag{50}$$

$$\nu^{CP} = \mathbf{0} \tag{51}$$

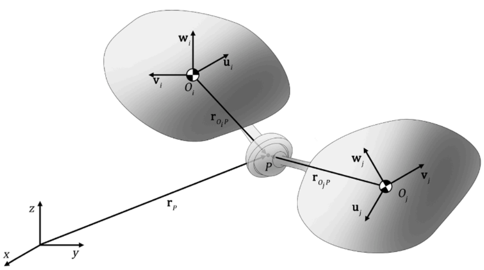

Fig. 3. Schematic representation of a spherical joint defined between rigid body *i* and *j* using the CP condition.

$$\gamma^{CP} = \mathbf{0} \tag{52}$$

The constant contributions for the Jacobian matrix and the null right-hand side vectors are a direct consequence of the constant nature of matrix $\mathbf{C}$. As its entries do not present a dependency of the time, the CP condition is scleronomic. Therefore, the contributions associated with the CP condition can be pre-assembled in the Jacobian matrix and right-hand side vectors of velocity and acceleration of the system at the beginning of the analysis. Being one of the conditions most applied during the modeling of multibody systems, as it is utilized to model all ideal joints that enable the relative rotation between two bodies, this result presents an advantage on the computational efficiency of the formulation, as no updates need to be performed during the kinematic and dynamic analysis of the system.

3.2.2. Coincident point between reduced rigid bodies

The procedure to define the CP condition between two rigid bodies established using the reduced form is similar to the one followed for fully-defined bodies. However, the equations presented in Section 3.1.2, which allow for the computation of the kinematics of a generic rigid body defined using its reduced form, should be utilized instead of the ones used for the fully-defined body. Therefore, the CP condition for a reduced rigid body can be expressed as

$$\mathbf{\Phi}^{CP}\left(\mathbf{q}_i, \mathbf{q}_j\right) = \mathbf{C}_i^P \mathbf{q}_{3v_i} - \mathbf{C}_j^P \mathbf{q}_{3v_j} = \mathbf{C}_i^P \mathbf{S} \mathbf{V}_i \mathbf{q}_i - \mathbf{C}_j^P \mathbf{S} \mathbf{V}_j \mathbf{q}_j = \mathbf{0} \tag{53}$$

Contrary to the fully-defined case, the CP condition for the reduced form presents a quadratic relation on the generalized coordinates. This results from the dependency of the transformation matrices $\mathbf{V}$ on the generalized coordinates of the system. Consequently, the contributions to the Jacobian matrix are not constant and the contributions to the vector $\boldsymbol{\gamma}$ have a quadratic dependency on the generalized coordinates, namely

$$\mathbf{\Phi}_{\mathbf{q}}^{CP} = \left[\mathbf{C}_i^P \mathbf{S}\left(\mathbf{V}_{\mathbf{q}_i}\mathbf{q}_i + \mathbf{V}_i\right) \quad -\mathbf{C}_j^P \mathbf{S}\left(\mathbf{V}_{\mathbf{q}_j}\mathbf{q}_j + \mathbf{V}_j\right)\right] = \left[\overbrace{\mathbf{C}_i^P \mathbf{V}_i}^{\mathbf{\Phi}_{\mathbf{q}_i}^{CP}} \quad \overbrace{-\mathbf{C}_j^P \mathbf{V}_j}^{\mathbf{\Phi}_{\mathbf{q}_j}^{CP}}\right]_{(3\times 18)} \tag{54}$$

$$\nu^{CP} = \mathbf{0} \tag{55}$$

$$\gamma^{CP} = -\left(\mathbf{C}_i^P \dot{\mathbf{V}}_i \dot{\mathbf{q}}_i - \mathbf{C}_j^P \dot{\mathbf{V}}_j \dot{\mathbf{q}}_j\right) \tag{56}$$

3.2.3. Coincident vectors between fully-defined rigid bodies

The coincident vectors (CV) condition ensures that two vectors belonging to two distinct rigid bodies have the same orientation and direction in space. This type of kinematic constraint restrains the DoFs associated with the rotations between the two bodies, allowing only for their translation and relative movement around a rotation axis defined by the orientation of the coincident vectors. Hence, the CV condition in tandem with the CP condition can be directly used to model the revolute joints of the system. CP condition can also be used to constrain the relative orientation between two bodies, helping in the definition of other kinematic pairs, such as the prismatic joint, decomposed spherical and universal joints, among others.

The CV condition in its explicit and homogenous form ($\mathbf{\Phi}^{CV}$) can be expressed by a set of algebraic equations that state that the differences between the coordinates of a generic vector $\mathbf{s}$ calculated from rigid body i ($\mathbf{s}_i$) and j ($\mathbf{s}_j$) are null (see Fig. 4). For that purpose, the methodology presented in Section 3.1 for the determination of the kinematics of a generic vector from the generalized coordinates of the system can be considered, yielding

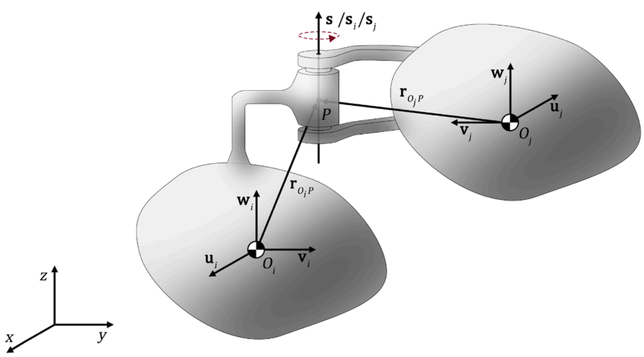

Fig. 4. Schematic representation of a revolute joint defined between rigid body i and j using the CP and CV condition.

$$\boldsymbol{\Phi}^{CV}\left(\mathbf{q}_i,\mathbf{q}_j\right)=\mathbf{s}_i-\mathbf{s}_j=\begin{Bmatrix}\mathbf{s}_{x_i}-\mathbf{s}_{x_j}\\ \mathbf{s}_{y_i}-\mathbf{s}_{y_j}\\ \mathbf{s}_{z_i}-\mathbf{s}_{z_j}\end{Bmatrix}=\mathbf{C}_i^{s}\mathbf{q}_i-\mathbf{C}_j^{s}\mathbf{q}_j=\mathbf{0} \tag{57}$$

Equation (57) describes a set of three constraint equations that restrain the relative rotation between bodies i and j in all directions with exception of the one defined by vector $\mathbf{s}$. This means that Eq. (57) generates only two independent equations, requiring proper methods to deal with redundant constraints [28,35].

As in the case of the CP condition, Eq. (57) represents a scleronomic constraint, which expresses a linear relation between the generalized coordinates of the bodies being constrained. Therefore, the contributions to the Jacobian matrix are constant and equal to the matrix $\mathbf{C}$, and null for the right-hand side vectors of velocity (ν^{CV}) and acceleration (γ^{CV}) as

$$\boldsymbol{\Phi}_{\mathbf{q}}^{CV}=\begin{bmatrix}\overbrace{\mathbf{C}_i^{s}}^{\boldsymbol{\Phi}_{\mathbf{q}_i}^{CV}} & \overbrace{-\mathbf{C}_j^{s}}^{\boldsymbol{\Phi}_{\mathbf{q}_j}^{CV}}\end{bmatrix}_{(3\times24)} \tag{58}$$

$$\boldsymbol{\nu}^{CV}=\mathbf{0} \tag{59}$$

$$\boldsymbol{\gamma}^{CV}=\mathbf{0} \tag{60}$$

The computational advantages associated with the CP formulation are also valid to the CV condition. Moreover, if the two vectors being constrained are rigid body vectors (e.g., $\mathbf{u}_i$ and $\mathbf{v}_j$), then Eqs. (57) and (58) can be simplified as

$$\boldsymbol{\Phi}^{CV}\left(\mathbf{q}_i,\mathbf{q}_j\right)=\mathbf{u}_i-\mathbf{v}_j=\mathbf{0} \tag{61}$$

$$\boldsymbol{\Phi}_{\mathbf{q}}^{CV}=\begin{bmatrix}\overbrace{\mathbf{I}_3}^{\boldsymbol{\Phi}_{\mathbf{u}_i}^{CV}} & \overbrace{-\mathbf{I}_3}^{\boldsymbol{\Phi}_{\mathbf{v}_j}^{CV}}\end{bmatrix}_{(3\times6)} \tag{62}$$

It should be noticed that, as in the case of the natural coordinates, the nature of the FCC formulation with a generic rigid body allows for an implicit definition of the CV condition, if the vectors that are being constrained are rigid body vectors. In this particular case, no algebraic equations are required to describe the CV condition, reducing the total number of kinematic constraints required to model the system.

3.2.4. Coincident vectors between reduced rigid bodies

The procedure for defining the CV condition for rigid bodies defined in their reduced form is similar to that used for fully-defined bodies. However, as with the CP condition for reduced rigid bodies, this procedure first requires computing the fully-defined equivalent vector using Eqs. (16) to (18), and then the method for computing the kinematics of a generic vector as presented in Section 3.1.2. Accordingly, the CV condition in its explicit and homogenous form ($\boldsymbol{\Phi}^{SV}$) can be expressed by

$$\boldsymbol{\Phi}^{CV}\left(\mathbf{q}_i,\mathbf{q}_j\right)=\mathbf{C}_i^{s}\mathbf{q}_{3v_i}-\mathbf{C}_j^{s}\mathbf{q}_{3v_j}=\mathbf{C}_i^{s}\mathbf{SV}_i\mathbf{q}_i-\mathbf{C}_j^{s}\mathbf{SV}_j\mathbf{q}_j=\mathbf{0} \tag{63}$$

Equation (63) is similar to the one obtained for the CP condition with reduced rigid bodies (see Eq. (53)), with the matrices $\mathbf{C}^{P}$ substituted by the matrices $\mathbf{C}^{s}$. Therefore, the contributions to the Jacobian matrix and the right hand-side vectors of velocity and acceleration are the same as the ones presented in Eqs. (54) to (56), also considering the referred substitution of matrices.

Once more, if the vectors being constrained are rigid body vectors or an implicit definition of the CV condition is used, the methods presented in the previous section for the fully-defined rigid bodies can be directly used, maintaining the same advantages reported for this case.

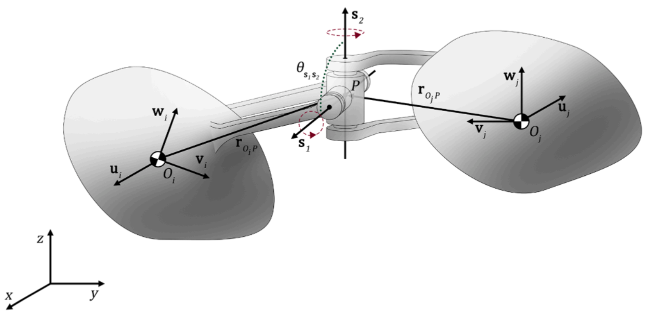

Fig. 5. Schematic representation of a universal joint defined between rigid body i and j using the CP and CAR condition.

3.2.5. Constant angular relation between two vectors belonging to fully-defined rigid bodies

The constant angular relation (CAR) condition ensures that two different generic vectors maintain their relative direction along the time. Besides ensuring the geometric relation between the two unit vectors, the CAR condition is also useful to define joints characterized by two rotational DoFs. By constraining the relative angle between the two vectors, the CAR condition allows for the free rotation of the bodies in the planes orthogonal to the axes defined by each vector. This condition, together with the CP condition, can be used to model universal-type joints. CAR condition can also be applied to define geometric relations that allow for the definition of prismatic joints, sliders, among other mechanical elements.

As in the case of the constant angle condition, the CAR relation can be expressed by one algebraic equation that forces the dot product of the two vectors to be equal throughout the period of analysis. Thus, let one consider two generic unit vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ belonging respectively to bodies i and j (see Fig. 5), it is possible to define the CAR constraint ($\mathbf{\Phi}^{CAR}$) in its homogenous form as

$$\begin{aligned}\Phi^{CAR}\left(\mathbf{q}_i, \mathbf{q}_j\right) = \mathbf{s}_1^{\mathrm{T}}\mathbf{s}_2 - \cos(\theta_{\mathbf{s}_1\mathbf{s}_2}) = \left(\mathbf{C}_i^{\mathbf{s}_1}\mathbf{q}_i\right)^{\mathrm{T}}\left(\mathbf{C}_j^{\mathbf{s}_2}\mathbf{q}_j\right) - \cos(\theta_{\mathbf{s}_1\mathbf{s}_2}) = 0 \\ \Phi^{CAR} = \mathbf{q}_i^{\mathrm{T}}\,\mathscr{C}^{\mathbf{s}_1\mathbf{s}_2}\mathbf{q}_j - \cos(\theta_{\mathbf{s}_1\mathbf{s}_2})\end{aligned} \tag{64}$$

with

$$\mathscr{C}^{\mathbf{s}_1\mathbf{s}_2} = \left[\mathbf{C}_i^{\mathbf{s}_1\,\mathrm{T}}\mathbf{C}_j^{\mathbf{s}_2}\right]_{(12\times12)} \tag{65}$$

Equation (64) represents one constraint equation that presents a quadratic relation between the generalized coordinates of bodies i and j, and, consequently, the contributions to the Jacobian matrix have a linear structure in the form, such that

$$\mathbf{\Phi}_{\mathbf{q}}^{CAR} = \left\{ \overbrace{\mathbf{q}_j^{\mathrm{T}}\,\mathscr{C}^{\mathbf{s}_1\mathbf{s}_2\,\mathrm{T}}}^{\mathbf{\Phi}_{\mathbf{q}_i}^{CAR}} \quad \overbrace{\mathbf{q}_i^{\mathrm{T}}\,\mathscr{C}^{\mathbf{s}_1\mathbf{s}_2}}^{\mathbf{\Phi}_{\mathbf{q}_j}^{CAR}} \right\}_{(1\times24)} \tag{66}$$

As the matrices $\mathbf{C}_i^{\mathbf{s}_1}$ and $\mathbf{C}_j^{\mathbf{s}_2}$ are constant and not time-dependent, matrix $\mathscr{C}^{\mathbf{s}_1\mathbf{s}_2}$ has also the same properties. Hence, Eq. (64) represents a scleronomic relation, meaning that its contribution to the right-hand side vector of velocity (ν^{CAR}) is null. In turn, the quadratic nature of the CAR condition implies that the contributions to the right-hand side vector of acceleration (γ^{CAR}) present a quadratic relation on the generalized velocities of the system, as presented below

$$\nu^{CAR} = 0 \tag{67}$$

$$\gamma^{CAR} = -2\dot{\mathbf{q}}_i^{\mathrm{T}}\,\mathscr{C}^{\mathbf{s}_1\mathbf{s}_2}\dot{\mathbf{q}}_j = -2\dot{\mathbf{s}}_1^{\mathrm{T}}\dot{\mathbf{s}}_2 \tag{68}$$

Although Eq. (64) presents a non-linear term, this only needs to be computed once during the analysis. Since matrix $\mathscr{C}^{\mathbf{s}_1\mathbf{s}_2}$ is constant, it must be evaluated one time at the beginning of the analysis. Thus, during the analysis of the mechanism, the CAR constraint condition is described only by algebraic equations that are null, linear, or quadratic in type, having a strong influence on the computational performance of the formulation. Moreover, Eqs. (64) to (68) can be further simplified if the vectors being constrained are rigid body vectors. Accordingly, let one consider two rigid body vectors (e.g., $\mathbf{u}_i$ and $\mathbf{v}_j$) belonging respectively to bodies i and j, the CAR constraint ($\mathbf{\Phi}^{CAR}$) in its homogenous form can be given by

$$\Phi^{CAR}\left(\mathbf{q}_i, \mathbf{q}_j\right) = \mathbf{u}_i{}^{\mathrm{T}}\mathbf{v}_j - \cos\left(\theta_{\mathbf{u}_i\mathbf{v}_j}\right) = 0 \tag{69}$$

$$\mathbf{\Phi}_{\mathbf{q}}^{CAR} = \left\{ \overbrace{\mathbf{v}_j^{\mathrm{T}}}^{\mathbf{\Phi}_{\mathbf{u}_i}^{CAR}} \quad \overbrace{\mathbf{u}_i^{\mathrm{T}}}^{\mathbf{\Phi}_{\mathbf{v}_j}^{CAR}} \right\}_{(1\times6)} \tag{70}$$

$$\gamma^{CAR} = -2\dot{\mathbf{u}}_i^{\mathrm{T}}\dot{\mathbf{v}}_j \tag{71}$$

In the case of rigid body vectors, the kinematic constraint equation, and the contributions to $\mathbf{\Phi}_{\mathbf{q}}$, $\boldsymbol{\nu}$ and $\boldsymbol{\gamma}$ for the CAR condition maintain the same degree of linearity as in the case with the generic vector, and hence it offers the same advantages. However, obtaining all the required values entails fewer mathematical operations, improving the computational efficiency of the formulation.

3.2.6. Constant angular relation between two vectors belonging to reduced rigid bodies

The definition of the CAR condition for the reduced rigid bodies can also use the dot product constraint explored in the previous section and the transformation in the fully-defined equivalent vector presented in Section 2.2.3. Accordingly, let one consider two generic unit vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ belonging respectively to bodies i and j defined in its reduced form, then, the CAR constraint (Φ^{CAR}) in its homogenous form can be expressed as:

$$\begin{aligned}\Phi^{CAR}\left(\mathbf{q}_i, \mathbf{q}_j\right) = \mathbf{s}_1^{\mathrm{T}}\mathbf{s}_2 - \cos(\theta_{\mathbf{s}_1\mathbf{s}_2}) = \left(\mathbf{C}_i^{\mathbf{s}_1}\mathbf{SV}_i\mathbf{q}_i\right)^{\mathrm{T}}\left(\mathbf{C}_j^{\mathbf{s}_2}\mathbf{SV}_j\mathbf{q}_j\right) - \cos(\theta_{\mathbf{s}_1\mathbf{s}_2}) = 0 \\ \Phi^{CAR} = \mathbf{q}_i^{\mathrm{T}}\mathbf{SV}_i\,\mathscr{C}^{\mathbf{s}_1\mathbf{s}_2}\mathbf{SV}_j\mathbf{q}_j - \cos(\theta_{\mathbf{s}_1\mathbf{s}_2})\end{aligned} \tag{72}$$

Due to the dependency of the matrices $\mathbf{V}$ on the generalized coordinates of the system, Eq. (58) represents one constraint equation of higher degree. It is important to note that its evaluation resorts only on algebraic multiplications involving linear matrices and vectors dependent on the generalized coordinates of the system. This higher complexity also results in more complex expressions for the contributions to the Jacobian matrix and vector $\boldsymbol{\gamma}$, in the form

$$\boldsymbol{\Phi}_{\mathbf{q}}^{CAR} = \left\{ \overbrace{\mathbf{q}_j^{\mathrm{T}}\mathbf{V}_j^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}\,\mathscr{C}^{\mathbf{s}_1\mathbf{s}_2\,\mathrm{T}}\mathbf{V}_i}^{\boldsymbol{\Phi}_{\mathbf{q}_i}^{CAR}} \quad \overbrace{\mathbf{q}_i^{T}\mathbf{V}_i^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}\,\mathscr{C}^{\mathbf{s}_1\mathbf{s}_2}\mathbf{V}_j}^{\boldsymbol{\Phi}_{\mathbf{q}_j}^{CAR}} \right\}_{(1\times 18)} \tag{73}$$

$$\nu^{CAR} = 0 \tag{74}$$

$$\begin{gathered}\gamma^{CAR} = -\left(\mathbf{q}_i^{\mathrm{T}}\mathbf{V}_i^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}\,\mathscr{C}^{\mathbf{s}_1\mathbf{s}_2}\dot{\mathbf{V}}_j\dot{\mathbf{q}}_j + \mathbf{q}_j^{\mathrm{T}}\mathbf{V}_j^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}\,\mathscr{C}^{\mathbf{s}_1\mathbf{s}_2\,\mathrm{T}}\dot{\mathbf{V}}_i\dot{\mathbf{q}}_i + 2\dot{\mathbf{q}}_i^{\mathrm{T}}\mathbf{V}_i^{\mathrm{T}}\,\mathscr{C}^{\mathbf{s}_1\mathbf{s}_2}\mathbf{V}_j\dot{\mathbf{q}}_j\right)\\ \gamma^{CAR} = -\left(\mathbf{s}_1^{\mathrm{T}}\mathbf{C}_j^{\mathbf{s}_2}\dot{\mathbf{V}}_j\dot{\mathbf{q}}_j + \mathbf{s}_2^{\mathrm{T}}\mathbf{C}_i^{\mathbf{s}_1}\dot{\mathbf{V}}_i\dot{\mathbf{q}}_i + 2\dot{\mathbf{s}}_1^{\mathrm{T}}\dot{\mathbf{s}}_2\right)\end{gathered} \tag{75}$$

3.3. Kinematic constraints of driving type

Often, the kinematic analysis of a multibody system requires the prescription of all the DoFs associated with the mechanical model. This information is included in the form of kinematic constraint equations of driving type, being usually divided into two major groups: (*i*) Linear driving constraints – comprise the expressions needed to prescribe the translations of specific points or the orientation of vectors; (*ii*) Angular driving constraints – include the equations required to depict the angular rotations between vectors/bodies.

The driving constraints are usually formulated using algebraic equations that relate the generalized coordinates and the kinematics of the prescribed movement. Consequently, this type of constraints presents a rheonomic nature, meaning that their expressions have an explicit dependency on the time vector. As for the topological constraints, several conditions can be applied to fully describe the kinematics of the mechanical elements to drive. In this work, particular focus is given to the formulation of the linear translation of one point, the orientation of one vector and the angular relation between two vectors.

It is noteworthy that, similar to the topological constraints, kinematic constraints of driving type can involve algebraic expressions to prescribe the relative movement between two elements within the system. Alternatively, they can encompass specific equations to describe the motion of one element of the system in relation to an exterior element. In the first group, one can consider the angular relation constraint to guide the angle between two system vectors, and in the second group the translation of one point, the orientation of one vector in space, or the relative angle between a system vector and an external vector.

3.3.1. Translation of one generic point belonging to a fully-defined body

The translation of one generic point (TP) condition ensures that one point of interest belonging to a given rigid body of the model follows the trajectory of one reference point along the period of analysis. Mathematically, this condition can be defined by stating that the point of the model shares the same position as the reference point in the global reference frame. Hence, the TP condition can be defined in a similar way to the approach used for modeling the CP condition. Accordingly, let one consider a point P belonging to body i and the respective reference point $P*$, in which the position ($\mathbf{r}_{P^*}$), velocity ($\dot{\mathbf{r}}_{P^*}$) and acceleration ($\ddot{\mathbf{r}}_{P^*}$) are known (see Fig. 6a), the TP constraint ($\boldsymbol{\Phi}^{TP}$) in its homogenous form can be stated as

$$\boldsymbol{\Phi}^{TP}(\mathbf{q}_i, t) = \mathbf{r}_{P_i} - \mathbf{r}_{P^*}(t) = \mathbf{C}_i^P\mathbf{q}_i - \mathbf{r}_{P^*}(t) = \mathbf{0} \tag{76}$$

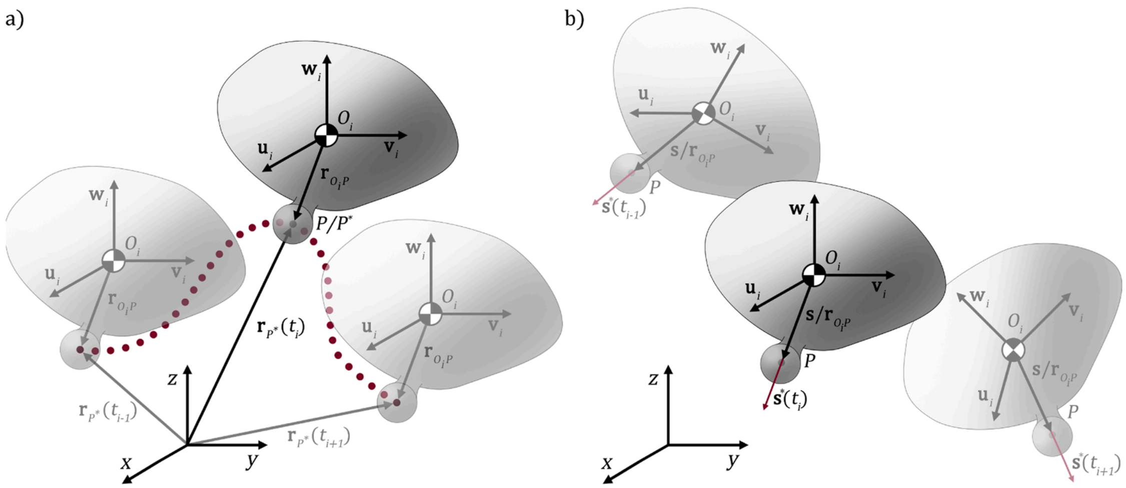

Fig. 6. Schematic representation of the linear kinematic constraints of driver type: a) TP condition; b) OV condition.

Equation (76) represents three linear equations that enforces the coordinates of the point P to be equal to those of the reference point $P*$. Hence, this constraint drives three DoFs, which are associated with the translation of the body in space. For that reason, the TP condition is usually applied to either prescribe the position of the entire model in space by guiding the position of one point in the model parent body, or to guide both the position and orientation of the rigid bodies of the system by minimizing the distance between the prescribed position of several reference points and their corresponding interest points [16].

As the second term of the Eq. (76) does not depend on the generalized coordinates of the system, the contributions of the TP constraint to the Jacobian matrix are constant and equal to the matrix $\mathbf{C}$

$$\mathbf{\Phi}_{\mathbf{q}}^{TP} = \left[\overbrace{\mathbf{C}_i^P}^{\mathbf{\Phi}_{\mathbf{q}_i}^{TP}} \right]_{(3\times12)} \tag{77}$$

Since the TP condition presents a rheonomic nature, the contributions to the right-hand side vectors of velocity ($\boldsymbol{\nu}^{TP}$) and acceleration ($\boldsymbol{\gamma}^{TP}$) are not null, being their values equal to the velocity ($\dot{\mathbf{r}}_{P^*}$) and acceleration ($\ddot{\mathbf{r}}_{P^*}$) of the reference point $P*$, respectively

$$\boldsymbol{\nu}^{TP} = \dot{\mathbf{r}}_{P^*}(t) \tag{78}$$

$$\boldsymbol{\gamma}^{TP} = \ddot{\mathbf{r}}_{P^*}(t) \tag{79}$$

3.3.2. Translation of one generic point belonging to a reduced rigid body

The formulation of the TP condition for a reduced rigid body is similar to that presented for a fully-defined body. Since this approach requires calculating the fully-defined equivalent vector by means of Eq. (16), both the kinematic constraint equations and the contributions to the Jacobian matrix and $\boldsymbol{\gamma}^{TP}$ reflect the quadratic nature of the constraint as a result of the multiplication of the transformation matrix $\mathbf{V}$ by the generalized coordinates of the body. Accordingly, the expression for the TP condition in its homogenous form and the contributions for the $\mathbf{\Phi}_{\mathbf{q}}$, $\boldsymbol{\nu}$ and $\boldsymbol{\gamma}$ are defined as

$$\mathbf{\Phi}^{TP}(\mathbf{q}_i, t) = \mathbf{C}_i^P \mathbf{q}_{3v_i} - \mathbf{r}_P^*(t) = \mathbf{C}_i^P \mathbf{S} \mathbf{V}_i \mathbf{q}_i - \mathbf{r}_P^*(t) = 0 \tag{80}$$

$$\mathbf{\Phi}_{\mathbf{q}}^{TP} = \left[\overbrace{\mathbf{C}_i^P \mathbf{V}_i}^{\mathbf{\Phi}_{\mathbf{q}_i}^{TP}} \right]_{(3\times9)} \tag{81}$$

$$\boldsymbol{\nu}^{TP} = \dot{\mathbf{r}}_P^*(t) \tag{82}$$

$$\boldsymbol{\gamma}^{TP} = \ddot{\mathbf{r}}_P^*(t) - \left(\mathbf{C}_i^P \dot{\mathbf{V}}_i \dot{\mathbf{q}}_i\right) \tag{83}$$

3.3.3. Orientation of one generic vector belonging to a fully-defined body

The orientation of a generic vector (OV) constraint is a kinematic driver condition that guides the Cartesian components of one vector in space. This constraint can be utilized to prescribe the direction of any generic vector, enabling the definition of the global orientation of the model segments or the rotation axes of the kinematic joints in space.

From a mathematical point of view, the OV constraint is similar to the TP condition. However, the expression uses the matrix $\mathbf{C}$ for a generic vector $\mathbf{s}$ instead of the one used for a generic point P. Hence, the OV kinematic constraint ($\mathbf{\Phi}^{OV}$) in its homogenous form can be stated as

$$\mathbf{\Phi}^{OV}(\mathbf{q}_i, t) = \mathbf{s}_i - \mathbf{s}^*(t) = \mathbf{C}_i^s \mathbf{q}_i - \mathbf{s}^*(t) = \mathbf{0} \tag{84}$$

where $\mathbf{s}_i$ is a generic vector belonging to body i, and $\mathbf{s}^*$ represents the prescribed Cartesian components of the reference vector to follow along the period of analysis (see Fig. 6b). Consequently, the respective contributions to the Jacobian matrix ($\mathbf{\Phi}_{\mathbf{q}}^{OV}$) and right-hand side vectors of velocity ($\boldsymbol{\nu}^{OV}$) and acceleration ($\boldsymbol{\gamma}^{OV}$) are similar to the ones presented in Eqs. (77) to (79), but considering instead the matrix substitution referred before and the velocity ($\dot{\mathbf{s}}^*$) and acceleration ($\ddot{\mathbf{s}}^*$) of the vector $\mathbf{s}^*$.

If the vector to drive is a rigid body vector, then the OV kinematic constraint can be simplified. Accordingly, let one consider the vector $\mathbf{u}$ from the rigid body i, the OV algebraic condition and the respective contributions to the Jacobian matrix are given by

$$\mathbf{\Phi}^{OV}(\mathbf{q}_i, t) = \mathbf{u}_i - \mathbf{s}^*(t) = \mathbf{0} \tag{85}$$

$$\mathbf{\Phi}_{\mathbf{q}}^{OV} = \left[\overbrace{\mathbf{I}_3}^{\mathbf{\Phi}_{\mathbf{u}_i}^{OV}} \right]_{(3\times3)} \tag{86}$$

In turn, the contributions to vectors $\boldsymbol{\nu}$ and $\boldsymbol{\gamma}$ are equal to ones presented for a generic vector $\mathbf{s}$.

3.3.4. Orientation of one generic vector belonging to a reduced rigid body

As in the case of the fully-defined body, the OV constraint equation for a reduced rigid body is similar to the one presented in the TP condition, considering the substitution of the constant matrix $\mathbf{C}$ for a generic point P by the equivalent matrix for a generic vector $\mathbf{s}$. As so, the OV kinematic constraint ($\mathbf{\Phi}^{OV}$) for a reduced rigid body in its homogenous form can be stated as

$$\mathbf{\Phi}^{OV}(\mathbf{q}_i, t) = \mathbf{C}_i^{\mathbf{s}}\mathbf{q}_{3v_i} - \mathbf{s}^*(t) = \mathbf{C}_i^{\mathbf{s}}\mathbf{S}\mathbf{V}_i\mathbf{q}_i - \mathbf{s}^*(t) = \mathbf{0} \tag{87}$$

Similarly, the contributions to the Jacobian matrix and vectors $\boldsymbol{\nu}$ and $\boldsymbol{\gamma}$ will be analogous to the ones presented in Eqs. (81) to (83), using the equivalent matrix $\mathbf{C}_i^{\mathbf{s}}$ and the velocity and acceleration vectors of the reference vector $\mathbf{s}^*$.

3.3.5. Angular relation between two vectors belonging to fully-defined rigid bodies

The angular relation (AR) kinematic driver enforces that the angular displacement between two vectors is equal to a prescribed angle θ^*. Consequently, this type of constraint can be used to guide the rotational DoFs associated with the kinematic joints, as it is schematically represented in Fig. 7.

Mathematically, the angular driver constraint can be defined using the dot product between two vectors as it was described in Section 3.2.5. However, since the angular displacement between the two vectors varies according to the prescribed angle, the AR condition becomes a rheonomic constraint, meaning that additional terms need to be included to the contributions to the right-hand side vectors of velocity and acceleration. Hence, let one consider the DoF associated with the relative angle between two generic unit vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ belonging respectively to bodies i and j, the AR kinematic driver constraint can be defined as

$$\begin{aligned}\Phi^{AR}\left(\mathbf{q}_i, \mathbf{q}_j, t\right) &= \mathbf{s}_1^{\mathrm{T}}\mathbf{s}_2 - \cos\left(\theta^*_{\mathbf{s}_1\mathbf{s}_2}(t)\right) = \left(\mathbf{C}_i^{\mathbf{s}_1}\mathbf{q}_i\right)^{\mathrm{T}}\left(\mathbf{C}_j^{\mathbf{s}_2}\mathbf{q}_j\right) - \cos\left(\theta^*_{\mathbf{s}_1\mathbf{s}_2}(t)\right) = 0 \\ \Phi^{AR} &= \mathbf{q}_i^{\mathrm{T}}\mathscr{C}^{\mathbf{s}_1\mathbf{s}_2}\mathbf{q}_j - \cos\left(\theta^*_{\mathbf{s}_1\mathbf{s}_2}(t)\right)\end{aligned} \tag{88}$$

where $\theta^*_{\mathbf{s}_1\mathbf{s}_2}$ is the prescribed angle between vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ along the time. As in the CAR case, Eq. (88) exhibit a quadratic relation between the generalized coordinates of the system, meaning that the computational advantages identified for the CAR condition are maintained. Moreover, as the second term of Eq. (88) is independent of the generalized coordinates, the contributions to the Jacobian matrix are, in fact, equal to the ones presented in Eq. (66). In contrast, the contributions to the vectors $\boldsymbol{\nu}$ and $\boldsymbol{\gamma}$ become dependent on the angular velocity ($\dot{\theta}^*_{\mathbf{s}_1\mathbf{s}_2}$) and angular acceleration ($\ddot{\theta}^*_{\mathbf{s}_1\mathbf{s}_2}$) of the prescribed angle $\theta^*_{\mathbf{s}_1\mathbf{s}_2}$, as presented hereafter

$$\nu^{AR} = -\sin\left(\theta^*_{\mathbf{s}_1\mathbf{s}_2}(t)\right)\dot{\theta}^*_{\mathbf{s}_1\mathbf{s}_2}(t) \tag{89}$$

$$\gamma^{AR} = -2\dot{\mathbf{q}}_i^{\mathrm{T}}\mathscr{C}^{\mathbf{s}_1\mathbf{s}_2}\dot{\mathbf{q}}_j - \left(\cos\left(\theta^*_{\mathbf{s}_1\mathbf{s}_2}(t)\right)\left(\dot{\theta}^*_{\mathbf{s}_1\mathbf{s}_2}(t)\right)^2 + \sin\left(\theta^*_{\mathbf{s}_1\mathbf{s}_2}(t)\right)\ddot{\theta}^*_{\mathbf{s}_1\mathbf{s}_2}(t)\right) \tag{90}$$

Equation (88) describes one constraint equation, which implies that each AR kinematic driver guides one DoF of the multibody system. Hence, one AR constraint equation needs to be added for each rotational DoF of the joint being guided to allow for its full kinematic description. It is important to note that, due to the use of the dot product, the AR condition presents limitations when driving angles near 0 and π rad. For these particular cases, the vectors become aligned, resulting in the appearance of linear dependent lines in the Jacobian matrix. Moreover, since the codomain of the cosine function is positive to values in the 1st and 4th quadrants and negative in the 2nd and 3rd quadrants, the AR constraint can generate kinematically consistent positions, which do not match the prescribed movement. In order to overcome the addressed issues, the AR kinematic driver should be defined in such a way that it only drives angles between 0 and π rad or between π and 2 π. An alternative approach to avoid the issues related to the domain of the dot product equation is to introduce additional driving constraints. Instead of relying on a single constraint equation, this method employs a set of AR equations with different vector combinations to describe the same DoF. This approach allows for expanding the range of applicability of the AR kinematic driver constraint to cover the entire RoM of the joint. However, this modeling strategy introduces redundant constraints, requiring numerical methods specifically designed to handle such systems. For inverse dynamics analysis, the kinematic problem can be efficiently solved using the Newton-Raphson method with a least squares approach, without significantly

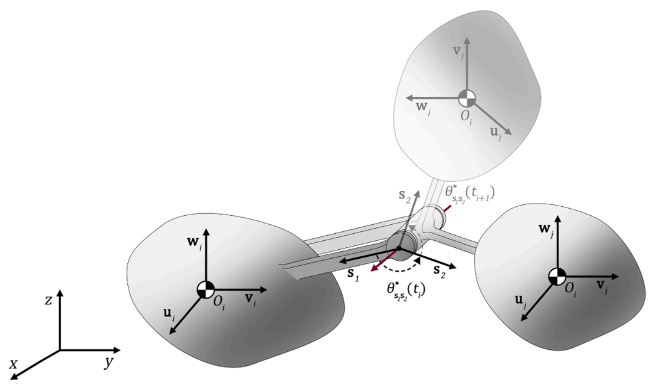

Fig. 7. Schematic representation of the AR kinematic driver defined between rigid body i and j.

impacting computational performance [35,51]. In contrast, forward dynamic analyses of guided systems may experience reduced computational efficiency due to the increased problem dimensions. Additionally, the presence of redundant constraints requires specialized methodologies to find a solution, such as those discussed in Section 2.1.2. A potential alternative to implementing these methodologies is to divide the simulation into multiple subproblems with shorter time intervals, ensuring that each subproblem employs a set of vectors that satisfies the AR condition domain.

Equations (88) to (90) can be further simplified, if the AR driver constraint is defined between two rigid body vectors. Accordingly, let one consider two rigid body vectors $\mathbf{u}$ and $\mathbf{v}$ belonging to bodies i and j, the AR driving constraint can be described using the dot product as follows

$$\Phi^{AR}\left(\mathbf{q}_i,\mathbf{q}_j\right) = \mathbf{u}_i{}^{\mathrm{T}}\mathbf{v}_j - \cos\left(\theta^*_{\mathbf{u}_i\mathbf{v}_j}(t)\right) = 0 \tag{91}$$

As in the generic case, the second term of the Eq. (91) does not have an explicit dependency on the generalized coordinates of the system. Hence, the contributions to the Jacobian matrix are equal to the ones presented in Eq. (70). In turn, the contributions to the right-hand side vector of velocity are the same as the ones presented in the generic case (see Eq. (89)), while the contributions to the vector $\boldsymbol{\gamma}$ include the dot product of the velocity of vectors $\mathbf{u}_i$ and $\mathbf{v}_j$, as expressed by Eq. (71), and the rheonomic terms presented on the generic case (see Eq. (90)).

3.3.6. Angular relation between two vectors belonging to reduced rigid bodies

The AR kinematic drive for a reduced body uses the dot product definition to describe the angular relation between two generic vectors. However, the method to compute the fully-defined equivalent vectors needs to be applied first to allow for the calculation of the components of the generic vectors. Consequently, the AR kinematic driver in its homogenous form can be stated as

$$\begin{aligned}\Phi^{AR}\left(\mathbf{q}_i,\mathbf{q}_j\right) &= \left(\mathbf{C}_i^{\mathbf{s}_1}\mathbf{SV}_i\mathbf{q}_i\right)^{\mathrm{T}}\left(\mathbf{C}_j^{\mathbf{s}_2}\mathbf{SV}_j\mathbf{q}_j\right) - \cos\left(\theta^*_{\mathbf{u}_i\mathbf{v}_j}(t)\right) = 0\\ \Phi^{AR} &= \mathbf{q}_i^{\mathrm{T}}\mathbf{SV}_i\,\mathscr{C}^{\mathbf{s}_1\mathbf{s}_2}\mathbf{SV}_j\mathbf{q}_j - \cos\left(\theta^*_{\mathbf{u}_i\mathbf{v}_j}(t)\right)\end{aligned} \tag{92}$$

As the rheonomic term is independent of the generalized coordinates, the contributions to the Jacobian matrix are equal to those obtained for the CAR condition and given by Eq. (73). Since this term is equal to the fully-defined case, the contributions to vector $\boldsymbol{\nu}$ are the same as the ones presented in Eq. (89). Finally, the contributions to the right-hand side vector of acceleration include the quadratic terms dependent on the velocity and position of the system as presented in Eq. (75) and the rheonomic terms given by Eq. (90).

3.3.7. Angular relation between one vector belonging to a fully-defined rigid body and one external vector

An additional angular relation driver to prescribe the relative motion of one vector belonging to a rigid body and an external vector is formulated. This condition can be used to guide the angular DoFs associated to the kinematic joints defined between a given body and an external element, such as the pinned joints introduced in Section 3.4. In terms of mathematical definition, this type of constraint can be formulated considering the dot product expression as presented for the generic case with two vectors (see Sections 3.3.5 and 3.3.6). Hence, let one consider a generic unit vector $\mathbf{s}_1$ belonging to a fully-defined rigid body i and an external unit vector $\mathbf{s}_2^*$, the AR* constraint driver in its homogeneous form can be established as

$$\Phi^{AR^*}(\mathbf{q}_i,t) = \mathbf{s}_1^{\mathrm{T}}\mathbf{s}_2^* - \cos\left(\theta^*_{\mathbf{s}_1\mathbf{s}_2^*}(t)\right) = \left(\mathbf{C}_i^{\mathbf{s}_1}\mathbf{q}_i\right)^{\mathrm{T}}\mathbf{s}_2^* - \cos\left(\theta^*_{\mathbf{s}_1\mathbf{s}_2^*}(t)\right) = 0 \tag{93}$$

where $\theta^*_{\mathbf{s}_1\mathbf{s}_2^*}$ is the prescribed angle between vector $\mathbf{s}_1$ and $\mathbf{s}_2^*$ along the period of analysis. Since only one of the vectors presents a dependency on the generalized coordinates of the system, the contributions to the Jacobian matrix include only the terms relative to its derivatives with respect to the $\mathbf{q}$, yielding

$$\boldsymbol{\Phi}_{\mathbf{q}}^{AR^*}(\mathbf{q}_i,t) = \left\{ \overbrace{\mathbf{s}_2^{*\mathrm{T}}\mathbf{C}_i^{\mathbf{s}_1}}^{\boldsymbol{\Phi}_{\mathbf{q}_i}^{AR^*}} \right\}_{(1\times12)} \tag{94}$$

When the vector to guide is a rigid body vector, Eqs. (93) and (94) simplifies, resulting in

$$\Phi^{AR^*}(\mathbf{q}_i,t) = \mathbf{u}_i^{\mathrm{T}}\mathbf{s}_2^* - \cos\left(\theta^*_{\mathbf{u}_i\mathbf{s}_2^*}(t)\right) = 0 \tag{95}$$

$$\boldsymbol{\Phi}_{\mathbf{q}}^{AR^*}(\mathbf{q}_i,t) = \left\{ \overbrace{\mathbf{s}_2^{*\mathrm{T}}}^{\boldsymbol{\Phi}_{\mathbf{q}_i}^{AR^*}} \right\}_{(1\times3)} \tag{96}$$

where $\mathbf{u}_i$ represents the vector $\mathbf{u}$ of rigid body i. The Jacobian contributions associated with Eqs. (94) and (96) do not present an explicit dependency on the generalized coordinates, which means that its value can be determined *a priori* if the time vector for the analysis is known. This independency of the $\mathbf{q}$ vector implies that the contributions to the right-hand side vector of velocity (ν^{AR^*}) and acceleration (γ^{AR^*}) have only the terms relative to the rheonomic term, as given by Eqs. (89) and (90).

3.3.8. Angular relation between one vector belonging to a reduced rigid body and one external vector

In the presence of a reduced rigid body definition, the AR* constraint driver can be established in a similar manner described above using the fully-defined equivalent vector, that is

$$\Phi^{AR^*}(\mathbf{q}_i, t) = \left(\mathbf{C}_i^{\mathbf{s}_1}\mathbf{S}\mathbf{V}_i\mathbf{q}_i\right)^{\mathrm{T}}\mathbf{s}_2^* - \cos\left(\theta^*_{\mathbf{s}_1\mathbf{s}_2^*}(t)\right) = 0 \tag{97}$$

with

$$\mathbf{\Phi}_{\mathbf{q}}^{AR^*}(\mathbf{q}_i, t) = \left\{ \overbrace{\mathbf{s}_2^{*\mathrm{T}}\mathbf{C}_i^{\mathbf{s}_1}\mathbf{V}_i}^{\mathbf{\Phi}_{\mathbf{q}_i}^{AR^*}} \right\}_{(1\times 9)} \tag{98}$$

As matrix $\mathbf{V}_i$ depends on the generalized coordinates, the contribution to the Jacobian matrix cannot be determined *a priori* as in the fully-defined case. This linear relation on the generalized coordinates implies that additional terms dependent on the generalized velocities of body *i* need to be included in the right-hand side vector of accelerations besides the rheonomic terms presented above, that is

$$\gamma^{AR^*} = -\left(\mathbf{s}_2^{*\mathrm{T}}\mathbf{C}_i^{\mathbf{s}_1}\dot{\mathbf{V}}_i\dot{\mathbf{q}}_i\right) - \left(\cos\left(\theta^*_{\mathbf{s}_1\mathbf{s}_2^*}(t)\right)\left(\dot{\theta}^*_{\mathbf{s}_1\mathbf{s}_2^*}(t)\right)^2 + \sin\left(\theta^*_{\mathbf{s}_1\mathbf{s}_2^*}(t)\right)\ddot{\theta}^*_{\mathbf{s}_1\mathbf{s}_2^*}(t)\right) \tag{99}$$

3.4. Kinematic constraints with respect to the global reference frame

The full definition of the multibody mechanical system may require additional constraints to define the topologic relations between the bodies and the global reference frame. These constraints allow for the establishment of specific joints, such as fixed or pinned joints, that relate the generalized coordinate of the system to external elements. Two main modeling approaches can be adopted to define these topologic relations.

The first one utilizes the definition of ground bodies, that is, virtual massless bodies that are used to constrain the system bodies. This approach has the main advantage of using the same kinematic constraint equations that are utilized to relate two rigid bodies (see Section 3.2), simplifying the modeling procedure. However, the definition of the ground bodies requires the incorporation of extra coordinates to the vector of generalized coordinates ($\mathbf{q}$), increasing the complexity of the analysis.

The second approach employs specific kinematic constraints to describe the geometric relations between points and vectors of the model and external elements. Consequently, no additional coordinates need to be added to the vector of generalized coordinates, being these relations fully-described by the use of algebraic equations. This approach requires defining specific equations based on the mechanical elements being modeled. Mathematically, these constraints are similar to those used to constrain or drive two rigid bodies (see Sections 3.2 and 3.3). As the constraint equations depend only on the generalized coordinates of one body, the contributions to the Jacobian matrix and right-hand side vector of velocity and acceleration include only the terms relative to that body. Therefore, the topological constraints with respect to global reference frame are not detailed in this work, however, the interested reader is referred to [30] for a derivation of this type of kinematic constraints.

4. Dynamics of 3D multibody systems with FCC-GRB formulation

The inverse and forward dynamic analysis of multibody systems requires the assembly of the EoM as described by Eq. (7) or Eq. (9), respectively. In both cases, beyond describing the system topology and prescribing the guided DoFs via the kinematic constraint equations presented in Section 3, the implementation of a multibody formulation also requires the definition of the system mass matrix and the development of methods for applying external forces and moments of force. Hence, this section explores the mathematical differences between using a fully-defined versus a reduced definition of a rigid body in the dynamic analysis of mechanical systems with FCC, presenting the main equations needed to evaluate inertial and external forces.

4.1. Mass matrix for a fully-defined generic rigid body

One of the distinctive features, which differentiates the FCC-GRB formulation from the natural coordinates formulation, is the simplicity in the definition of the system mass matrix. This straightforwardness is a direct result of the kinematic structure adopted for the generic rigid body, which generates sparse and uncoupled mass matrices [30].

The rigid body mass matrix can be defined using the principle of the virtual power, as described in Jalón and Bayo [29]. In fact, the derivation of the mass matrix for a spatial rigid body follows the same approach applied in the 2D formulation [30] and, for that reason, this work focuses on presenting only the main differences. The interested reader is referred to these two works for a detailed derivation of the rigid body mass matrices.

By applying the principle of the virtual power and the method for describing the kinematics of a generic point *P* (see Section 3.1.1), it is possible to obtain an expression that relates the mass matrix for a given rigid body *i* ($\mathbf{M}_i$) and the constant matrix $\mathbf{C}_i^P$ for a generic point *P* [29,30,53]

$$\mathbf{M}_i = \rho_i \int_{\Omega_i} \mathbf{C}_i^{P^{\mathrm{T}}} \mathbf{C}_i^P d\Omega \tag{100}$$

where ρ_i and Ω_i are respectively the mass density and the geometric domain of the rigid body i. By solving the integral presented in Eq. (100), and considering that the reference point is strategically located at the CoM and the rigid bodies vectors are aligned with the inertia principal axes of the rigid body being modeled, the final form of the mass matrix for a generic rigid body is obtained as [29,53]

$$\mathbf{M}_i = \begin{bmatrix} m_i\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \bar{I}_{\mathbf{u}_i}\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \bar{I}_{\mathbf{v}_i}\mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \bar{I}_{\mathbf{w}_i}\mathbf{I}_3 \end{bmatrix}_{(12\times12)} \tag{101}$$

with

$$\begin{aligned} \bar{I}_{\mathbf{u}_i} &= \frac{1}{2}\left(I_{\eta\eta_i} + I_{\zeta\zeta_i} - I_{\xi\xi_i}\right) \\ \bar{I}_{\mathbf{v}_i} &= \frac{1}{2}\left(I_{\xi\xi_i} + I_{\zeta\zeta_i} - I_{\eta\eta_i}\right) \\ \bar{I}_{\mathbf{w}_i} &= \frac{1}{2}\left(I_{\xi\xi_i} + I_{\eta\eta_i} - I_{\zeta\zeta_i}\right) \end{aligned} \tag{102}$$

where m_i is the mass of rigid body i and I_{ii} the components of its inertia tensor. It is important to note that by locating the reference point at the CoM and aligning the vectors with the principal axes of inertia, the mass matrix of a generic rigid body becomes diagonal, with its entries equal to the mass and the inertia relations along the rigid body vectors ($\bar{I}_{\mathbf{u}_i}, \bar{I}_{\mathbf{v}_i}, \bar{I}_{\mathbf{w}_i}$). If the reference point is located at another point or the vectors are not aligned, the off-diagonal entries of the mass matrix will present a dependency on the products of inertia.

From a modeling approach, the use of a generic rigid body approximates the definition of the mass matrix from the one utilized in the Cartesian coordinates, i.e., the matrix becomes constant and independent of the topology of the system. Moreover, its entries present a higher physical meaning than the ones obtained in the natural coordinates formulation, as they directly represent the inertial properties of the body.

As mentioned above, the possibility of sharing vectors between bodies implies that the FCC formulation with a generic rigid body allows for both an explicit and implicit definition of some kinematic joints. Considering a full explicit model, the assembly of the system mass matrix is a straightforward process, and its entries are directly the rigid body mass matrices. Considering the vector of the generalized coordinates presented in Eq. (1), the global mass matrix of the system will be constant and diagonal, as shown below

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_1 & & & & \\ & \ddots & & & \\ & & \mathbf{M}_i & & \\ & & & \ddots & \\ & & & & \mathbf{M}_{nb} \end{bmatrix}_{(nq\times nq)} \tag{103}$$

Despite generating more coordinates, this modeling approach has the main advantage of generating sparser and uncoupled mass matrices, which is a computational advantage if the proper numerical methods are used. In turn, the use of an implicit modeling approach results in coupled mass matrices, and consequently, less sparse matrices. Thus, let one consider a vector $\mathbf{u}$ shared by the generic rigid bodies i and j, the contributions of each body to the global mass matrix are given by

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_1 &&&&&&&&&& \\ & \ddots &&&&&&&&& \\ && \mathbf{M}_{\mathbf{r}_{O_i}} &&&&&&&& \\ &&& \mathbf{M}_{\mathbf{u}_i} + \mathbf{M}_{\mathbf{u}_j} &&&&&&& \\ &&&& \mathbf{M}_{\mathbf{v}_i} &&&&&& \\ &&&&& \mathbf{M}_{\mathbf{w}_i} &&&&& \\ &&&&&& \mathbf{M}_{\mathbf{r}_{O_j}} &&&& \\ &&&&&&& \mathbf{M}_{\mathbf{v}_j} &&& \\ &&&&&&&& \mathbf{M}_{\mathbf{w}_j} && \\ &&&&&&&&& \ddots & \\ &&&&&&&&&& \mathbf{M}_{nb} \end{bmatrix}_{(nq\times nq)} \tag{104}$$

It is important to note that in both cases, as the rigid body mass matrices are independent of the generalized coordinates, the mass matrix of the system is constant. This fact implies that matrix $\mathbf{M}$ only needs to be evaluated once at the beginning of the analysis, reducing the number of calculations required for each evaluation of the EoM.

4.2. Mass matrix for a reduced generic rigid body

The mass matrix given by Eq. (101) describes mathematically the inertial properties of the generic rigid body i, namely the mass and the mass moments of inertia around the three axes that constitute the local reference frame, which, accordingly with the kinematic structure adopted for the generic rigid body, are defined by the three rigid body vectors. Therefore, when in the presence of a reduced rigid body, the mass matrix needs to also include the inertia moment around a third virtual vector $\mathbf{w}$ to fully describe the inertial properties of the body.

Once more, the procedure presented in Section 2.2.3 can be utilized to determine the respective fully-defined equivalent vector, and then define the mass matrix for the reduced rigid body i. As for the fully-defined case, the principle of virtual power can be used to determine the equivalent mass matrix. Thus, let one consider the fully-defined equivalent vector of the reduced rigid body i, the virtual power generated within this body by the inertial forces (W_i^*) is given by

$$W_i^* = -\rho \int \dot{\mathbf{q}}_{3v_i}^{*\mathrm{T}} \mathbf{C}_i^{P\mathrm{T}} \mathbf{C}_i^P \ddot{\mathbf{q}}_{3v_i} d\Omega = -\rho \int \dot{\mathbf{q}}_i^{*\mathrm{T}} \mathbf{V}_i^{\mathrm{T}} \mathbf{C}_i^{P\mathrm{T}} \mathbf{C}_i^P \left(\dot{\mathbf{V}}_i \dot{\mathbf{q}}_i + \mathbf{V}_i \ddot{\mathbf{q}}_i \right) d\Omega \tag{105}$$

where $\dot{\mathbf{q}}_i^*$ represents the generalized virtual velocities of the rigid body i and $\dot{\mathbf{q}}_{3v_i}^*$ denotes the virtual velocities described in terms of the fully-defined equivalent form. As the transformations matrices $\mathbf{V}_i$ and $\dot{\mathbf{V}}_i$ are independent of the volume of body i, these can be moved outside the integral. In this way, the integral in Eq. (105) is equal to the one in Eq. (100), which in turn is equal to the mass matrix of a fully-defined rigid body. Hence, Eq. (105) can be further simplified as

$$\begin{aligned} W_i^* &= -\dot{\mathbf{q}}_i^{*\mathrm{T}} \mathbf{V}_i^{\mathrm{T}} \rho \left(\int_\Omega \mathbf{C}_i^{P\mathrm{T}} \mathbf{C}_i^P d\Omega \right) \left(\dot{\mathbf{V}}_i \dot{\mathbf{q}}_i + \mathbf{V}_i \ddot{\mathbf{q}}_i \right) \\ W_i^* &= -\dot{\mathbf{q}}_i^{*T} \left(\mathbf{V}_i^{\mathrm{T}} \mathbf{M}_i \mathbf{V}_i \ddot{\mathbf{q}}_i + \mathbf{V}_i^{\mathrm{T}} \mathbf{M}_i \dot{\mathbf{V}}_i \dot{\mathbf{q}}_i \right) \end{aligned} \tag{106}$$

Equation (106) can be divided into two main terms. The first one, dependent on the generalized accelerations of the system, corresponds to the equivalent mass matrix of the reduced rigid body ($\mathbf{M}_{2v_i}$) as

$$\mathbf{M}_{2v_i} = \left[\mathbf{V}_i^{\mathrm{T}} \mathbf{M}_i \mathbf{V}_i \right]_{(9\times9)} \tag{107}$$

Matrix $\mathbf{M}_{2v_i}$ can be assembled in the global mass matrix of the system, following the same approach presented in Eqs. (103) or (104). However, since matrix $\mathbf{V}_i$ is dependent on the generalized coordinates, the global mass matrix becomes dependent on the state of the system. This fact implies that for each evaluation of the EoM, the global mass matrix needs to be updated with the contributions relative to the reduced rigid bodies.

The second term of Eq. (106) can be treated as a velocity-dependent generalized inertial force ($\mathbf{g}_{2v_i}$), which needs to be added to the vector of the generalized forces, such that

$$\mathbf{g}_{2v_i} = \mathbf{V}_i^{\mathrm{T}} \mathbf{M}_i \dot{\mathbf{V}}_i \dot{\mathbf{q}}_i \tag{108}$$

The velocity-dependent inertial force is dependent on the state of the system, meaning that the vector of the generalized force vectors needs to be also updated for each time step.

4.3. Application of external forces and moments of force

Since the external forces and external moments of force are not necessarily applied in the generalized coordinates of the system, those must be transformed into equivalent generalized forces. It is important to note that the methodology proposed hereafter is generic and can be applied to any external force or moment applied to the system. This method is also valid for state-dependent forces or moments that can be represented by an equivalent external force or moment of force. However, in this particular case, the state of the system needs to be firstly determined to allow for the calculation of the magnitude, orientation and application point of these forces.

4.3.1. External forces for a fully-defined rigid body

The principle of virtual power can be utilized to derive the equivalent generalized force of any external force applied in the system. This principle states that the virtual power produced by an external force $\mathbf{f}$ applied in point P belonging to body i is equal to the virtual power produced by the generalized equivalent force, i.e., the product of the generalized equivalent force $\mathbf{g}_i^{\mathbf{f}}$ by the vector of virtual velocities of rigid body i [29,53], as presented below

$$\dot{\mathbf{r}}_P^{*\mathrm{T}} \mathbf{f} = \dot{\mathbf{q}}_i^{*\mathrm{T}} \mathbf{g}_i^{\mathbf{f}} \tag{109}$$

where $\dot{\mathbf{r}}_P^*$ represents the vector of the virtual velocities of point P. By applying the methodology presented in Section 3.1.1, it is possible to express the algebraic relation presented in Eq. (109) as a function of the generalized coordinates of the system, such that

$$\dot{\mathbf{q}}_i^{*\mathrm{T}} \mathbf{C}_i^{P\mathrm{T}} \mathbf{f} = \dot{\mathbf{q}}_i^{*\mathrm{T}} \mathbf{g}_i^{\mathrm{f}} \Rightarrow \mathbf{g}_i^{\mathrm{f}} = \mathbf{C}_i^{P\mathrm{T}} \mathbf{f} \tag{110}$$

Equation (110) shows that the equivalent generalized force can be expressed as the product of the vector of the external force **f** and the constant transformation matrix **C** of the point P described in relation to body i (see Fig. 8a). This relation implies that if both the external force **f** and its application point are constant, as in the case of the conservative gravitational force, the equivalent generalized force is also constant, meaning that its contribution to the generalized force vector only needs to be computed once at the beginning of the analysis. Similarly, if only the application point of force **f** is constant in the local reference of the body (e.g., forces generated by linear springs, dampers, or actuator forces with fixed attachment points), matrix **C** becomes constant along the period of analysis, implying that it can also be defined *a priori*.

4.3.2. External moment of force for a fully-defined rigid body

As the FCC-GRB formulation does not utilize angular coordinates to describe the rigid bodies orientations, the external moments of force cannot be directly applied to the system as a generalized moment. Thus, each external moment of force needs to be first converted into the equivalent forces couple, which is then applied to the system using the same approach presented in the previous section. Accordingly, let one consider the external moment of force $\boldsymbol{\tau}$ applied to body i, and the equivalent force couple $\mathbf{f}_\tau$ and $\mathbf{f}_{-\tau}$, such that

$$\begin{cases} \boldsymbol{\tau} = \mathbf{f}_\tau \times \mathbf{b}_\tau + \mathbf{f}_{-\tau} \times \mathbf{b}_{-\tau} \\ \quad \mathbf{f}_\tau + \mathbf{f}_{-\tau} = \mathbf{0} \end{cases} \tag{111}$$

where $\mathbf{b}_\tau$ and $\mathbf{b}_{-\tau}$ are respectively the moment arms of force vectors $\mathbf{f}_\tau$ and $\mathbf{f}_{-\tau}$.

As the virtual power produced by the generalized equivalent moment of force ($\mathbf{g}_i^\tau$) is equal to the sum of the virtual power produced by the equivalent force couple [29,53], the following expression yields

$$\dot{\mathbf{q}}_i^{*\mathrm{T}} \mathbf{g}_i^\tau = \dot{\mathbf{r}}_{P_{\mathrm{f}_\tau}}^{*\ \mathrm{T}} \mathbf{f}_\tau - \dot{\mathbf{r}}_{P_{\mathrm{f}_{-\tau}}}^{*\ \mathrm{T}} \mathbf{f}_{-\tau} \tag{112}$$

Defining the position of point $P_{\mathrm{f}_{-\tau}}$ at the origin of the local reference frame of body i and of point P_{f_τ} at the tip of vector **s** (see Fig. 8b) and recalling Eqs. (36) and (40), Eq. (112) can be written in terms of the generalized coordinates of the system as

$$\dot{\mathbf{q}}_i^{*\mathrm{T}} \mathbf{g}_i^\tau = \dot{\mathbf{q}}_i^{*\mathrm{T}} \left(\mathbf{C}_i^{P\mathrm{T}} - \mathbf{C}_i^{O_i\mathrm{T}} \right) \mathbf{f}_\tau \Rightarrow \mathbf{g}_i^\tau = \mathbf{C}_i^{\mathrm{s}\mathrm{T}} \mathbf{f}_\tau \tag{113}$$

By locating the point $P_{\mathrm{f}_{-\tau}}$ at the origin of the reference, the contribution of the force $\mathbf{f}_{-\tau}$ to the moment is null, being only responsible by counteracting the translation produced by the force $\mathbf{f}_\tau$. Therefore, the vector **s** should be defined in such a way that it is contained in the plane normal to vector $\boldsymbol{\tau}$ and the norm of the product $\mathbf{C}_i^{\mathrm{s}\mathrm{T}} \mathbf{f}$ is equal to the magnitude of the moment of force $\boldsymbol{\tau}$.

It must be noted that this linear dependency on the matrix **C** implies that if the direction of the external moment of force $\boldsymbol{\tau}$ does not change along the time, the matrix $\mathbf{C}_i^{\mathrm{s}}$ is constant and consequently only the product presented in Eq. (113) needs to be computed each time the EoM are evaluated.

4.3.3. External forces for a reduced rigid body

The application of the external forces for a rigid body defined in the reduced form follows the same approach presented for the fully-defined rigid body. However, the expressions presented in Section 4.3.1 need to be revised considering the fully-defined equivalent vector ($\mathbf{q}_{3v}$) introduced in Section 2.2.3. Hence, and recalling the principle of virtual power for an external force **f** (see

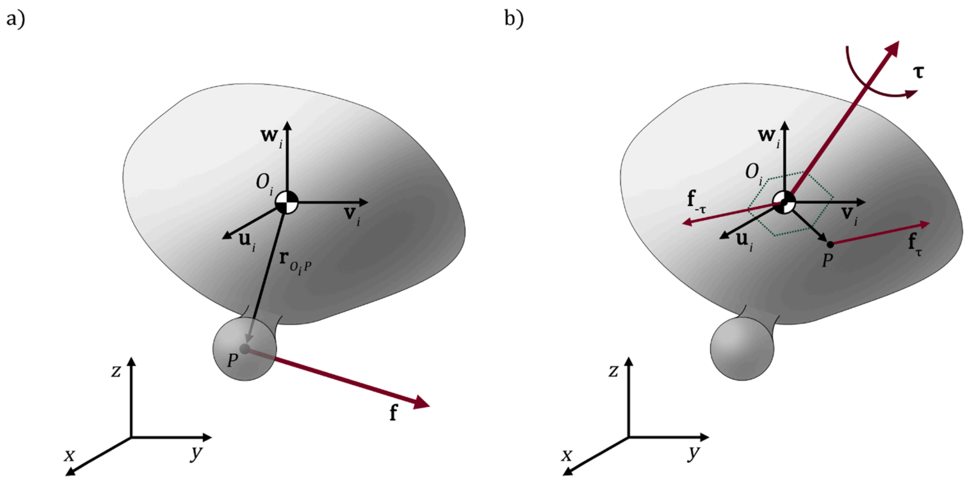

Fig. 8. a) Application of an external force **f** at point P belonging to the generic rigid body i; b) Application of an external moment of force $\boldsymbol{\tau}$ to the generic rigid body i and respective transformation into the equivalent force couple $\mathbf{f}_\tau$ and $\mathbf{f}_{-\tau}$.

Eq. (109)), the generalized equivalent force for a reduced rigid body can be described as

$$\dot{\mathbf{q}}_i^{*\mathrm{T}}\mathbf{V}_i^{\mathrm{T}}\mathbf{C}_i^{P\mathrm{T}}\mathbf{f} = \dot{\mathbf{q}}_i^{*\mathrm{T}}\mathbf{g}_i^{\mathbf{f}} \;\Rightarrow\; \mathbf{g}_i^{\mathbf{f}} = \mathbf{V}_i^{\mathrm{T}}\mathbf{C}_i^{P\mathrm{T}}\mathbf{f} \tag{114}$$

In contrast to the fully-defined case, the generalized equivalent force for the reduced rigid body depends on the generalized coordinates of the system, meaning that each time the EoM are evaluated, the generalized force vector needs to be updated, even when constant forces are applied.

4.3.4. *External moment of force for a reduced rigid body*

The same principles presented in Section 4.3.2 can be applied to determine the generalized moment of force for an external moment $\boldsymbol{\tau}$. However, the fully-defined equivalent vector needs to be used to allow for the systematic calculation of the contributions to the generalized force vector ($\mathbf{g}_i^{\tau}$).

$$\dot{\mathbf{q}}_i^{*\mathrm{T}}\mathbf{g}_i^{\tau} = \dot{\mathbf{q}}_i^{*\mathrm{T}}\mathbf{V}_i^{\mathrm{T}}\left(\mathbf{C}_i^{P\mathrm{T}} - \mathbf{C}_i^{O_i\mathrm{T}}\right)\mathbf{f}_{\tau} \;\Rightarrow\; \mathbf{g}_i^{\tau} = \mathbf{V}_i^{\mathrm{T}}\mathbf{C}_i^{s\mathrm{T}}\mathbf{f}_{\tau} \tag{115}$$

It should be noticed that Eq. (115) presents an explicit dependency on the generalized coordinates of the system, which implies that, even for constant moments of force, the contributions to vector $\mathbf{g}$ need to be calculated for each instant of time. Nevertheless, as in this particular case, the vector $\mathbf{s}$ is constant in the local reference frame of the body i, matrix $\mathbf{C}_i^{\mathbf{s}}$ is also constant and can be defined *a priori* during the pre-analysis procedure.

4.4. *Internal reaction forces and moments of force for a fully-defined rigid body*

One of the major objectives, when performing a dynamic analysis of mechanical systems, is the determination of the internal forces and moments of force generated during the motion of the system. From a mechanical standpoint, these represent the forces and moments that need to be generated within the system to comply with the imposed kinematic constraints. Their physical meaning is related to the type of kinematic constraint they are obtained. Algebraically, the internal forces of the system ($\mathbf{g}^{\boldsymbol{\Phi}}$) can be calculated by solving Eqs. (7) or (9) in order to the internal forces, such that

$$\mathbf{g}^{\boldsymbol{\Phi}} = \left\{\; \mathbf{g}_1^{\boldsymbol{\Phi}\mathrm{T}} \quad \cdots \quad \mathbf{g}_i^{\boldsymbol{\Phi}\mathrm{T}} \quad \cdots \quad \mathbf{g}_{nb}^{\boldsymbol{\Phi}\mathrm{T}} \;\right\}^{\mathrm{T}} = \boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\lambda} \tag{116}$$

with

$$\mathbf{g}_i^{\boldsymbol{\Phi}} = \mathbf{g}_i^{\boldsymbol{\Phi}^{RB}} + \mathbf{g}_i^{\boldsymbol{\Phi}^{KJ}} + \mathbf{g}_i^{\boldsymbol{\Phi}^{DC}} \tag{117}$$

where $\mathbf{g}_i^{\boldsymbol{\Phi}}$ represents the vector of the internal forces acting on body i, which, in turn, is equal to the sum of all contributions from kinematic constraints and drivers related to this body, including rigid body topological constraints (RB), kinematic joint topological constraints (KJ), and driving constraints (DC).

The physical meaning of Eqs. (116) and (117) can be better interpreted if the contributions of each kinematic driver are evaluated separately. In fact, the Lagrangian multipliers associated with each kinematic constraint represent the magnitude of the internal forces, while the respective lines of the Jacobian matrix their direction. This relevant characteristic of the Lagrange multipliers is of particular relevance, as it allows for the determination of the joint reaction forces and moments directly from the EoM.

Taking as an example the case of a spherical joint, defined by the CP condition (see Sections 3.2.1 and 3.2.2), the generalized internal forces generated by this joint are given by

$$\mathbf{g}^{\boldsymbol{\Phi}^{CP}} = \boldsymbol{\Phi}_{\mathbf{q}}^{CP^{\mathrm{T}}}\lambda^{CP} = \begin{bmatrix} \mathbf{C}_i^{P\mathrm{T}} \\ -\mathbf{C}_j^{P\mathrm{T}} \end{bmatrix}\lambda^{CP} = \begin{Bmatrix} \mathbf{C}_i^{P\mathrm{T}}\lambda^{CP} \\ \mathbf{C}_j^{P\mathrm{T}}(-\lambda^{CP}) \end{Bmatrix}_{(24\times1)} \tag{118}$$

By comparing Eq. (118) with Eq. (110) and having in mind that λ^{CP} is a 3D vector, it can be observed that it expresses the application of two concentrated forces of equal magnitude and opposite direction (λ^{CP}, $-\lambda^{CP}$) at point P. These two forces represent the action-reaction force couple generated at the joint. Hence, the joint reaction forces ($\mathbf{f}_R^{CP}$) are obtained directly from the EoM, as they are equal to the Lagrangian multipliers associated with the CP condition as

$$\mathbf{f}_R^{CP} = \mathbf{f}_{R_i}^{CP} = -\mathbf{f}_{R_j}^{CP} = \left\{\lambda^{CP}\right\}_{(3\times1)} \tag{119}$$

In turn, the internal forces generated by the AR kinematic drivers describe the equivalent forces that produce the internal driving moments of force. Therefore, the determination of the moments of force at the joints in an FCC formulation is not a direct procedure, requiring additional calculations. Accordingly, let one consider an angular driving constraint between vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ belonging to bodies i and j, and recalling the contributions to the Jacobian matrix presented in Sections 3.3.5 and 3.3.6, the generalized internal forces for an AR driver constraint ($\mathbf{g}^{\boldsymbol{\Phi}^{AR}}$) can be given by

$$\mathbf{g}^{\Phi^{AR}} = \Phi_{\mathbf{q}}^{AR^{\mathrm{T}}} \lambda^{AR} = \begin{bmatrix} \mathscr{C}^{\mathbf{s}_1\mathbf{s}_2}\mathbf{q}_j \\ \mathscr{C}^{\mathbf{s}_1\mathbf{s}_2{}^T}\mathbf{q}_i \end{bmatrix} \lambda^{AR} = \begin{Bmatrix} \mathbf{C}_i^{\mathbf{s}_1\mathrm{T}} \lambda^{AR} \mathbf{s}_2 \\ \mathbf{C}_i^{\mathbf{s}_2\mathrm{T}} \lambda^{AR} \mathbf{s}_1 \end{Bmatrix}_{(24\times 1)} \tag{120}$$

A comparison of Eq. (120) with Eq. (113) allows one to conclude that the internal forces associated to an AR driver constraint represent the application of two symmetric concentrated moments of force, one applied in body i and the other in body j. As the two vectors being constrained are not necessarily orthogonal, the determination of the joint moment of force requires an additional procedure, namely the orthogonalization of one of the vectors in the plane defined by them. Accordingly, the joint moment of force for the joint k ($\boldsymbol{\tau}_k$) along the time is given by

$$\boldsymbol{\tau}_k^{AR}(t) = \lambda^{AR} \sin\left(\theta^{*}_{\mathbf{s}_1\mathbf{s}_2}(t)\right) \mathbf{s}^{\mathbf{s}_1 \perp \mathbf{s}_2} \tag{121}$$

where $\mathbf{s}^{\mathbf{s}_1 \perp \mathbf{s}_2}$ is a unitary vector orthogonal to the plane defined by vectors $\mathbf{s}_1$ and $\mathbf{s}_2$. In the particular case where several AR driving constraints are utilized simultaneously to describe a given angular DoF (see Section 3.3.5), the corresponding joint moment of force is equal to the sum of the contributions of all constraints, such as

$$\boldsymbol{\tau}_k^{AR}(t) = \sum_{m=1}^{n_{\tau_k}} \boldsymbol{\tau}_{k_m}^{AR} \tag{122}$$

where $\boldsymbol{\tau}_{km}$ is the internal moment of force of joint k associated with the *m-th* angular driving constraint and n_{τ_k} is the number of AR driver constraints used to describe the angular DoF in analysis.

4.5. Internal reaction forces and moments of force for a reduced rigid body

The same approach proposed for the fully-defined rigid bodies can be utilized to determine the generalized internal forces for the reduced approach. However, the linear transformation presented in Section 2.2.3 needs to be considered to define the rigid body reference frame. In fact, this procedure arises naturally from the kinematic modeling of the system, as this transformation was already performed while evaluating the system Jacobian matrix. Hence, no additional computations are required to determine the vector of the generalized internal forces of the system ($\mathbf{g}^{\Phi}$), beyond those expressed in Eqs. (116) and (117).

Likewise, the procedure to determine the internal forces and moments of force for the reduced approach is the same as the one utilized for the fully-defined case. The physical meaning of the Lagrangian multipliers for a given kinematic constraint is also the same and, once more, no further calculations need to be performed, as these were already obtained when the Jacobian matrix was determined.

It is important to note that although the reduced approach does not require additional calculations, the dependency on the generalized coordinates increases. This issue is a direct consequence of adopting the transformation matrices $\mathbf{V}$ that present an explicit dependency on the generalized coordinates or generalized velocities of the system. As example, the internal forces associated with the CP condition, which for the fully-defined case depend only on the constant matrix $\mathbf{C}^P$ and the Lagrangian multipliers, are dependent on the generalized coordinates. Similarly, the generalized internal forces for the AR driving constraint, which has a linear dependency on the generalized coordinates for the fully-defined case, will present a higher degree dependency if the reduced approach is considered.

5. Demonstrative examples of application

The validation of the formulation was evaluated by comparing the outcomes of forward dynamic simulations of different mechanical systems using the two rigid body modeling approaches presented in this work. The analysis focused on comparing the time evolution of the coordinates of relevant points of the model, the norm of the position constraints violation, and variation of the mechanical energy between the two modeling approaches and against benchmark data. For this purpose, a set of benchmark problems proposed by [54] and available in the Library of Computational Benchmark Problems of the IFToMM [55] were analyzed. These problems included the simple pendulum, spatial slider-crank mechanism, and double four-bar mechanism. Additional mechanisms, namely the free-falling of a rigid body, spatial double pendulum, and n-four-bar mechanism, were also analyzed to evaluate the differences in computational performance between using a fully-defined and reduced definition of a rigid body. To assess the similarity of the simulation outcomes between the two modeling approaches and the benchmark data, the intraclass correlation coefficient (ICC) for data consistency and absolute agreement (two-way random, confidence interval of 95 %) was computed recurring to the software IBM SPSS® Statistics v28 (IBM Corp.©, Armonk, NY).

The FCC formulation was implemented in an in-house software developed using MATLAB language. The direct integration method presented in [28] was used to solve the equations of motion (EoM), employing the ODE45 numerical integrator, which is based on an explicit fourth-order Runge-Kutta method. The default solver parameters were applied: relative error tolerance of 10^{-3}; absolute error tolerance of 10^{-6}; and a maximum step size of 10 % of the simulation time. To control the violation of constraints during the numerical integration of the EoM [62], the Baumgarte stabilization method was implemented with the stabilization coefficients α and β selected according to [61]. The Baumgarte stabilization coefficients $\alpha = 20$ and $\beta = 40$ were used for the simpler models, including the free-falling of a rigid body, the simple pendulum, and the spatial double pendulum. For the remaining models, the set of coefficients $\alpha = 50$ and $\beta = 100$ was employed.

5.1. Benchmark cases

The geometric and inertial properties of the models were established according to the benchmark specifications available at the Library of Computational Benchmark Problems of IFToMM [55]. For the sake of brevity, only the models and results for the double four-bar mechanism and spatial slider-crank mechanism are presented in this section (see Fig. 9). The description and the results for the remaining models are compiled in Tables A1 to A7 and Figs. A1 to A5 of Supplementary Materials A and B. Except for the spatial slider-crank mechanism, which was simulated for 5 s, a simulation time of 10 s was defined for each model to comply with the benchmark specifications [54,55].

The forward dynamics simulations of the models in this study yielded kinematic responses similar to those from the analytical formulation and the benchmark database. The coordinates of relevant points or angles, shown in Fig. 10 and Fig. 11, exhibit the same cyclic behavior over time as the benchmark data. This idea is statistically supported by the high ICC scores obtained for both consistency and absolute agreement across all the analyzed models (see Table A7).

A detailed comparison between the results obtained using the fully-defined and reduced modeling approach indicates that the methodology followed to define the generic rigid body does not influence the kinematic response. The patterns were consistent, presenting the same differences to the benchmark data at the same time instants. No significant differences were observed for the models characterized by the presence of singular configurations for the set of selected Baumgarte coefficients (double four-bar, *n*-four-bar, and spatial slider-crank mechanisms), exhibiting the same motion behavior. A minor delay, progressively increasing over time, was observed only in the spatial crank mechanism (see Fig. 10). Nevertheless, the relative difference in the cumulative crank angle at the end of the simulation to the benchmark data was approximately 0.11 % for the reduced approach, while the fully-defined approach presented a difference of 0.04 %.

It is important to note that the selected problems included models characterized by the existence of singular positions [54,88]. Although no specific methodologies were applied to address these particular configurations, no convergence issues were encountered under the initial simulation conditions. However, when the Baumgarte stabilization coefficients were reduced ($\alpha, \beta < 10$) or when different initial conditions were used, numerical problems arose as the simulations passed through these configurations. These issues led to deviations from the expected solution or, in the worst cases, the occurrence of singular matrices, making it impossible to obtain a feasible solution.

A comparison of the magnitude of constraint violations between the fully-defined and reduced modeling approaches reveals that both stay within the specified tolerances for the simulation, with values of the same order over time (see Supplementary Materials B – Figs. A4b and A5b). The curves representing the violation of kinematic constraints over time exhibit similar behavior for both types of rigid bodies.

Similarly, the analysis of the variation of the mechanical energy yields outcomes consistent with the benchmark data, displaying a noisy, cyclic curve without significant peaks indicative of local constraint violations. No significant differences were observed between the two modeling methods, both exhibiting similar behavior over time (see Supplementary Materials B – Figs. A4a and A5a). The higher difference was observed in the double pendulum, with the reduced approach showing a 7 % reduction in the mechanical energy at the end of the simulation when compared to the fully-defined method (see Appendix B – Fig. A3d). Except for the free-falling rigid body model, all models showed a slight loss of energy over the simulation time, with the more complex models exhibiting a greater energy loss.

It is worth mentioning that in this work, the integration procedures employed the default tolerances of the MATLAB integrator utilized (relative error tolerance: 10^{-3}; absolute error tolerance 10^{-6}). Stricter error tolerances were tested, leading to a reduction in both energy drift and the magnitude of constraint violations, but this improvement came at the cost of increased computational time.

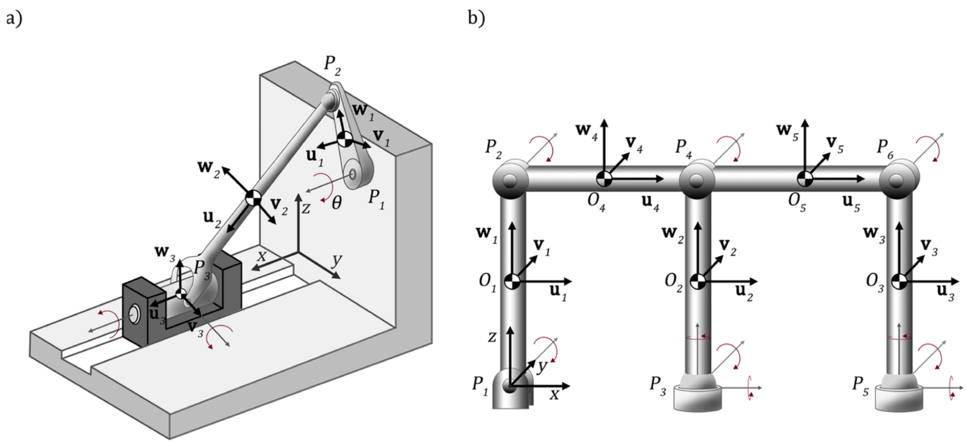

Fig. 9. Schematic representation of the multibody models for the: a) Spatial slider-crank mechanism (adapted from Masoudi [55] (based on Haug [40]); b) Double four-bar mechanism.

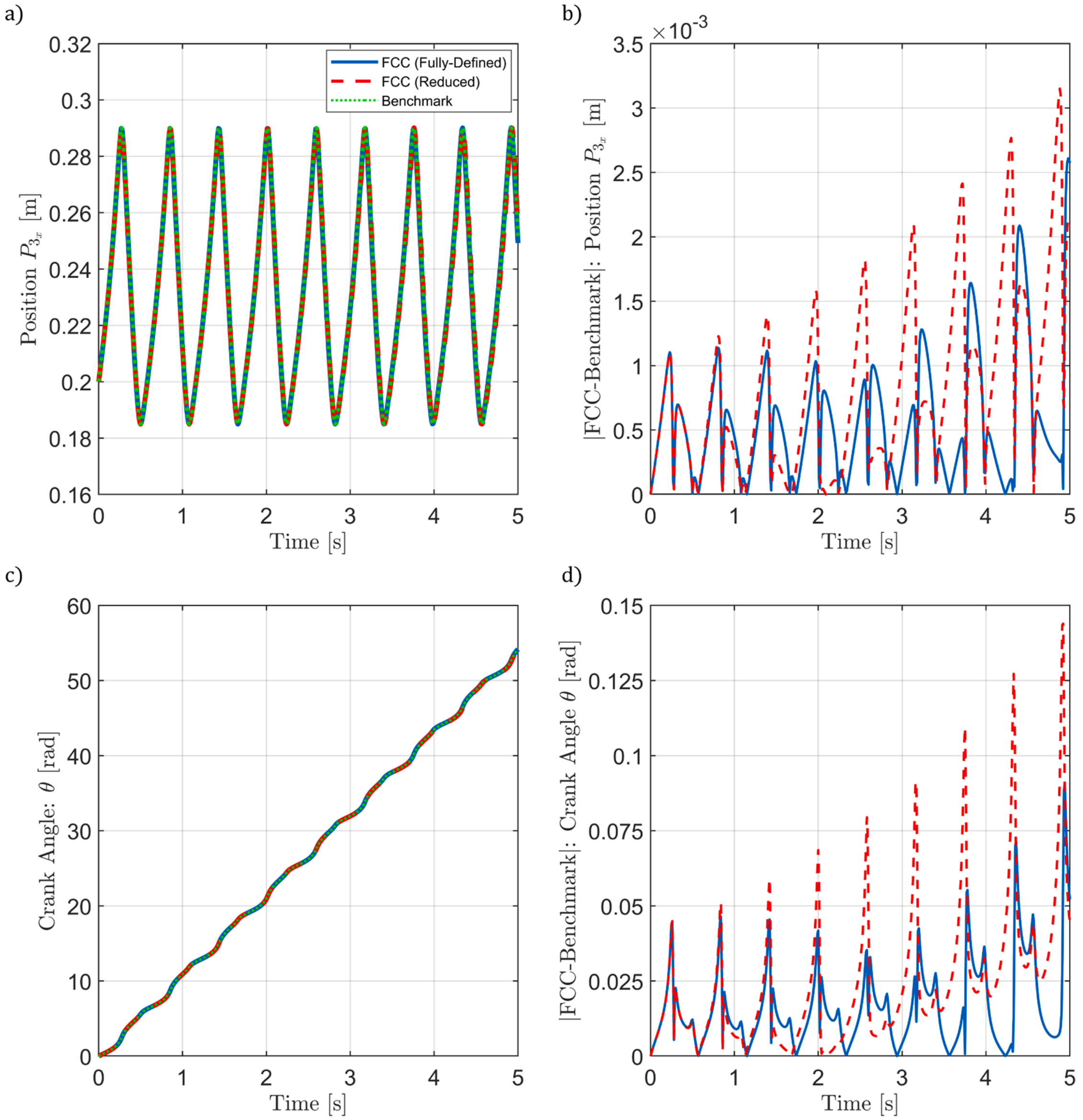

Fig. 10. Comparison between the fully-defined approach (solid blue), reduced approach (dashed red) and benchmark data (dot green; IFToMM Masarati [55]) for the FD simulation of the spatial slider-crank: a) Time-history of x coordinate of point P_3; b) Absolute difference between the FCC simulations and the benchmark data for the x coordinate of point P_3; c) Cumulative crank angle; d) Absolute difference between the FCC simulations and the benchmark data for the cumulative crank angle.

5.2. Comparison of the computational performance

In order to assess the influence of the size of the mechanical system on the simulation time and compare the computational performance of the two rigid body modeling approaches, the computational times for the benchmark problems analyzed in the previous section were measured. For the evaluation of complex models, different configurations of the n-four-bar mechanism were studied, varying the number of multiple four-bar mechanisms from 2 (5 GRBs) to 19 (39 GRBs). A detailed description of the generic n-four-bar mechanism model can be found in Table A6 of Supplementary Materials A. The simulation of the different mechanisms was run on the same computer (CPU: Intel® Core™ i7–8700, 3.2 to 4.6 GHz), and each condition was repeated 10 times to minimize errors related to fluctuations in computational performance.

Although the two modeling approaches produced similar kinematic outcomes, the differences in computational time varied with the complexity of the simulated models (see Fig. 12). For models with a small number of rigid bodies, such as the free-falling rigid body and the simple and double pendulum, the FCC formulation with a reduced generic model took less time to reach a solution. This difference was most pronounced in the simplest model, with a reduction of 39.2 % in computational time, and decreased as model

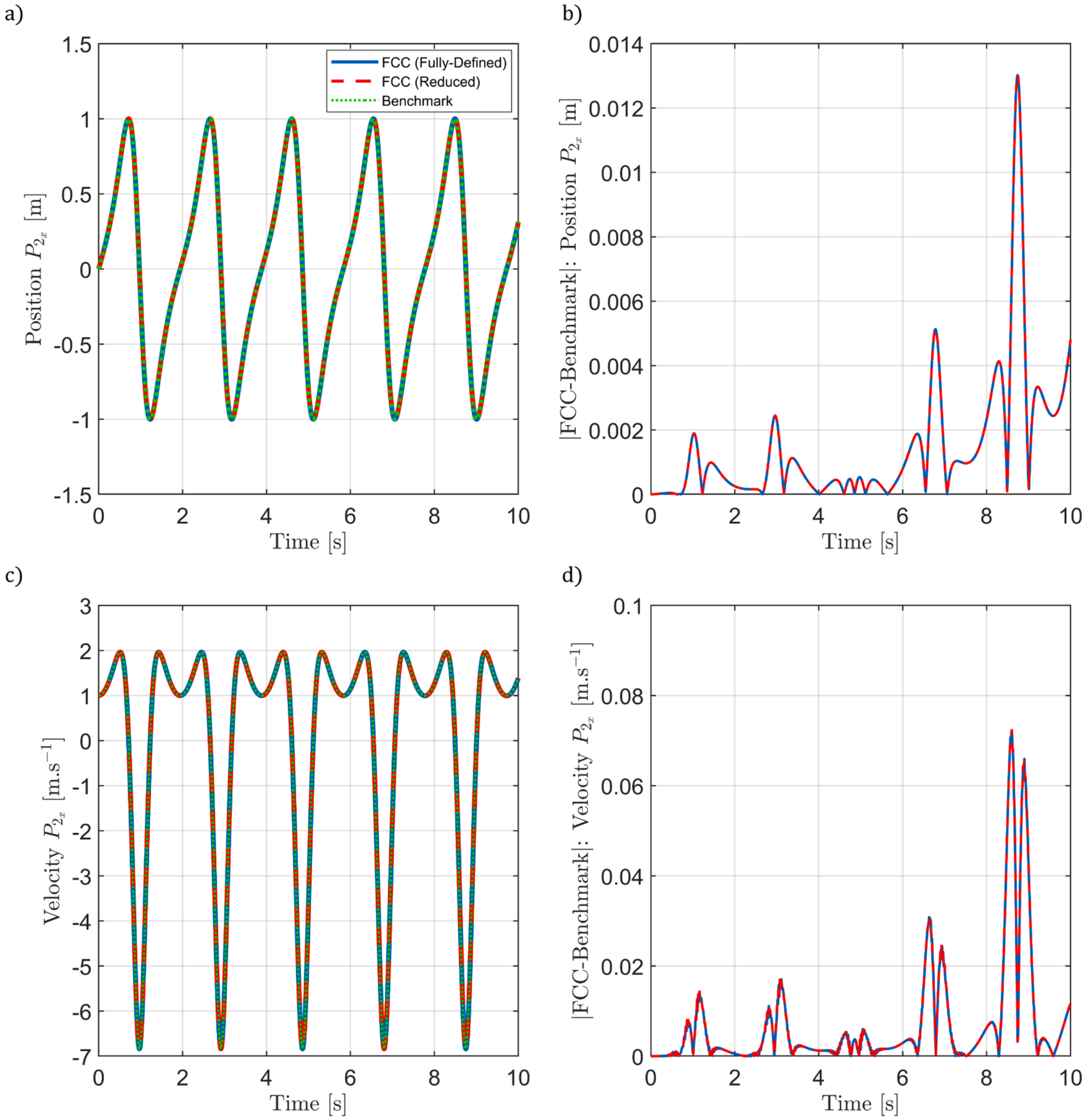

Fig. 11. Comparison between the fully-defined approach (solid blue), reduced approach (dashed red) and benchmark data (dot green; IFToMM Cuadrado [55]) for the FD simulation of a double four-bar mechanism: a) Time-history of x coordinate of point P_2; b) Absolute difference between the FCC simulations and the benchmark data for the x coordinate of point P_2; c) Time-history of x velocity of point P_2; d) Absolute difference between the FCC simulations and the benchmark data for the x velocity of point P_2.

complexity increased.

However, for the n-four-bar models comprising 2 and 3 four-bar mechanisms, the reduced approach required more time to find a solution than the fully-defined approach. This trend reversed as more rigid bodies were added to the model. As the number of segments in the n-four-bar mechanism model increased, the relative difference between the reduced and fully-defined approaches decreased, indicating that the reduced approach became more computationally efficient for more complex models. Notably, a 20.6 % reduction in computational time was observed for the most complex model analyzed in this work.

A detailed analysis of the integrator solver outcomes indicates that the temporal differences observed between the two rigid body modeling approaches are not due to differences in the number of integration steps required to achieve a solution but rather stem from the varying number of calculations performed within the solver due to the different number of integration variables. Table 1 demonstrates that the number of time steps, failed iterations, and function evaluations is similar for both modeling approaches. The minor differences observed in the number of integration steps and function evaluations can be attributed to variations in the number of failed iterations. Furthermore, the integration outcomes suggest that the number of steps is more closely related to the topology of the system than to the number of rigid bodies. This is particularly evident in the results for the n-four-bar mechanisms, which indicate that the

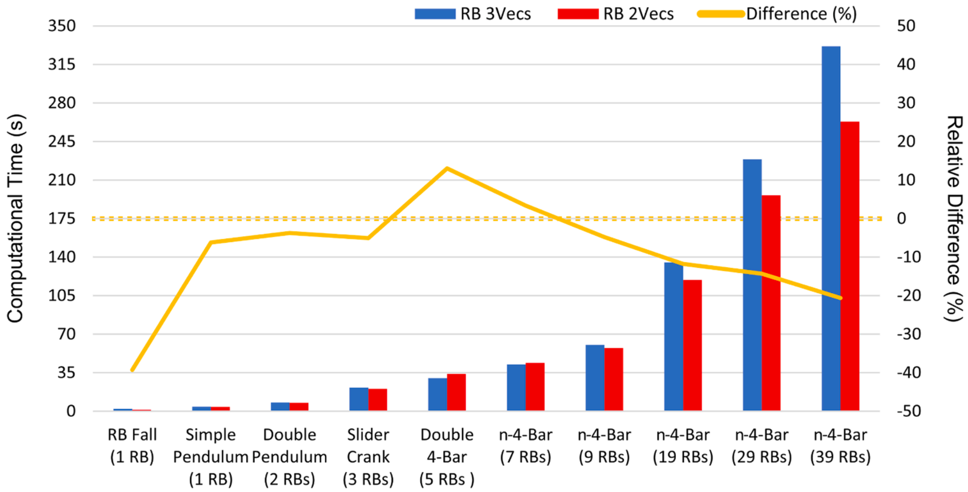

Fig. 12. Comparison of the simulation times between the fully-defined (blue) and reduced (red) modeling approach and respective relative differences for the different models.

Table 1
Integrator solver outcomes.

	N. of Integration Variables ($\|\dot{\mathbf{y}}\|$)		N. of Integration Steps		N. of Failed Iterations		N. of Function Evaluations	
Model	RGRB	GRB	RGRB	GRB	RGRB	GRB	RGRB	GRB
RB fall (1 RB)	18	24	112	112	30	33	853	871
Simple Pendulum (1 RB)	18	24	350	350	22	22	2233	2233
Double Pendulum (2 RBs)	36	48	406	414	12	9	2509	2539
Slider Crank (3 RBs)	54	72	458	489	26	29	2905	3109
Double 4-Bar (5 RBs)	90	120	704	704	0	1	4225	4231
n-4-Bar (7 RBs)	126	168	704	704	0	0	4225	4225
n-4-Bar (9 RBs)	162	216	704	704	1	0	4231	4225
n-4-Bar (19 RBs)	342	456	706	705	3	2	4255	4243
n-4-Bar (29 RBs)	522	696	704	704	0	1	4225	4231
n-4-Bar (39 RBs)	702	936	703	703	0	0	4219	4219

number of integration steps and function evaluations required to reach a solution is similar regardless of the number of rigid bodies included in the model.

As in the natural coordinates formulation, the FCC-GRB formulation allows for the implicit sharing of vectors between adjacent bodies. By adopting this approach and depending on the model topology, the number of generalized coordinates and kinematic constraint equations in the system can be significantly reduced, potentially decreasing the simulation time. This is particularly applicable when two adjacent bodies are constrained using kinematic pairs that can be mathematically defined using the CV conditions. However, this modeling procedure has the disadvantage of generating coupled and less sparse mass matrices, not being explored in this work.

6. Concluding remarks

The present work describes the implementation and validation of the FCC-GRB formulation to the spatial analysis of mechanical systems. The intrinsic differences to the planar case [30,89] are presented in detail, so that the spatial formulation can be easily implemented by students and researchers with different levels of knowledge on this topic. The formulation is also compared with other global multibody formulations, focusing on its main advantages and limitations.

The expansion of the formulation to 3D maintains the same theoretical concepts identified in the planar case. In its fully-defined form, a rigid body is modeled with a fixed and predetermined kinematic structure composed of one reference point and three unit vectors. Consequently, the multibody system is described using solely Cartesian/rectangular coordinates, thereby avoiding the use of angular-related variables as seen in the Cartesian coordinates formulation.

One of the main characteristics of the FCC-GRB formulation is the use of the transformation matrix **C** to define the system kinematics. Computationally, matrix **C** allows for the calculation of the position, velocity, and acceleration of any point or vector of interest directly from the generalized coordinates of the system. Moreover, using matrix **C** simplifies the definition of the kinematic pairs, mass matrices, and the application of forces and moments in the system, as this method can also be applied to describe the kinematics of relevant points and vectors that constitute them. This intrinsic property is a direct consequence of the kinematic structure adopted for the rigid body. As the rigid body vectors directly describe the local reference frame of the body, the transformation coefficients that define matrix **C** are the coordinates of the points or vectors in the local reference frame of the body to which they belong. This relation implies that matrix **C** is constant if the position of the point or the orientation of the vector is fixed in the local reference frame of the body and independent of the generalized coordinates. Therefore, the calculation of the kinematics of points and vectors becomes a straightforward procedure, described by a linear relation on the generalized coordinates. Additionally, and contrarily to the natural coordinates formulation that also adopts a similar representation, the determination of the matrix **C** in the FCC-GRB formulation does not require additional calculation steps.

These two features, namely the use of only Cartesian coordinates to describe the multibody system and the definition of the system kinematics through matrix **C**, directly impact the efficiency of the FCC-GRB multibody formulation, as they imply that the common kinematic constraints are described by at the most second-degree equations without resorting to non-linear terms. Consequently, the contributions to the Jacobian matrix are mainly linear or constant, leading to expressions for the right-hand side vector of velocity and acceleration that are typically null, linear, or quadratic.

To decrease the total number of generalized coordinates in the system, an alternative modeling approach using a reduced generic rigid body was introduced in this work. In this approach, the generic rigid body is defined by one point and two non-collinear vectors, leading to a reduction of three generalized coordinates and three rigid body-type constraint equations per segment. The transition to a reduced rigid body modeling is a straightforward process, as the principles derived for the fully-defined form are retained. This process involves adopting the definition of the fully-defined equivalent vector ($\mathbf{q}_{3v}$), as presented in the Section 2.2.3. With this approach, the kinematic constraint equations deduced for the fully-defined form can be easily converted to the reduced rigid body case by means of a set of transformation matrices ($\mathbf{V}$ and $\dot{\mathbf{V}}$). However, it should be noted that the transformation matrices $\mathbf{V}$ present an explicit dependency on the generalized coordinates of the system, implying that the degree of the kinematic constraints increase. This fact means that some of the contributions to the Jacobian matrix and right-hand side vectors of velocity and acceleration that were constant become dependent on the generalized coordinates, as well as others that presented a linear or quadratic dependency increase the respective degree. Moreover, the mass matrix of the system becomes also dependent on the generalized coordinates and terms related to velocity-dependent inertial forces appear in the system.

From a modeling perspective, since the two types of rigid bodies exhibit a fixed kinematic structure, the modeling procedure becomes independent of the system's topology. As a result, the model can be assembled using the same building blocks and requires the same type of rigid body kinematic constraints to be fully defined, closely resembling the modeling procedure followed in the Cartesian coordinates formulation. Furthermore, if the reference point is located at the CoM of the body, the rigid body mass matrix becomes diagonal, with its entries corresponding to its inertial properties. Additionally, when a full-definition approach is used to describe all the bodies in the model, the system mass matrix becomes constant and diagonal.

The validity and accuracy of the formulation with both modeling approaches was tested by performing a forward dynamic simulation of a set of benchmark problems. No convergence problems were found for the two cases, presenting excellent agreement scores with the analytical solutions and benchmark data. Similarly, no relevant differences were found between the kinematic outcomes of the two modeling approaches, presenting similar patterns for all the analyzed cases. It is worth noting that the purpose of using benchmark cases was to directly compare the FCC-GRB formulation with other established formulations, such as the natural coordinates, cartesian coordinates, relative, and linear graph formulations, as well as with other multibody software (e.g., OpenSim, MBDyn, Altair MotionSolve) [55]. Despite the conceptual differences among these formulations, the FCC-GRB formulation demonstrated excellent agreement in all cases, validating its applicability.

Despite generating 25 % less coordinates than the fully-defined case, the dependency of the transformation matrices **V** on the generalized coordinates and the appearance of variable mass matrices in the reduced case imply that more calculations need to be performed each time the EoM are solved, not necessarily generating faster simulation times. The results obtained for computational times reveal that selecting the reduced approach is not always preferable from a computational perspective. However, when more complex models, typically characterized by longer computational times, are considered, the reduced modeling approach becomes the preferable choice, as the difference in computational times decreases significantly.

Being based on both the Cartesian coordinates and natural coordinate formulations, the FCC-GRB formulation inherits several of the advantages typically associated with these two methodologies. To summarize, the use of a generic rigid body allows the FCC-GRB to retain the straightforward modeling procedure characteristic of the Cartesian coordinates formulation. Additionally, placing the reference point at the body's CoM and aligning the rigid body vectors with the principal axes of inertia leads to diagonal mass matrices [28,40,56]. This feature enables the application of numerical methods optimized for sparse matrices, improving computational efficiency [29]. Furthermore, as joints are explicitly defined in the FCC-GRB formulation, internal forces are directly obtained from solving the EoM, eliminating the need for post-processing steps as required in the natural coordinates with implicit joints.

In turn, adopting the same type of coordinates used in the natural coordinates introduces its associated benefits into the FCC-GRB formulation. Specifically, the system kinematics are described through linear relations between the transformation matrices (**C** and **V**) and the generalized coordinates. This characteristic results in linear or quadratic kinematic constraint equations, which are generally more efficient to evaluate than the nonlinear expressions generated by the Cartesian coordinates formulation. This linearity also extends to the contributions to the Jacobian matrix and the right-hand side vectors of velocity and acceleration, which exhibit a null, linear, or quadratic relation in the generalized coordinates [29,30]. However, unlike the natural coordinates formulation, the determination of the matrix **C** is a direct procedure, with its coefficients corresponding to the local coordinates of the points and vectors. This relation simplifies the computational implementation of the formulation, including the definition of the system topology, mass matrix, and the calculation of internal forces. Another shared advantage with the natural coordinates is the possibility to align the rigid body vectors with joint axes, simplifying the definition of system topology and enabling the generation of more efficient mathematical expressions [33]. Additionally, the constant nature of matrix **C** for points and vectors fixed in the local reference frame allows it to be computed in a pre-kinematic step and assembled into the Jacobian matrix without requiring continuous updates during EoM evaluations.

Nevertheless, the FCC-GRB formulation is not without limitations. Compared to both the Cartesian coordinates and natural coordinates formulations, the FCC-GRB typically generates a larger number of generalized coordinates and kinematic constraint equations. Specifically, even when using the reduced approach, the number of generalized coordinates is larger than in the classical angular-based global formulations. For instance, when compared to a Cartesian coordinates formulation with Euler parameters, the FCC-GRB formulation requires approximately two additional generalized coordinates, and, consequently, two additional kinematic constraints, per rigid body [28]. On the other hand, while the FCC-GRB generates the same number of generalized coordinates per rigid body as the natural coordinates, the ability of this formulation to share points and vectors between adjacent bodies can reduce the total number of coordinates and constraints in the system [29,53]. However, the FCC-GRB also supports vector sharing, which implicitly defines shared vector SV conditions. Although not explored in this study, this capability could further reduce the total number of generalized coordinates and kinematic constraints. For highly constrained systems with a large number of revolute joints, the FCC-GRB may result in a lower total coordinate count compared to the Cartesian coordinates formulation.

Finally, one potential application identified for the planar FCC-GRB formulation is its use in teaching multibody dynamics topics in advanced courses [30]. The main features supporting this idea were its greater intuitiveness and simplicity in implementing and modeling multibody systems compared to other common global formulations. These premises are still valid for the spatial formulation with the fully-defined approach, namely: i) the introduction of the generic rigid body simplifies the modeling of the multibody system, allowing for easier systematization of this procedure. This is particularly more noticeable when compared with the natural coordinates formulation, where the model is strongly dependent on the topology of the system; ii) the rigid body mass matrix remains diagonal and constant, with its structure independent of the system topology. Moreover, its entries are directly the inertial properties of the body being modeled, resulting in a mass matrix with a higher physical meaning; iii) the rigid bodies are defined resorting only to Cartesian coordinates, meaning that no background knowledge on the parametrization of rotations in spatial models is required, simplifying the implementation of the formulation. This last point was precisely one of the major advantages attributed to the natural coordinates formulation, supporting its use in educational applications [33]. Considering that the FCC-GRB formulation is even more intuitive, it can be more easily applied in the teaching of multibody topics in advanced courses. For that reason, the authors opted for presenting in detail the specificities of the formulation in 3D, so that, together with the planar work [30], the undergraduate and graduate students can easily implement the formulation and understand the results it provides. Hence, these two works can be directly used as an education tool to support courses in the STEM areas, providing insights of how to model and analyze simple and complex mechanical systems in 3D.

CRediT authorship contribution statement

Sérgio B. Gonçalves: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Methodology, Investigation, Formal analysis, Conceptualization. **Ivo Roupa:** Writing – review & editing, Validation, Conceptualization. **Paulo Flores:** Writing – review & editing, Supervision. **Miguel Tavares da Silva:** Writing – review & editing, Validation, Supervision, Project administration, Conceptualization.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

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Supplementary materials

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.mechmachtheory.2025.105955.

Data availability

No data was used for the research described in the article.

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