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Linear transformation semigroups

Abstract

According to Clifford and Preston (1961), the theory of semigroups began with a paper by Suschkewitsch in 1928. In this work, Suschkewitsch used transformations of a finite set to describe the structure of a certain algebraic semigroup, thus as observed by Sullivan in 2000, "at the very beginning, transformation semigroups were recognised as being useful" by providing interesting examples to illustrate concepts in the theory of semigroups. Since then, important results on ideals, morphisms, congruences and generators of semigroups of transformations have been obtained, and some of these results have corresponding ones in the context of linear algebra. As noted by Sullivan in 1991, results were first proved for transformations of a set, and then analogous statements were studied for the vector space setting. But the ideas and techniques used in these two areas are quite different, and often there are 'puzzling' differences between the results obtained. Thus, there is an ongoing interest in linear transformation semigroups and, in the last two decades, many papers have been written on this topic. Throughout this thesis, we will study several semigroups of linear transformations.

In chapter 2, we consider the semigroup $GS(m,n)$ consisting of all one-to-one linear transformations of a vector space V into itself whose ranges have codimension n, where $\aleph_0 \leq n \leq m = \dim V$. This is a linear version of the well-known Baer-Levi semigroup $BL(p,q)$ defined on a set X, where $\aleph_0 \leq q \leq p = |X|$. We show that these two types of Baer-Levi semigroup are never isomorphic, although they have the same basic properties; and we prove that the semigroups $GS(m,n)$ act as a model for right simple, right cancellative semigroups without idempotents. Moreover, we determine the left ideals of $GS(m,n)$ and some of its maximal subsemigroups.

In chapter 3, we study the semigroup $KN(m,n)$ of all one-to-one linear transformations of V into itself whose ranges have codimension at least n, where $\aleph_0 \leq n \leq m = \dim V$. For $n < m$, we describe the Green's relations and ideals of $KN(m,n)$, and show that it has infinitely many bi-ideals which are not quasi-ideals. Also, we prove that $KN(m,n)$ acts as a model for right cancellative semigroups without idempotents

whose right ideals form a chain and which have no maximal principal left ideals.

The semigroups considered in chapters 2 and 3 are defined by restricting certain cardinalities: namely, the dimension of the kernels and the codimension of the ranges of their elements. Analogously, in chapter 4 we consider four semigroups of linear transformations defined by fixing an infinite upperbound or lowerbound on one of these cardinalities. We determine whether these four semigroups belong to the class of all semigroups whose sets of bi-ideals and quasi-ideals coincide, and, for each semigroup, we describe its maximal regular subsemigroup, Green's relations and ideals. We also determine all the maximal right simple subsemigroups of one of the semigroups, and, using the work by Reynolds and Sullivan in 1985 and by Sullivan in 1994, we discuss products of idempotents and products of nilpotents.

In chapter 5, we show that the symmetric inverse semigroup $I(X)$ defined on a set X is almost never isomorphic to the semigroup $I(V)$ consisting of all injective partial linear transformations of a vector space V, and prove that any inverse semigroup can be embedded in some $I(V)$. In 1999, Yang classified the maximal inverse subsemigroups of all the ideals of $I(X)$ for finite X, and we do the same for $I(V)$ where V is finite-dimensional. In 1984, Howie and Marques-Smith showed that, if $p = q$, then $BB^{-1} = I(X)$, where $B = BL(p, q)$ with $|X| = p \geq q \geq \aleph_0$, and they described the subsemigroup of $I(X)$ generated by $B^{-1}B$. A decade later, Lima extended that work to 'strong independence algebras', and thus also to vector spaces. Here, we answer the natural question: what happens when $p > q$? We also show that, in this case, BB^{-1} and its analogue for vector spaces are never isomorphic.

In chapter 6, we consider the problem of describing all isomorphisms between certain semigroups of linear transformations. We extend a result by Sullivan on the automorphisms of subsemigroups of $T(X)$ covering X to isomorphisms of semigroups of transformations covering the underlying sets, and we compare this result with the work by Schwachhöfer and Stroppel in 1997. Although the problem of describing all isomorphisms between linear Baer-Levi semigroups is as yet unsolved, we present some results in that direction. Finally, we extend Baer's First Fundamental Theorem of Projective Geometry to certain posets of subspaces which may be important for a complete solution of this problem.

Semigrupos de transformações lineares

Resumo

Segundo Clifford e Preston (1961), a teoria de semigrupos começou com um artigo de Suschkewitsch em 1928. Nesse trabalho, Suschkewitsch descreve a estrutura de um certo semigrupo algébrico usando transformações de um conjunto finito. Assim, como foi observado por Sullivan em 2000, os semigrupos de transformações foram tidos como úteis desde o início, por fornecerem interessantes exemplos de conceitos da teoria de semigrupos. Desde então, obtiveram-se importantes resultados sobre ideais, morfismos, congruências e geradores de semigrupos de transformações, alguns deles admitindo resultados correspondentes no contexto da álgebra linear. Segundo Sullivan (1991), primeiro eram provados resultados para transformações de um conjunto e, depois, estudadas afirmações análogas no âmbito dos espaços vectoriais. Porém, as ideias e técnicas usadas nestas duas áreas diferem bastante e, por vezes, encontram-se intrigantes diferenças entre os resultados obtidos. Surge, assim, um interesse crescente na teoria de semigrupos de transformações lineares e, nas últimas duas décadas, publicaram-se diversos artigos nesta área. Neste trabalho, estudaremos vários semigrupos de transformações lineares.

No capítulo 2, consideramos o semigrupo $GS(m,n)$ de todas as aplicações lineares injectivas de um espaço vectorial V nele próprio cujas imagens têm codimensão n, onde $\aleph_0 \leq n \leq m = \dim V$. Este semigrupo é uma versão linear do semigrupo de Baer-Levi $BL(p,q)$ definido num conjunto X, com $\aleph_0 \leq q \leq p = |X|$. Provamos que, apesar de terem as mesmas propriedades básicas, estes dois tipos de semigrupo de Baer-Levi nunca são isomorfos, e que $GS(m,n)$ é um modelo para semigrupos que não têm idempotentes e que são simples à direita e cancelativos à direita. Além disso, descrevemos todos os seus ideais esquerdos e alguns subsemigrupos maximais.

No capítulo 3, estudamos o semigrupo $KN(m,n)$ das aplicações lineares injectivas de V em V cujas imagens têm codimensão não inferior a n, com $\aleph_0 \leq n \leq m$. Descrevemos as relações de Green e os ideais de $KN(m,n)$, com $n < m$, e vemos que este semigrupo tem uma infinidade de bi-ideais que não são quasi-ideais. Vemos também que é um modelo para semigrupos cancelativos à direita que não têm idempotentes

nem ideais esquerdos principais maximais e cujos ideias direitos formam uma cadeia.

Os semigrupos considerados nos capítulos 2 e 3 definem-se pela restrição de certas cardinalidades: nomeadamente, a dimensão dos núcleos e a codimensão das imagens dos seus elementos. De modo análogo, no capítulo 4 definimos quatro semigrupos de transformações lineares ao fixar um limite superior ou inferior infinito para uma dessas cardinalidades. Determinamos se esses semigrupos pertencem à classe de todos os semigrupos cujos conjuntos de bi-ideais e de quasi-ideais coincidem e descrevemos o subsemigrupo regular maximal, as relações de Green e os ideais de cada um dos semigrupos. Também determinamos todos os subsemigrupos simples à direita maximais de um dos semigrupos e descrevemos produtos de idempotentes e de nilpotents, baseando-nos no trabalho de Reynolds e Sullivan em 1985 e de Sullivan em 1994.

No capítulo 5, provamos que o semigrupo inverso simétrico $I(X)$ definido num conjunto X quase nunca é isomorfo ao semigrupo $I(V)$ de todas as aplicações lineares parciais injectivas de um espaço vectorial V, e que todo o semigrupo inverso é mergulhável em algum $I(V)$. Em 1999, Yang caracterizou os subsemigrupos inversos maximais de todos os ideais de $I(X)$, para X finito: aqui, fazemos o mesmo para $I(V)$, onde V tem dimensão finita. Em 1984, Howie e Marques-Smith provaram que, se $p = q$, então $BB^{-1} = I(X)$, onde $B = BL(p, q)$ e $|X| = p \geq q \geq \aleph_0$, e descreveram o subsemigrupo de $I(X)$ gerado por $B^{-1}B$. Em 1994, Lima estendeu este trabalho para 'álgebras de independência forte' e, portanto, para espaços vectoriais. Neste capítulo, respondemos à seguinte questão: o que acontece para $p > q$? Provamos também que, neste caso, BB^{-1} nunca é isomorfo ao seu análogo linear.

No capítulo 6, estudamos o problema de descrever os isomorfismos entre certos semigrupos de transformações lineares. Estendemos um resultado de Sullivan sobre os automorfismos de subsemigrupos de $T(X)$ que cobrem X para isomorfismos de semigrupos de transformações que cobrem os conjuntos subjacentes, e comparamos este resultado com um de Schwachhöfer e Stroppel. Apesar do problema de descrever todos os isomorfismos entre semigrupos de Baer-Levi lineares ainda não estar resolvido, apresentamos alguns resultados nessa direcção. Finalmente, estendemos o Primeiro Teorema Fundamental da Geometria Projectiva de Baer para certos c.p.o.'s de subespaços, o que julgamos ser importante para a resolução integral deste problema.

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Overall notation

$\emptyset$	the empty set	17
	or the empty mapping of a set X	5
0	the zero element of a semigroup with zero	2
	or the (total) linear transformation with range $\{0\}$	10
	or the partial linear transformation with domain $\{0\}$	10
1	the identity element of a monoid	2
a^{-1}	the inverse of an element a of an inverse semigroup S	7
a^n	the nth power of an element a of a semigroup S	2
$\|A\|$	the cardinal number of the set A	5
$\langle A \rangle$	the subspace of a vector space V generated by A	10
$\langle\langle A \rangle\rangle$	the inverse subsemigroup of a semigroup S generated by a set A	119
A_x	the constant partial transformation of X with domain $A \subseteq X$ that maps every element of A onto $x \in X$	5
$a\alpha$	the image of an element a under a mapping α	5
$A\alpha$	the set of all images of elements of $A \subseteq \text{dom}\ \alpha$ under α	13
$A\alpha^{-1}$	the set of all elements x in dom α such that $x\alpha \in A$	5
$A \dot\cup B$	the disjoint union of the sets A and B	5
$A \oplus B$	the direct sum of vector spaces A and B	8
$AE(p,q)$	the semigroup of all linear transformations of a vector space V with defect smaller than q	69
$AE(X)$	the semigroup of all almost onto transformations of a set X	69
$AM(p,q)$	the semigroup of all linear transformations of a vector space V with nullity smaller than q	69
$AM(X)$	the semigroup of all almost one-to-one transformations of a set X	69
$\text{Aut}(S)$	the automorphism group of a semigroup S	7
$\aleph_0$	the cardinal of infinite countable sets	13
$\alpha \circ \beta$	the composition of two partial transformations α and β	5
	or the composition of two partial linear transformations α and β	8
B^A	the set of all mappings $g : A \to B$	26
$B(G,J)$	the Brandt semigroup (over the group with zero G^0)	109

$BL(p,q)$	the Baer-Levi semigroup of type (p,q) defined on a set X	13
BQ	the class of all semigroups whose sets of bi-ideals and quasi-ideals coincide	49
$c(\alpha)$	the collapse of a partial transformation α	5
codim A	the codimension of a subspace A in a vector space V	8
$\mathrm{Cong}(S)$	the lattice of congruences on a semigroup S	23
$\mathcal{D}$	Green's relation	4
D_a	the $\mathcal{D}$–class of a semigroup S which contains $a \in S$	4
D_r	the $\mathcal{D}$–class of $I(V)$ consisting of all α with rank r	110
$d(\alpha)$	the defect of a partial transformation α	5
	or the defect of a partial linear transformation α	9
$D(\alpha)$	the complement of ran α in X	5
$\dim V$	the dimension of a vector space V over a field F	9
dom α	the domain of a partial transformation α	4
	or the domain of a partial linear transformation α	8
$E(S)$	the set of idempotents of a semigroup S	2
$E(V)$	the semigroup of all onto (total) linear transformations of a vector space V	51
$\mathcal{E}(X)$	the semigroup generated by the non-identity idempotents of $T(X)$	70
$F(\alpha)$	the subset of X of all fixed points of $\alpha \in T(X)$	32
$\mathrm{Fix}\,(\alpha)$	the subspace of V of all fixed points of $\alpha \in T(V)$	9
$\phi\|A$	the restriction of a transformation ϕ to $A \subseteq \mathrm{dom}\ \phi$	32
G_m	the general linear group of degree m over F	109
$G(V)$	the group of all invertible linear transformations of a vector space V	102
$G(X)$	the symmetric group on a set X	5
$g(\alpha)$	the gap of a partial transformation α	5
	or the gap of a partial linear transformation α	9
$GS(m,n)$	the linear Baer-Levi semigroup of type (m,n) defined on a vector space V	13
$\mathcal{H}$	Green's relation	4
H_a	the $\mathcal{H}$–class of a semigroup S containing $a \in S$	98

I_r	the ideal of $I(V)$ consisting of all α with rank smaller than r	109
id_X	the identity mapping of a set X	5
$I(V)$	the inverse semigroup of all one-to-one partial linear transformations of a vector space V	9
$I(X)$	the symmetric inverse semigroup on a set X	5
$\mathcal{J}$	Green's relation	4
J_a	the $\mathcal{J}$–class of a semigroup S containing $a \in S$	4
$J(a)$	the principal ideal of a semigroup S generated by $a \in S$	85
$K(S)$	the set of all constant mappings in $S \subseteq P(X)$ with $\emptyset$ adjoined	139
$K(V)$	the subset of $T(V)$ consisting of zero and all α with rank at most 1	142
$\ker \alpha$	the kernel of a partial linear transformation α	9
$KN(p,q)$	the semigroup of all injective linear transformations of V with defect at least q	49
$\mathcal{L}$	Green's relation	3
L_a	the $\mathcal{L}$–class of a semigroup S containing $a \in S$	3
$M(V)$	the semigroup of all injective (total) linear transformations of a vector space V	49
$M(X)$	the semigroup of all injective (total) transformations of a set X	49
$\mathbb{N}$	the set of all non-negative integers	28
$n(\alpha)$	the nullity of a partial linear transformation α	9
$OE(p,q)$	the semigroup of all linear transformations of a vector space V with defect at least q	69
$OE(X)$	the semigroup of all opposite onto transformations of a set X	69
$OM(p,q)$	the semigroup of all linear transformations of a vector space V with nullity at least q	69
p'	the successor of the cardinal p	6
$P(V)$	the semigroup of all partial linear transformations of a vector space V	8
$P(X)$	the partial transformation semigroup on a set X	4
$\mathbb{Q}$	the set of all rational numbers	29
$\mathcal{R}$	Green's relation	4

R_a	the $\mathcal{R}$–class of a semigroup S containing $a \in S$	4
$r(\alpha)$	the rank of a partial transformation α	5
	or the rank of a partial linear transformation α	9
$\mathrm{Ran}(S)$	the family of all ranges of elements of a semigroup of transformations S	18
ran α	the range of a partial transformation α	4
	or the range of a partial linear transformation α	8
ρ_x	the inner right translation $a \mapsto ax$ of a semigroup S, where x is a fixed element of S	40
S^0	the semigroup obtained from S by adjoining a zero	2
S^1	the monoid obtained from S by adjoining an identity	2
S^{-1}	the set of all inverses of elements of S	110
S/I	the Rees quotient of S by an ideal I	57
$s(\alpha)$	the shift of a transformation α	5
	or the shift of a linear transformation α	9
$S(\alpha)$	the complement of $F(\alpha)$ in X	5
T_ξ	the ideal of $T(V)$ consisting of all α with rank smaller than ξ	10
$T(V)$	the semigroup of all (total) linear transformations of a vector space V	9
$T(X)$	the total transformation semigroup on a set X	5
V/A	the quotient space of a vector space V by a subspace A	8
$(X)_B$	the bi-ideal of a semigroup S generated by a subset X of S	54
$(X)_Q$	the quasi-ideal of a semigroup S generated by a subset X of S	54

Chapter 1

Preliminaries

It is well-known that every semigroup can be regarded as a semigroup of transformations, and semigroups of transformations provide interesting examples of certain concepts in semigroup theory. Also, there are results in the theory of transformation semigroups which are of general importance. Throughout this thesis, we will study the structure of several semigroups of (linear) transformations, and often we will show that these semigroups act as a 'model' for specific classes of algebraic semigroups. In this chapter, we give a short account of notation and results of the theory of arbitrary semigroups, as well as the theories of transformation and linear transformation semigroups, and we assume the reader has a basic knowledge of these topics. Unless stated otherwise, we use the notation and terminology of [5].

1.1. Semigroups

A *semigroup* is a non-empty set S with a binary operation $*$ which is *associative*, that is to say, for all $a, b, c \in S$,

$$a * (b * c) = (a * b) * c.$$

Usually, we write the operation as *multiplication*, and we often write $a \cdot b$ more simply as ab. A non-empty subset T of a semigroup S is called a *subsemigroup of* S if, for all $a, b \in T$, we have $ab \in T$.

Let a be an element of a semigroup S. We say that a is *left* [*right*] *cancellable in* S if, for all $b, c \in S$,

$$ab = ac \ [ba = ca] \quad \text{implies} \quad b = c.$$

The semigroup S is said to be *left* [*right*] *cancellative* if every $a \in S$ is left [right] cancellable in S. If S is both left and right cancellative, it is called *cancellative*. In chapters 2 and 3 of this thesis, we study two examples of semigroups in the class of right cancellative semigroups.

If S is a semigroup, we say that $a \in S$ is *regular* if $a = axa$ for some x in S, and S is called *regular* if every $a \in S$ is regular. An element b of S is said to be an *inverse* of

$a \in S$ if $a = aba$ and $b = bab$. In fact, it is not difficult to see that $a \in S$ is regular if and only if it has at least one inverse in S. By an *inverse semigroup* we mean a semigroup in which every element has a unique inverse. All the semigroups studied in chapters 2, 3 and 4 are non-regular, and some even have no regular elements. On the other hand, chapter 5 is dedicated to the study of one particular inverse semigroup.

A semigroup S with an identity element is said to be a *monoid*. If S has no identity, then we can easily adjoin an identity 1 to S by extending the binary operation on S to one on the set $S \cup \{1\}$ as follows:

$$1 \cdot 1 = 1, \quad 1 \cdot a = a \cdot 1 = a, \quad \text{for every} \quad a \in S.$$

In fact, $S \cup \{1\}$ is a monoid (since the extended operation is still associative and 1 is an identity of $S \cup \{1\}$). Thus, we can define the *monoid* S^1 *obtained from* S by

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity element} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

Analogously, if S has no zero element, then we can adjoin an extra element 0 to S and define a binary operation on $S \cup \{0\}$ by extending the one on S with

$$0 \cdot 0 = 0 \cdot a = a \cdot 0 = 0 \quad \text{for every} \quad a \in S.$$

Also, we define the *semigroup with zero* S^0 *obtained from* S by

$$S^0 = \begin{cases} S & \text{if } S \text{ has a zero element} \\ S \cup \{0\} & \text{otherwise.} \end{cases}$$

By a *group with zero* we mean G^0 where G is a group.

An element a in a semigroup S with zero is a *nilpotent* with *index* $r \geq 2$ if $a^r = 0$ and $a^{r-1} \neq 0$. If S is a semigroup and $e \in S$ is such that $e^2 = e$, then we say e is an *idempotent*. The set of all idempotents of S is usually denoted by $E(S)$. Clearly, if S is a semigroup with identity 1, then $1 \in E(S)$. Also, if S has a zero element 0, then $0 \in E(S)$. An idempotent $e \in E(S) \setminus \{1\}$ is called a *proper* idempotent of S, and we say that $f \in E(S)$ is a *primitive* idempotent if $f \neq 0$ and the only idempotents of S under f, with respect to the natural partial order $\leq$ on $E(S)$, are 0 (if S has a zero) and f itself: that is, for every $e \in E(S)$, $e \leq f$ implies $e = 0$ or $e = f$. A

semigroup S is said to be a *band* if all elements of S are idempotents, that is to say, if $S = E(S)$. In chapter 4, we describe products of nilpotents and idempotents in the semigroups studied, whereas the two semigroups considered in chapters 2 and 3 have no idempotents or nilpotents.

Given two subsets A and B of a semigroup S, the *set product* AB of A and B consists of all products ab where $a \in A$ and $b \in B$. If $A = \{a\}$ then we write $\{a\}B$ more simply as aB. Analogously, Ab denotes the set product $A\{b\}$. A non-empty subset A of S is called a *left* [*right*] *ideal of* S if $SA \subseteq A$ [$AS \subseteq A$]. We say that A is a (*two-sided*) *ideal of* S if it is both a left and a right ideal. Obviously, S is an ideal of itself, and among the ideals of a semigroup with zero is also the set $\{0\}$. By a *proper two-sided* [*left*, *right*] *ideal of* S we mean a two-sided [respectively: left, right] ideal properly contained in S. A semigroup S is said to be *left* [*right*] *simple* if it is the only left [right] ideal of itself. Similarly, we say that S is *simple* if S has no proper ideals. We also have the corresponding concepts of 0-simplicity: namely, a semigroup S with zero element 0 is called 0-*simple* [*left* 0-*simple*, *right* 0-*simple*] if $S^2 \neq \{0\}$ and $\{0\}$ is the unique proper two-sided [respectively: left, right] ideal of S. A *completely* [0-]*simple* semigroup is a [0-]simple semigroup containing a primitive idempotent. In chapters 2, 3, 4 and 5 of this thesis, we study several semigroups and we describe the ideal structure of each one. In fact, we show that the semigroup studied in chapter 2 is in the class of right simple semigroups, whereas the semigroup studied in chapter 3 is almost never in that class.

If A is a non-empty subset of S, then the intersection of all left ideals of S containing A is a left ideal of S and we call it the *left ideal of* S *generated by* A. Analogously, we define the *right* [*two-sided*] *ideal of* S *generated by* A. It is easy to see that the left [right, two-sided] ideal generated by A is $SA \cup A = S^1A$ [respectively: $AS \cup A = AS^1$, $A \cup SA \cup AS \cup SAS = S^1AS^1$]. If A is a singleton, say $A = \{a\}$, then we call S^1a [aS^1, S^1aS^1] the *principal left* [respectively: *right*, *two-sided*] *ideal of* S *generated by* a. Given a semigroup S and two elements a and b in S, we write $a \mathcal{L} b$, and say that a and b are $\mathcal{L}$*–related*, if and only if a and b generate the same principal left ideal of S: that is, $S^1a = S^1b$. Clearly, $\mathcal{L}$ is an equivalence relation on S and, for every $a \in S$, we denote the $\mathcal{L}$–class of S containing a by L_a. Dually, we define the relation

$\mathcal{R}$ on S by

$$a \mathcal{R} b \text{ if and only if } aS^1 = bS^1 \qquad (a, b \in S).$$

It is easy to see that $\mathcal{R}$ is also an equivalence relation on S and we denote the $\mathcal{R}$–class of S which contains $a \in S$ by R_a. We let $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \vee \mathcal{R}$. It is well-known that the relations $\mathcal{L}$ and $\mathcal{R}$ commute under composition (see [5] Vol. 1, Lemma 2.1), hence $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$. Finally, we write $a \mathcal{J} b$, and say that a and b are *$\mathcal{J}$–related*, if and only if they generate the same principal (two-sided) ideal of S, that is to say,

$$a \mathcal{J} b \text{ if and only if } S^1aS^1 = S^1bS^1.$$

Since $\mathcal{L} \subseteq \mathcal{J}$ and $\mathcal{R} \subseteq \mathcal{J}$, it follows that $\mathcal{D} \subseteq \mathcal{J}$, but the converse is not always true. In fact, for the three semigroups considered in chapters 2, 3 and 5, we show that the relations $\mathcal{D}$ and $\mathcal{J}$ coincide, as well as for two of the semigroups studied in chapter 4. But for the other two semigroups studied in chapter 4, we see that the relation $\mathcal{D}$ is properly contained in $\mathcal{J}$. As stated by Clifford and Preston in [5] Vol. 1, p. 47, these "fundamental" equivalence relations were first studied by Green in 1951, and they are a crucial tool for the study of the structure of semigroups in general. We call them *Green's relations*. We note that a semigroup S is left [right] simple if and only if it consists of a single $\mathcal{L}$–class [$\mathcal{R}$–class]. Also, S is simple if and only if it consists of a single $\mathcal{J}$–class.

1.2. Semigroups of transformations

It is well-known that every semigroup can be embedded in a transformation semigroup. Thus, as observed by Sullivan in [44] p. 195, transformation semigroups are for semigroup theory as permutation groups are for group theory. And, according to Sullivan [44] p. 191, "the theory of transformation semigroups now constitutes a substantial body of knowledge in its own right". In this section, we introduce some notation and well-known results in this field.

Let X be an arbitrary non-empty set and let $P(X)$ denote the set of all *partial transformations* of X: that is, all transformations α whose *domain*, dom α, and *range*, ran α, are subsets of X. We denote the image of $x \in X$ under a transformation α

by $x\alpha$. As usual, the *composition* $\alpha \circ \beta$ of $\alpha, \beta \in P(X)$ is the transformation with domain $Y = (\text{ran } \alpha \cap \text{dom } \beta)\alpha^{-1}$ such that, for all $x \in Y$,

$$x(\alpha \circ \beta) = (x\alpha)\beta,$$

and we often write $\alpha \circ \beta$ more simply as $\alpha\beta$ (compare [5] Vol. 1, p. 29). Obviously, Y is a subset of X and $\alpha \circ \beta \in P(X)$ if $\alpha, \beta \in P(X)$. In fact, $(P(X), \circ)$ is a semigroup: we call $P(X)$ the *partial transformation semigroup on* X. We observe that $P(X)$ contains a zero element and an identity: namely, the empty mapping $\emptyset$ and the identity transformation id_X on X, respectively.

Now let $T(X)$ denote the *total transformation semigroup on* X: that is, the subsemigroup of $P(X)$ consisting of all $\alpha \in P(X)$ with domain X. Note that we use the 'P' and the 'T' to indicate that we are considering *partial* and *total* transformations, respectively. The *symmetric inverse semigroup* $I(X)$ *on* X consists of all injective elements of $P(X)$, and the *symmetric group* $G(X)$ *on* X is the set of all permutations of X.

If $\alpha \in P(X)$, we define the *rank* of α to be $r(\alpha) = |\text{ran } \alpha|$. We also write

$$\begin{aligned} C(\alpha) &= \bigcup\{y\alpha^{-1} : |y\alpha^{-1}| \geq 2\}, & c(\alpha) &= |C(\alpha)|, \\ G(\alpha) &= X \setminus \text{dom } \alpha, & g(\alpha) &= |G(\alpha)|, \\ D(\alpha) &= X \setminus \text{ran } \alpha, & d(\alpha) &= |D(\alpha)|. \end{aligned}$$

The cardinal numbers $c(\alpha)$, $g(\alpha)$ and $d(\alpha)$ are called the *collapse*, *gap* and *defect* of α, respectively. Given $\alpha \in T(X)$, we define $S(\alpha) = \{x \in X : x\alpha \neq x\}$, and let $s(\alpha) = |S(\alpha)|$, and we call this the *shift* of $\alpha \in T(X)$.

We extend the convention introduced in [5] Vol. 2, p. 241: namely, if $\alpha \in P(X)$ is non-zero then we write

$$\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}$$

and take as understood that the subscript i belongs to some (unmentioned) index set I, that the abbreviation $\{x_i\}$ denotes the set $\{x_i : i \in I\}$, and that $\text{ran } \alpha = \{x_i\}$, $x_i\alpha^{-1} = A_i$ and $\text{dom } \alpha = \bigcup\{A_i : i \in I\}$. If $\text{dom } \alpha = A$ and α maps every element of A onto a fixed $x \in X$, then we write $\alpha = A_x$. Throughout this thesis, $X = A \,\dot\cup\, B$ means X is a disjoint union of A and B.

It is well-known that $T(X)$ is regular (see [5] Vol. 1, p. 33, Exercise 1), and the same is true for $P(X)$. For, clearly $\emptyset$ is a regular element of $P(X)$ and, given

$$\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}$$

in $P(X) \setminus \{\emptyset\}$, we can choose $a_i \in A_i$ for every i, and define $\beta \in P(X)$ by

$$\beta = \begin{pmatrix} x_i \\ a_i \end{pmatrix},$$

which satisfies $\alpha\beta\alpha = \alpha$.

Clearly, if $\alpha \in T(X)$ is one-to-one, then it is right cancellable in $T(X)$. In fact, the converse is also true. For, suppose $\alpha \in T(X)$ is right cancellable and let $x, y \in X$ be such that $x\alpha = y\alpha$. Then, $X_x\alpha = X_y\alpha$ and this implies $X_x = X_y$, hence $x = y$. Likewise, one can show that the left cancellable elements in $T(X)$ are exactly all the surjective $\alpha \in T(X)$.

According to Sullivan [41] p. 431, in 1952 Malcev determined the ideals of $T(X)$: they are precisely the sets

$$T_\xi(X) = \{\alpha \in T(X) : r(\alpha) < \xi\},$$

where $1 < \xi \leq |X|'$ and $|X|'$ denotes the *successor* of $|X|$ (that is, the least cardinal greater than $|X|$). Also, Liber (in 1953) and Šutov (in 1961) obtained analogous results for $I(X)$ and $P(X)$, respectively.

For every $\alpha \in P(X)$, we define a relation $\alpha \circ \alpha^{-1}$ on dom α as follows: given $x, y \in$ dom α,

$$(x, y) \in \alpha \circ \alpha^{-1} \text{ if and only if } x\alpha = y\alpha.$$

It is not difficult to see that $\alpha \circ \alpha^{-1}$ is an equivalence relation on dom α. As stated by Clifford and Preston in [5] Vol. 1, p. 51, in 1955 Miller and Doss characterised the Green's relations on $T(X)$: namely, given $\alpha, \beta \in T(X)$,

(a) $\alpha \mathcal{L} \beta$ if and only if ran $\alpha =$ ran β,

(b) $\alpha \mathcal{R} \beta$ if and only if $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$,

(c) $\alpha \mathcal{D} \beta$ if and only if $r(\alpha) = r(\beta)$,

and $\mathcal{D} = \mathcal{J}$ in $T(X)$.

As observed by Sullivan in [44] p. 201, Munn (in 1970) obtained essentially the same result for $I(X)$. Clearly, the characterisation of the relation $\mathcal{R}$ on $I(X)$ differs from the one in (b): namely, given $\alpha, \beta \in I(X)$,

$$\alpha \ \mathcal{R} \ \beta \ \text{ if and only if } \ \operatorname{dom} \alpha = \operatorname{dom} \beta.$$

According to Marques-Smith and Sullivan in [28] p. 307, in 1971 FitzGerald and Preston described the relations $\mathcal{L}$ and $\mathcal{R}$ on $P(X)$. In [28], the authors give a complete description of the Green's relations on $P(X)$. In fact, by [28] Theorem 10, it follows that, for every $\alpha, \beta \in P(X)$,

(a) $\alpha \ \mathcal{L} \ \beta$ if and only if $\operatorname{ran} \alpha = \operatorname{ran} \beta$,

(b) $\alpha \ \mathcal{R} \ \beta$ if and only if $\operatorname{dom} \alpha = \operatorname{dom} \beta$ and $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$,

(c) $\alpha \ \mathcal{D} \ \beta$ if and only if $r(\alpha) = r(\beta)$,

and $\mathcal{D} = \mathcal{J}$ in $P(X)$.

Given a subsemigroup S of $P(X)$, we denote the automorphism group of S by $\operatorname{Aut}(S)$. We say that $\phi \in \operatorname{Aut}(S)$ is *inner* if there exists some $g \in G(X)$ such that $\alpha\phi = g^{-1}\alpha g$ for every $\alpha \in S$. As observed by Sullivan in [40] p. 82, Malcev (in 1952) showed that the automorphisms of $T(X)$ and of its ideals are all inner and, a decade later, Šutov extended Malcev's results to $P(X)$ and its ideals. In [40] Theorem 1, Sullivan unified these and similar results by showing that if S is a semigroup of partial transformations of X which *covers* X (that is, such that for every $x \in X$ there exists an idempotent map in S with range $\{x\}$), then all automorphisms of S are inner. Note that, in this case, it is possible that S does not contain the symmetric group $G(X)$.

We observe that $T(X)$ is isomorphic to $T(Y)$ if and only if $|X| = |Y|$: this follows from the fact that the right zero elements of $T(X)$ are precisely the constant transformations of X (see [5] Vol. 1, p. 6, Exercise 7). Moreover, each isomorphism $\phi : T(X) \to T(Y)$ is *induced* by a bijection $g : X \to Y$ in the sense that $\alpha\phi = g^{-1}\alpha g$ for every $\alpha \in T(X)$. And it is not difficult to see that this result can be extended to $P(X)$ and $I(X)$, and their ideals. In chapter 2, we prove a similar result for a well-known semigroup of transformations.

1.3. Semigroups of linear transformations

As first observed by Sullivan in [42], there are significant results in the theory of transformation semigroups that have corresponding results in the context of linear algebra. Despite the similarity between some results in these two areas, the ideas and techniques are quite different, thus a theory of linear transformation semigroups has emerged and, during the last two decades, many papers have been written on this field. Throughout this thesis, we will focus essentially on semigroups of linear transformations. In this section, we give a brief summary of basic concepts and some results in this topic.

Let V be a vector space over a field F. In essence, we will follow the notation of [37]: namely, we will usually denote the vectors of V by $a, b, c, \ldots$ or $u, v, w, \ldots$, *scalars* (that is, elements of F) by $x, y, z, \ldots$ or $r, s, t, \ldots$, and subspaces of V by capital letters $A, B, \ldots$ or $U, W, \ldots$. Such notation will be used wherever possible, *but it may be disregarded when the context requires other notation.*

If A and B are subspaces of V such that $A \cap B = \{0\}$, then we say that $A + B$ is a *direct sum* of A and B, and we write it as $A \oplus B$. It is well-known that, for every subspace A of V, there exists some subspace B such that $V = A \oplus B$. Such a subspace B is called a *(vector space) complement of A in V*. We note that, whereas the complement of a subset Y in a set X is unique, this is not true for a complement of a subspace A of V. To see this, suppose $\{e_i\} \dot\cup \{a\} \dot\cup \{b\}$ is a basis for V and let A be the subspace of V generated by $\{e_i\} \dot\cup \{a\}$: clearly, the subspaces generated by $\{b\}$ and by $\{a + b\}$, respectively, are distinct complements of A in V. Despite this, it can be shown that all complements of a subspace A in V are isomorphic to the quotient space V/A, thus they all have the same dimension: namely, the dimension of V/A, which is called the *codimension of A in V* and is denoted by $\operatorname{codim} A$.

Let $P(V)$ denote the set of all *partial linear transformations* of V: that is, all linear transformations α whose *domain*, $\operatorname{dom}\alpha$, and *range*, $\operatorname{ran}\alpha$, are subspaces of V. The elements of $P(V)$ are denoted by $\alpha, \beta, \gamma, \ldots$ and (as in the set case) written on the right of the argument. As for partial transformations of a set, we define the *composition* $\alpha \circ \beta$ of $\alpha, \beta \in P(V)$ to be the linear transformation with domain

$A = (\text{ran } \alpha \cap \text{dom } \beta)\alpha^{-1}$ such that, for all $a \in A$,

$$a(\alpha \circ \beta) = (a\alpha)\beta,$$

and we often write $\alpha \circ \beta$ more simply as $\alpha\beta$. Clearly, A is a subspace of V and $\alpha \circ \beta \in P(V)$ if $\alpha, \beta \in P(V)$. Also, $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$ for all $\alpha, \beta, \gamma \in P(V)$, so $(P(V), \circ)$ is a semigroup. Note that we use the 'V' in $P(V)$ to denote the fact that we are considering *linear* transformations. Similarly, we let $T(V)$ denote the subsemigroup of $P(V)$ consisting of all total linear transformations from V to V.

If $\alpha \in P(V)$, we write $\ker \alpha$ for the *kernel* of α: that is, the subspace of V consisting of all vectors in dom α which are mapped to 0. Also, we put

$$n(\alpha) = \dim \ker \alpha, \quad r(\alpha) = \dim \text{ran } \alpha,$$

$$g(\alpha) = \text{codim dom } \alpha, \quad d(\alpha) = \text{codim ran } \alpha.$$

As usual, these are called the *nullity*, *rank*, *gap* and *defect* of α, respectively. For each $\alpha \in T(V)$, we define

$$\text{Fix}\,(\alpha) = \{u \in V : u\alpha = u\}.$$

Clearly this is a subspace of V, hence we can let $s(\alpha) = \text{codim Fix}\,(\alpha)$, and we call this the *shift* of α.

There is another natural subsemigroup of $P(V)$ – namely, $I(V)$ – which consists of all *one-to-one* partial linear transformations of V (that is, all $\alpha \in P(V)$ with nullity 0); and it is easy to see that $I(V)$ is an inverse semigroup. In fact, $I(V)$ can be regarded as a linear version of the symmetric inverse semigroup $I(X)$ defined on a set X, and we adopt this approach in this thesis.

It is well-known that the semigroup $T(V)$ is regular (see [5] Vol. 1, p. 57, Exercise 6) and, as observed by Sullivan in [45] p. 414, the same is true for $P(V)$. In chapters 2 and 3, we consider semigroups of linear transformations which have no idempotents, and hence no regular elements. On the other hand, in chapter 4 we describe all the regular elements of each of the subsemigroups of $T(V)$ studied and, using this, we determine the maximal regular subsemigroups of such semigroups. In chapter 5, we characterise all the maximal inverse subsemigroups of $I(V)$ where V is finite-dimensional and, for the infinite case, we determine some inverse subsemigroups of $I(V)$.

It is easily verified that the identity mapping id_V is an identity in $P(V)$, but this semigroup has no zero element if $V \neq \{0\}$. For, in this case, if $\delta \in P(V)$ is such that $\delta\alpha = \delta = \alpha\delta$ for every $\alpha \in P(V)$, then $\mathrm{dom}\ \delta = \mathrm{dom}\,(\alpha\delta) = (\mathrm{ran}\ \alpha \cap \mathrm{dom}\ \delta)\alpha^{-1}$ and this implies $\mathrm{dom}\ \delta \subseteq \mathrm{dom}\ \alpha$ for every $\alpha \in P(V)$. Therefore $\mathrm{dom}\ \delta = \{0\}$, hence δ is the partial linear transformation of V with domain $\{0\}$, which we denote by 0. If we choose $\beta \in P(V) \setminus I(V)$ (note that this set is non-empty if $V \neq \{0\}$), then $\ker \beta \neq \{0\}$ and $\mathrm{dom}\,(\beta\delta) = \{0\}\beta^{-1} = \ker \beta \neq \mathrm{dom}\ \delta$. Thus, $\beta\delta \neq \delta$, which contradicts our assumption on δ. Nevertheless, it is not difficult to see that 0 is a zero element for $I(V)$ and that the (total) linear transformation of V with range $\{0\}$ is a zero element in $T(V)$. We also denote this mapping by 0 and allow the context to determine which one we mean. In passing, we observe that if $V = \{0\}$, then $P(V)$ consists of a single element, namely the mapping 0.

As usual, the subspace of V generated by a subset $\{e_i\}$ of V is denoted by $\langle e_i \rangle$. Sometimes we write $A = \langle e_i \rangle$ and assume (with no mention of this fact) that the set $\{e_i\}$ is a basis for the subspace A.

Often it is necessary to construct some $\alpha \in P(V)$ by first choosing some linearly independent subset $\{e_i\}$ of V and some $\{a_i\} \subseteq V$, and then letting $e_i\alpha = a_i$ for each $i \in I$ and extending this action by linearity to the whole of $\mathrm{dom}\ \alpha = \langle e_i \rangle$. To abbreviate matters, we simply say, given $\{e_i\}$ and $\{a_i\}$ within context, that $\alpha \in P(V)$ is defined by letting

$$\alpha = \begin{pmatrix} e_i \\ a_i \end{pmatrix}.$$

Sometimes our argument starts by choosing a basis for $\ker \alpha$ and expanding it to one for a subspace containing $\ker \alpha$: provided no confusion will arise, we use this expression even if α is one-to-one (in which case, $\ker \alpha = \{0\}$ and so it has no basis).

In the previous section, we presented the well-known results on the Green's relations and ideals of the total transformation semigroup on a set X. As observed by Clifford and Preston in [5] Vol. 1, p. 51, similar results hold for $T(V)$ where V is an arbitrary vector space over a field F: namely, the ideals of $T(V)$ are precisely the sets

$$T_\xi = \{\alpha \in T(V) : r(\alpha) < \xi\},$$

where $1 \leq \xi \leq m'$ and $m = \dim V$. Also, for every $\alpha, \beta \in T(V)$,

(a) $\alpha \mathcal{L} \beta$ if and only if ran $\alpha =$ ran β,

(b) $\alpha \mathcal{R} \beta$ if and only if ker $\alpha =$ ker β,

(c) $\alpha \mathcal{D} \beta$ if and only if $r(\alpha) = r(\beta)$,

and $\mathcal{D} = \mathcal{J}$ in $T(V)$ (compare [5] Vol. 1, p. 57, Exercise 6).

As for the set case, the characterisation of the Green's relations on $P(V)$ and on $I(V)$ is essentially the same: it only differs in (b), where one must add the condition 'dom $\alpha =$ dom β' to the right-hand side of the equivalence stated. In chapters 2, 3 and 4, we characterise the Green's relations on several semigroups of (total) linear transformations and, in most cases, the results obtained differ substantially from the ones above for $T(V)$.

By analogy with the set case, it is natural to ask which linear transformation semigroups act as a 'model' for certain algebraic semigroups. In chapters 2 and 3, we obtain two results of this type: namely, we show that the semigroup considered in chapter 2 is a model for right simple, right cancellative semigroups without idempotents and, similarly, the semigroup studied in chapter 3 is a model for right cancellative semigroups without idempotents whose right ideals form a chain and which have no maximal principal left ideals.

Chapter 2

Baer-Levi semigroups of linear transformations

In 1932 Baer and Levi constructed a right simple, right cancellative semigroup which is not a group: they considered the semigroup (under composition) of all one-to-one transformations α of a countably infinite set X into itself such that $X \setminus X\alpha$ is not finite (here, for historical background, we rely on comments by Clifford and Preston in [5] Vol. 2, p. 82). Clifford and Preston also state that in 1953 Teissier considered a natural generalisation of Baer and Levi's semigroup which we define as follows. Let X be an infinite set with cardinal p and, for each infinite cardinal $q \leq p$, let $BL(p,q)$ be the set of all one-to-one transformations α of X into itself such that $X \setminus X\alpha$ has cardinal q: following Clifford and Preston, we call this the *Baer-Levi semigroup of type* (p,q) *on the set* X. It is well known that $BL(p,q)$ is a right simple, right cancellative semigroup without idempotents; and, according to Clifford and Preston [5] Vol. 2, p. 82, Teissier showed that any semigroup with these properties can be embedded in some Baer-Levi semigroup. In 1961 Šutov characterised the left ideals of the Baer-Levi semigroup of type (p,p) and many years later Sullivan generalised Šutov's work to any Baer-Levi semigroup of type (p,q). In 1963, Šutov described all congruences on $BL(p,p)$ and a decade later Mielke studied special congruences on Baer-Levi semigroups. A few years later, Lindsey and Madison provided more detailed information about the lattice of congruences on $BL(p,q)$. In 1983, Schein, Sullivan and Wood, together with Inessa Levi (different from the afore-mentioned Levi), showed that every automorphism φ of $BL(p,q)$ is "inner": that is, there exists $g \in G(X)$, the symmetric group on X, such that $\alpha\varphi = g\alpha g^{-1}$ for all $\alpha \in BL(p,q)$. Levi and Wood proved in 1984 that every non-empty subset Y of X whose complement in X has cardinal at least q gives rise to a maximal subsemigroup of $BL(p,q)$. And, in 1995, Hotzel determined other families of maximal subsemigroups of $BL(p,q)$.

In this chapter we consider a related semigroup, which we now proceed to describe. Let V be a vector space over a field F with dimension $m \geq \aleph_0$ and, for each cardinal n such that $\aleph_0 \leq n \leq m$, let $GS(m,n)$ denote the set of all one-to-one linear

transformations α from V into itself whose range has codimension n: we call this the *linear Baer-Levi semigroup* on V. In section 2.1, we present a summary of the afore-mentioned results concerning the Baer-Levi semigroups defined on sets and we show that every isomorphism between such semigroups is induced by a bijection between the underlying sets (this extends the result mentioned above regarding the automorphism group of a Baer-Levi semigroup defined on a set). Also, to obtain more information on the left ideals of $BL(p,q)$, we use a different approach than that used by Šutov and Sullivan: in doing so, we describe the principal left ideals of $BL(p,q)$ and we show that $BL(p,q)$ has no minimal or maximal (proper) left ideals. In section 2.2, we show $GS(m,n)$ is indeed a semigroup with the same properties as $BL(p,q)$: this fact extends work by Lima in 1994 on $GS(m,m)$. More importantly however, in section 2.3 we show these two types of Baer-Levi semigroup – one defined on sets, the other on vector spaces – are never isomorphic; and, as for the set case, we prove that every right simple, right cancellative semigroup without idempotents can be embedded in some linear Baer-Levi semigroup. In section 2.4, we transfer results on the left ideals of $BL(p,q)$ to the vector space setting and in section 2.5 we initiate the study of maximal subsemigroups of $GS(m,n)$ by using ideas taken from the work by Levi and Wood. All the work in sections 2.2–2.5 has appeared in [30].

2.1. Baer-Levi semigroups defined on sets

Let X be an infinite set with cardinal p. For every cardinal q such that $\aleph_0 \leq q \leq p$, write

$$BL(p,q) = \{\alpha \in T(X) : c(\alpha) = 0,\ d(\alpha) = q\}$$

and call this the *Baer-Levi semigroup of type* (p,q) *on the set* X. For clarity, we sometimes write $BL(X,p,q)$ in place of $BL(p,q)$. As shown in [5] Vol. 2, Theorem 8.2, $BL(p,q)$ is a right simple, right cancellative semigroup without idempotents. According to Clifford and Preston [5] Vol. 2, p. 82, this semigroup was first studied by Baer and Levi in 1932 for $p = q = \aleph_0$; and later, in 1953, Teissier considered the above generalisation (that is, for cardinals p and q such that $\aleph_0 \leq q \leq p$). Clifford and Preston also state that Teissier proved the following result (see [5] Vol. 2, Theorem 8.5).

Theorem 2.1.1. Any right simple, right cancellative semigroup S without idempotents can be embedded in some $BL(p,p)$, where $|S| = p$.

In [46], Šutov showed that the left ideals of $BL(p,p)$ are precisely the subsets L of $BL(p,p)$ which satisfy the following condition:

$$[\alpha \in L,\ \beta \in BL(p,p),\ \text{ran}\ \beta \subseteq \text{ran}\ \alpha,\ |\,\text{ran}\ \alpha \setminus \text{ran}\ \beta| = p] \ \text{implies}\ \beta \in L.$$

Several years later, Sullivan generalised this result to any Baer-Levi semigroup by showing the following (see [41] Theorem 3).

Theorem 2.1.2. Let $\alpha, \beta \in BL(p,q)$. Then, $\beta = \lambda\alpha$ for some $\lambda \in BL(p,q)$ if and only if $\text{ran}\ \beta \subseteq \text{ran}\ \alpha$ and $|\,\text{ran}\ \alpha \setminus \text{ran}\ \beta| = q$.

To obtain more information about the left ideals of $BL(p,q)$ we proceed as follows.

Let Y be a non-empty subset of $BL(p,q)$ and let $L_Y^{\oplus} = Y \cup L_Y$, where

$$L_Y = \{\beta \in BL(p,q) : \text{ran}\ \beta \subseteq \text{ran}\ \alpha,\ |\,\text{ran}\ \alpha \setminus \text{ran}\ \beta| = q \ \text{ for some } \ \alpha \in Y\}.$$

We first show that L_Y is non-empty. To do this, choose any $\alpha \in Y$. If $X = \{x_i\}$ with $|I| = p$, then write $x_i\alpha = a_i$ for each i and put $X \setminus \{a_i\} = \{a_j\}$. Since $d(\alpha) = q$, we have $|J| = q \leq p$. Now write $\{a_i\} = \{b_i\} \dot\cup \{b_j\}$ (this is possible since p is infinite) and define $\beta \in T(X)$ by

$$\beta = \begin{pmatrix} x_i \\ b_i \end{pmatrix}.$$

Clearly, $\beta \in BL(p,q)$ since it is one-to-one and $d(\beta) = |\{b_j\} \dot\cup \{a_j\}| = q + q = q$. Also, we have $\text{ran}\ \beta = \{b_i\} \subseteq \{a_i\} = \text{ran}\ \alpha$ and $|\,\text{ran}\ \alpha \setminus \text{ran}\ \beta| = |\{b_j\}| = q$. Hence $\beta \in L_Y$, so L_Y is a non-empty subset of $BL(p,q)$.

Theorem 2.1.3. If Y is a non-empty subset of $BL(p,q)$, then $L_Y^{\oplus}$ is a left ideal of $BL(p,q)$ and conversely, if I is a left ideal of $BL(p,q)$, then $L_I^{\oplus} = I$.

Proof. Suppose Y is a non-empty subset of $BL(p,q)$. Let $\alpha \in L_Y^{\oplus}$ and $\beta \in BL(p,q)$. Then both mappings α and β are elements of $BL(p,q)$ and so is $\beta\alpha$. Clearly $\beta\alpha = \beta.\alpha$ with $\beta \in BL(p,q)$, hence $\text{ran}\,(\beta\alpha) \subseteq \text{ran}\ \alpha$ and $|\,\text{ran}\ \alpha \setminus \text{ran}\,(\beta\alpha)| = q$ (by Theorem 2.1.2). If $\alpha \in Y$, then $\beta\alpha \in L_Y$. If not, then $\alpha \in L_Y$ and so there exists some $\lambda \in Y$

such that $\operatorname{ran}\alpha \subseteq \operatorname{ran}\lambda$ and $|\operatorname{ran}\lambda \backslash \operatorname{ran}\alpha| = q$. Therefore, $\operatorname{ran}(\beta\alpha) \subseteq \operatorname{ran}\alpha \subseteq \operatorname{ran}\lambda$ and $q = |\operatorname{ran}\lambda \setminus \operatorname{ran}\alpha| \leq |\operatorname{ran}\lambda \setminus \operatorname{ran}(\beta\alpha)| \leq |X \setminus \operatorname{ran}(\beta\alpha)| = q$. Thus $\beta\alpha \in L_Y$. Hence $\beta\alpha \in L_Y^{\oplus}$, so $L_Y^{\oplus}$ is a left ideal of $BL(p,q)$, as required.

Suppose I is a left ideal of $BL(p,q)$. Next we show that $I = L_I^{\oplus}$. If $\beta \in L_I$, then there exists some $\alpha \in I$ such that $\operatorname{ran}\beta \subseteq \operatorname{ran}\alpha$ and $|\operatorname{ran}\alpha \setminus \operatorname{ran}\beta| = q$. From Theorem 2.1.2, there exists $\lambda \in BL(p,q)$ such that $\beta = \lambda\alpha$. Since I is a left ideal of $BL(p,q)$, it follows that $\beta \in I$. Therefore, $L_I \subseteq I$ and so $L_I^{\oplus} = I$. □

Remark 2.1.4. We note that the left ideals of $BL(p,q)$ do not form a chain under $\subseteq$. Suppose $X = \{x_i\}$ with $|I| = p$ and let $\alpha \in BL(p,q)$. For each $i \in I$, write $x_i\alpha = a_i$. Then, $X = \{a_i\} \dot{\cup} \{a_j\}$ for some $\{a_j\} \subseteq X$ with $|J| = d(\alpha) = q$. Now write $\{x_i\} = \{y_i\} \dot{\cup} \{y_k\}$ and $\{a_j\} = \{b_j\} \dot{\cup} \{b_k\}$ with $|K| < q$ (possible since p and q are infinite). Define β in $T(X)$ by

$$\beta = \begin{pmatrix} y_i & y_k \\ a_i & b_k \end{pmatrix}.$$

Then $\beta \in BL(p,q)$, since it is one-to-one and $X \setminus \operatorname{ran}\beta = \{b_j\}$ with $|J| = q$. Clearly, $\alpha \notin L_{\{\beta\}}^{\oplus}$ and $\beta \notin L_{\{\alpha\}}^{\oplus}$ (since $|\operatorname{ran}\beta \setminus \operatorname{ran}\alpha| < q$ and $\operatorname{ran}\beta \not\subseteq \operatorname{ran}\alpha$). Thus $L_{\{\alpha\}}^{\oplus} \not\subseteq L_{\{\beta\}}^{\oplus}$ and $L_{\{\beta\}}^{\oplus} \not\subseteq L_{\{\alpha\}}^{\oplus}$.

All of the following results about the left ideals of $BL(p,q)$ are new. However, since their proofs are entirely similar to those for corresponding results in section 2.4 (and the latter have appeared in [30]), we sometimes omit the details. We start by determining when one left ideal of $BL(p,q)$ is contained in another.

Theorem 2.1.5. Let A, B be non-empty subsets of $BL(p,q)$. Then $L_A^{\oplus} \subseteq L_B^{\oplus}$ if and only if $A \setminus B \subseteq L_B$.

Proof. If $L_A^{\oplus} \subseteq L_B^{\oplus}$, then $A \subseteq B \cup L_B$ and so $A \setminus B \subseteq L_B$. Now suppose that $A \setminus B \subseteq L_B$ and let $\alpha \in L_A^{\oplus}$. Since $A = (A \cap B) \dot{\cup} A \setminus B$, if $\alpha \in A$ then $\alpha \in B \cup L_B = L_B^{\oplus}$. If $\alpha \in L_A$, then there exists $\gamma \in A$ such that $\operatorname{ran}\alpha \subseteq \operatorname{ran}\gamma$ and $|\operatorname{ran}\gamma \setminus \operatorname{ran}\alpha| = q$. If $\gamma \in B$, then $\alpha \in L_B$. If not, then $\gamma \in A \setminus B \subseteq L_B$. Thus, $\operatorname{ran}\gamma \subseteq \operatorname{ran}\delta$ and $|\operatorname{ran}\delta \setminus \operatorname{ran}\gamma| = q$ for some $\delta \in B$. Therefore, $\operatorname{ran}\alpha \subseteq \operatorname{ran}\delta$ and $q = |\operatorname{ran}\delta \setminus \operatorname{ran}\gamma| \leq |\operatorname{ran}\delta \setminus \operatorname{ran}\alpha| \leq |X \setminus \operatorname{ran}\alpha| = q$. Hence $\alpha \in L_B$. In other words, we have shown that $\alpha \in L_B^{\oplus}$, so $L_A^{\oplus} \subseteq L_B^{\oplus}$. □

Hence, $A \subseteq B$ implies $L_A^{\oplus} \subseteq L_B^{\oplus}$. Next we show that the converse does not hold. For, suppose $X = \{x_i\}$ and write $\{x_i\} = \{a_i\} \dot{\cup} \{a_j\}$ and $\{a_i\} = \{b_i\} \dot{\cup} \{b_j\}$ with $|J| = q$. Now define $\alpha, \beta \in T(X)$ by

$$\alpha = \begin{pmatrix} x_i \\ a_i \end{pmatrix}, \qquad \beta = \begin{pmatrix} x_i \\ b_i \end{pmatrix}.$$

Since α, β are one-to-one and $d(\alpha) = |\{a_j\}| = q = |\{a_j, b_j\}| = d(\beta)$, it follows that $\alpha, \beta \in BL(p,q)$. Let $A = \{\beta\}$ and $B = \{\alpha\}$: clearly, $L_A^{\oplus} \subseteq L_B^{\oplus}$ but $A \not\subseteq B$.

Corollary 2.1.6. Let A, B be non-empty subsets of $BL(p,q)$. Then $L_A^{\oplus} \cup L_B^{\oplus} = L_{A \cup B}^{\oplus}$.

Proof. By Theorem 2.1.5, $A, B \subseteq A \cup B$ implies $L_A^{\oplus} \cup L_B^{\oplus} \subseteq L_{A \cup B}^{\oplus}$. Now let $\alpha \in L_{A \cup B}^{\oplus}$. If $\alpha \in A \cup B$, then $\alpha \in A$ or $\alpha \in B$. On the other hand, if $\alpha \in L_{A \cup B}$, then there exists $\beta \in A \cup B$ such that $\operatorname{ran} \alpha \subseteq \operatorname{ran} \beta$ and $|\operatorname{ran} \beta \setminus \operatorname{ran} \alpha| = q$. Thus $\alpha \in L_A \cup L_B$. Hence $\alpha \in L_A^{\oplus} \cup L_B^{\oplus}$ and the result follows. □

By analogy with the linear case in section 2.4, it is not difficult to see that there are non-empty subsets A, B of $BL(p,q)$ such that $A \cap B \neq \emptyset$ and $L_{A \cap B}^{\oplus} \subsetneq L_A^{\oplus} \cap L_B^{\oplus}$, hence a similar result to the one above does not hold for the intersection of two non-empty subsets of $BL(p,q)$. In addition, there are distinct non-empty subsets C, D of $BL(p,q)$ such that $L_C^{\oplus} = L_D^{\oplus}$.

The semigroup $BL(p,q)$ is not a principal left ideal of itself. To see this, we note that by [5] Vol. 2, p. 85, Exercise 3, if S is a right simple semigroup without idempotents then every element x of S belongs to (at least) two distinct principal left ideals. For, given $a \in S \setminus \{x\}$, there exist $b, c \in S$ such that $ab = x$ and $xc = a$ (since S is right simple), hence $x \in S^1 b$; also, $bcb \in S \setminus \{b\}$ (by [5] Vol. 2, Lemma 8.3) and $a(bcb) = (xc)b = ab = x$, so $x \in S^1(bcb)$. Again by [5] Vol. 2, Lemma 8.3, $b \notin S^1(bcb)$ and so $S^1 b \neq S^1(bcb)$. Therefore, if $S = Sx \cup \{x\}$ then x belongs to two distinct left ideals L_1 and L_2 of S, and clearly S is contained in both. Thus $L_1 = L_2$, a contradiction. Next we determine when other left ideals of $BL(p,q)$ are principal. To do this, we first observe that the principal left ideal generated by $\alpha \in BL(p,q)$ is $L_{\{\alpha\}}^{\oplus}$. For, clearly $\alpha \in L_{\{\alpha\}}^{\oplus}$, and this implies $BL(p,q)^1 \alpha \subseteq L_{\{\alpha\}}^{\oplus}$ (since $L_{\{\alpha\}}^{\oplus}$ is a left ideal of $BL(p,q)$). Conversely, if $\beta \in L_{\{\alpha\}}$, then $\beta = \delta\alpha$ for some $\delta \in BL(p,q)$ and so $L_{\{\alpha\}} \subseteq BL(p,q)\alpha$. Hence, $L_{\{\alpha\}}^{\oplus} \subseteq BL(p,q)^1 \alpha$ and the assertion follows.

Corollary 2.1.7. Let A be a non-empty subset of $BL(p,q)$ and $\alpha \in BL(p,q)$. Then $L_A^{\oplus} = L_{\{\alpha\}}^{\oplus}$ if and only if $\alpha \in L_A^{\oplus}$ and $A \setminus \{\alpha\} \subseteq L_{\{\alpha\}}$.

To decide whether a left ideal of $BL(p,q)$ is proper we have the following result (compare Theorem 2.4.6 below).

Theorem 2.1.8. Let A be a non-empty subset of $BL(p,q)$. Then $L_A^{\oplus} = BL(p,q)$ if and only if for each $\alpha \in BL(p,q)$ there exists $\gamma \in A$ such that ran $\alpha \subseteq$ ran γ.

We note that there exists a proper subset A of $BL(p,q)$ such that $L_A^{\oplus} = BL(p,q)$. To see this, fix $\alpha, \beta \in BL(p,q)$. Write $\gamma = \beta\alpha$ and let $A = BL(p,q) \setminus \{\gamma\}$. Clearly, $\alpha \neq \beta\alpha$ (otherwise, the injectivity of α implies $\beta = \mathrm{id}_X$, a contradiction). Therefore, we have $\alpha \in A$. If $\lambda \in BL(p,q)$, then $\lambda \in A$ or $\lambda = \beta\alpha$, and in this case, ran $\lambda \subseteq$ ran α with $\alpha \in A$. By Theorem 2.1.8, it follows that $L_A^{\oplus} = BL(p,q)$.

By [5] Vol. 2, p. 85, Exercise 4, if S is a right simple semigroup without idempotents then Syx is a proper subset of Sx for each $x, y \in S$. For, $Syx \subseteq Sx$ always and, by [5] Vol. 2, Lemma 8.3, $yx \notin Syx$. Now suppose M is a minimal left ideal of S and let $a, b \in M$. Then, by minimality of M, we have $Sba = M = Sa$, a contradiction. Therefore S does not contain minimal left ideals; in particular, we conclude that $BL(p,q)$ has no minimal left ideals. In our next result, we show that $BL(p,q)$ has no maximal left ideals either (compare Theorem 2.4.7 below).

Theorem 2.1.9. The semigroup $BL(p,q)$ has no maximal (proper) left ideals.

In [24], Levi, Schein, Sullivan and Wood together showed that the automorphisms of a Baer-Levi semigroup are inner. To do this, the authors first proved that every automorphism $\phi : BL(p,q) \to BL(p,q)$ gives rise, in a natural way, to an order-preserving automorphism of the family $\mathrm{Ran}(BL(p,q))$ consisting of the ranges of all $\alpha \in BL(p,q)$. Next we extend their result to isomorphisms between Baer-Levi semigroups.

We first observe that Green's $\mathcal{L}$ relation on $BL(p,q)$ equals the identity relation on $BL(p,q)$ and the $\mathcal{R}$ relation equals the universal relation on $BL(p,q)$ (recall that $BL(p,q)$ is right simple). In addition, $BL(p,q)$ is not regular (since it contains no

idempotents). In this situation, it can be useful to study Green's $*$-relations instead. That is, following [7], if S is any semigroup and $a, b \in S$, we say $a \ \mathcal{L}^* \ b$ if and only if

$$\text{for all } x, y \in S^1, \quad ax = ay \quad \text{if and only if} \quad bx = by,$$

and we define $\mathcal{R}^*$ on S dually. Clearly these relations are equivalences on S. In fact, $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$ always, so $\mathcal{R}^*$ is universal on $BL(p,q)$. However the characterisation of $\mathcal{L}^*$ on $BL(p,q)$ is comparable with that of $\mathcal{L}$ on $T(X)$ [5] Vol. 1, Lemma 2.5: namely, from the next result (taken from [30] Lemma 3.1), we deduce that $\alpha \ \mathcal{L}^* \ \beta$ on $BL(p,q)$ if and only if $\operatorname{ran} \alpha = \operatorname{ran} \beta$.

Lemma 2.1.10. If $\alpha, \beta \in BL(p,q)$ then the following are equivalent.

(a) $\operatorname{ran} \beta \subseteq \operatorname{ran} \alpha$,

(b) for each $\lambda, \mu \in BL(p,q)^1$, $\alpha\lambda = \alpha\mu$ implies $\beta\lambda = \beta\mu$,

(c) for each $\lambda \in BL(p,q)$, $\alpha\lambda = \alpha$ implies $\beta\lambda = \beta$.

Proof. Assume $\alpha, \beta \in BL(p,q)$ are such that $\operatorname{ran} \beta \subseteq \operatorname{ran} \alpha$. Then $\beta = \beta_1\alpha$ for some $\beta_1 \in T(X)$. Let $\lambda, \mu \in BL(p,q)^1$. Then, $\alpha\lambda = \alpha\mu$ implies $\beta\lambda = (\beta_1\alpha)\lambda = \beta_1(\alpha\lambda) = \beta_1(\alpha\mu) = (\beta_1\alpha)\mu = \beta\mu$. Hence (a) implies (b). It is obvious that (b) implies (c). To prove (c) implies (a), assume that, for each $\lambda \in BL(p,q)$, $\alpha\lambda = \alpha$ implies $\beta\lambda = \beta$. Write $X = \{x_i\}$ and

$$\alpha = \begin{pmatrix} x_i \\ a_i \end{pmatrix}, \quad \beta = \begin{pmatrix} x_i \\ b_i \end{pmatrix}.$$

If $X = \{a_i\} \,\dot\cup\, \{a_j\} = \{b_i\} \,\dot\cup\, \{b_j\}$ where $|J| = q$, write $\{a_j\} = \{c_j\} \,\dot\cup\, \{d_j\}$ and define $\lambda, \mu \in T(X)$ by

$$\lambda = \begin{pmatrix} a_i & a_j \\ a_i & c_j \end{pmatrix}, \quad \mu = \begin{pmatrix} a_i & a_j \\ a_i & d_j \end{pmatrix}.$$

Clearly, $\lambda, \mu \in BL(p,q)$ and $\alpha\lambda = \alpha = \alpha\mu$. Consequently $\beta\lambda = \beta = \beta\mu$, and this implies $\operatorname{ran} \beta \subseteq \operatorname{ran} \lambda = \{a_i\} \,\dot\cup\, \{c_j\}$ and $\operatorname{ran} \beta \subseteq \operatorname{ran} \mu = \{a_i\} \,\dot\cup\, \{d_j\}$. Hence $\operatorname{ran} \beta \subseteq \{a_i\} = \operatorname{ran} \alpha$, as required. $\square$

As in [30] Theorem 3.2, we now decide when $BL(X,p,q)$ and $BL(Y,m,n)$ are isomorphic for infinite cardinals p, q, m, n with $q \leq p$ and $n \leq m$: although the proof of the next result closely follows the arguments in [24], we provide all the details since

similar ideas will be used later. However, first note that if $\psi : \mathcal{A} \to \mathcal{B}$ is an order-preserving isomorphism between two families of sets then $(A_1 \cap A_2)\psi = A_1\psi \cap A_2\psi$ whenever $A_1, A_2 \in \mathcal{A}$ and $A_1 \cap A_2 \in \mathcal{A}$. This is because order-preserving isomorphisms preserve infima.

Theorem 2.1.11. The semigroups $BL(X, p, q)$ and $BL(Y, m, n)$ are isomorphic if and only if $p = m$ and $q = n$. Moreover, for each isomorphism θ, there is a bijection $h : X \to Y$ such that $\alpha\theta = h^{-1}\alpha h$ for each $\alpha \in BL(X, p, q)$.

Proof. Clearly, if the cardinals are equal as stated, then any bijection from X onto Y will induce an isomorphism between the semigroups. So we assume there is an isomorphism $\theta : BL(X, p, q) \to BL(Y, m, n)$ and aim to find a bijection $h : X \to Y$. We begin by noting that Lemma 2.1.10 says: for $\alpha_1, \alpha_2 \in BL(p, q)$, $\operatorname{ran} \alpha_1 \subseteq \operatorname{ran} \alpha_2$ if and only if for each β such that $\alpha_2\beta = \alpha_2$, we have $\alpha_1\beta = \alpha_1$. Since θ is an isomorphism, it follows that $\operatorname{ran} \alpha_1 = \operatorname{ran} \alpha_2$ if and only if $\operatorname{ran}(\alpha_1\theta) = \operatorname{ran}(\alpha_2\theta)$. Hence, if $\mathcal{B}(X, q)$ is the family of all subsets A of X such that $|A| = p$ and $|X \setminus A| = q$, and $\mathcal{B}(Y, n)$ the family of all subsets B of Y such that $|B| = m$ and $|Y \setminus B| = n$, then $\psi_\theta : \mathcal{B}(X, q) \to \mathcal{B}(Y, n)$, defined by letting $A\psi_\theta = \operatorname{ran}(\alpha\theta)$ where $\alpha \in BL(p, q)$ is such that $\operatorname{ran} \alpha = A$, is a well-defined order-preserving isomorphism of $\mathcal{B}(X, q)$ onto $\mathcal{B}(Y, n)$. In passing, we observe that $\mathcal{B}(X, q) = \operatorname{Ran}(BL(X, p, q))$ and $\mathcal{B}(Y, n) = \operatorname{Ran}(BL(Y, m, n))$.

Next we show that every order-preserving isomorphism ψ of $\mathcal{B}(X, q)$ onto $\mathcal{B}(Y, n)$ is induced by a bijection of X onto Y. Let $A \in \mathcal{B}(X, q)$ and $x \in X \setminus A$. We write $A \cup \{x\}$ as $A \cup x$. Clearly, $A \cup x \in \mathcal{B}(X, q)$ and $A \cup x$ covers A. Hence $(A \cup x)\psi$ covers $A\psi$, that is, $(A \cup x)\psi = A\psi \cup y$ for some $y \in Y \setminus A\psi$. Write $y = xh_A$. We proceed to show that $xh_{A_1} = xh_{A_2}$ for all $A_1, A_2 \in \mathcal{B}(X, q)$ not containing x. Let $A_1, A_2 \in \mathcal{B}(X, q)$ with $x \notin A_1 \cup A_2$. If $A_1 \cap A_2 \in \mathcal{B}(X, q)$, then

$$\begin{aligned}
(A_1\psi \cap A_2\psi) \cup xh_{A_1 \cap A_2} &= (A_1 \cap A_2)\,\psi \cup xh_{A_1 \cap A_2} \\
&= ((A_1 \cap A_2) \cup x)\,\psi \\
&= ((A_1 \cup x) \cap (A_2 \cup x))\,\psi \qquad (2.1.1) \\
&= (A_1 \cup x)\,\psi \cap (A_2 \cup x)\,\psi \\
&= (A_1\psi \cup xh_{A_1}) \cap (A_2\psi \cup xh_{A_2})\,.
\end{aligned}$$

Thus,

$$\{xh_{A_1\cap A_2}\} = (A_1\psi \cap \{xh_{A_2}\}) \cup (\{xh_{A_1}\} \cap A_2\psi) \cup (\{xh_{A_1}\} \cap \{xh_{A_2}\}).$$

Suppose $xh_{A_2} \in A_1\psi$. Then, $xh_{A_2} = xh_{A_1\cap A_2}$ and so $((A_1 \cap A_2) \cup x)\,\psi \subseteq A_1\psi$ by (2.1.1). Since ψ preserves order, $(A_1 \cap A_2) \cup x \subseteq A_1$ and this implies $x \in A_1$, a contradiction. Therefore, $xh_{A_2} \notin A_1\psi$. Similarly, we conclude that $xh_{A_1} \notin A_2\psi$ and hence $\{xh_{A_1\cap A_2}\} = \{xh_{A_1}\} \cap \{xh_{A_2}\}$. Thus $xh_{A_1} = xh_{A_2} = xh_{A_1\cap A_2}$. On the other hand, if $A_1 \cap A_2 \notin \mathcal{B}(X,q)$ then, since $|X \setminus (A_1 \cap A_2)| = q$, we have $|A_1 \cap A_2| \neq p$ and thus p must equal q. In addition, $|A_1| = |A_1 \setminus A_2| = p = |A_2 \setminus A_1| = |A_2|$. We write $A_2 \setminus A_1$ as the disjoint union of two sets M and N, with $|M| = |N| = p$ and let $A_3 = (A_1 \setminus A_2) \cup M$. By construction, both M and A_3 belong to $\mathcal{B}(X,q)$. Moreover, $x \notin A_1 \cup A_3$, $A_1 \cap A_3 \in \mathcal{B}(X,q)$ and $x \notin A_2 \cup A_3$, $A_2 \cap A_3 \in \mathcal{B}(X,q)$. From the first case, we may conclude that $xh_{A_1} = xh_{A_3} = xh_{A_2}$.

We now define $h : X \to Y$ as follows: $xh = xh_A$, where $A \in \mathcal{B}(X,q)$ satisfies $x \notin A$. The foregoing argument shows h is well-defined. Suppose $x_1h = x_2h$ for $x_1, x_2 \in X$ and take $A \in \mathcal{B}(X,q)$ with $x_1, x_2 \in X \setminus A$. Then $(A \cup x_1)\,\psi = A\psi \cup x_1h_A = A\psi \cup x_2h_A = (A \cup x_2)\,\psi$ and hence $A \cup x_1 = A \cup x_2$ since ψ is one-to-one. Therefore $x_1 = x_2$ and thus h is one-to-one. In order to show that h is onto, let $y \in Y$ and $B \in \mathcal{B}(Y,n)$, with $y \in B$. Let $A_1, A_2 \in \mathcal{B}(X,q)$ be such that $A_1\psi = B \setminus y$ and $A_2\psi = B$. Then A_2 covers A_1 and so there exists $x \in X \setminus A_1$ such that $A_2 = A_1 \cup x$. Thus $B = (B \setminus y) \cup xh_{A_1}$ and $y = xh_{A_1}$. Hence h is a bijection and $|X| = |Y|$.

Next we show that ψ is induced by h, that is, $A\psi = Ah$ for each $A \in \mathcal{B}(X,q)$. Let $y \in Ah$. Then there exists $x \in A$ with $y = xh$. Since $A \setminus x \in \mathcal{B}(X,q)$ and A covers $A \setminus x$, we have $A\psi = (A \setminus x)\,\psi \cup xh_{A\setminus x}$ which equals $(A \setminus x)\,\psi \cup y$ by the definition of h. Hence $y \in A\psi$. Conversely, if $y \in A\psi$ then $A\psi$ covers $A\psi \setminus y$. Let $A_1 \in \mathcal{B}(X,q)$ be such that $A\psi \setminus y = A_1\psi$. Then, A covers A_1 since ψ preserves order, and so there exists $x \in X \setminus A_1$ with $A = A_1 \cup x$. Thus $A\psi = (A\psi \setminus y) \cup xh$ (again by definition of h) and hence $y = xh \in Ah$. Therefore $A\psi = Ah$.

Finally, we prove that, for each $\alpha \in BL(p,q)$, $\alpha\theta = h_\theta^{-1}\alpha h_\theta$ where h_θ is the bijection corresponding to the order-preserving isomorphism ψ_θ. Let $\alpha \in BL(p,q)$, $x_1 \in X$ and $x_2 = x_1\alpha$. We may choose A_1, A_2 in $\mathcal{B}(X,q)$ such that $A_1 \subseteq A_2$ and $A_2 \setminus A_1 =$

$\{x_1\}$, together with $\beta, \gamma \in BL(X,q)$ such that $\operatorname{ran}\beta = A_1$ and $\operatorname{ran}\gamma = A_2$. Now $\operatorname{ran}\gamma \setminus \operatorname{ran}\beta = \{x_1\}$ and so

$$\begin{aligned}
\operatorname{ran}((\gamma\alpha)\theta) \setminus \operatorname{ran}((\beta\alpha)\theta) &= \operatorname{ran}((\gamma\theta)(\alpha\theta)) \setminus \operatorname{ran}((\beta\theta)(\alpha\theta)) \\
&= (\operatorname{ran}(\gamma\theta) \setminus \operatorname{ran}(\beta\theta))(\alpha\theta) \\
&= (A_2\psi_\theta \setminus A_1\psi_\theta)(\alpha\theta) \\
&= \{x_1 h_\theta\}\alpha\theta.
\end{aligned}$$

On the other hand, $\operatorname{ran}(\gamma\alpha) \setminus \operatorname{ran}(\beta\alpha) = (A_2 \setminus A_1)\alpha = \{x_2\}$ and so

$$\begin{aligned}
\operatorname{ran}((\gamma\alpha)\theta) \setminus \operatorname{ran}((\beta\alpha)\theta) &= (\operatorname{ran}(\gamma\alpha))\psi_\theta \setminus (\operatorname{ran}(\beta\alpha))\psi_\theta \\
&= \operatorname{ran}(\gamma\alpha)h_\theta \setminus \operatorname{ran}(\beta\alpha)h_\theta \\
&= \{x_2 h_\theta\}.
\end{aligned}$$

Thus $x_1 h_\theta \alpha\theta = x_2 h_\theta = x_1 \alpha h_\theta$ for all $x_1 \in X$ and so $\alpha\theta = h_\theta^{-1}\alpha h_\theta$. Finally, since $\alpha\theta \in BL(Y,n)$ implies that $|Y \setminus Y\alpha\theta| = n$ and, on the other hand, $|Y \setminus Yh^{-1}\alpha h| = |(X \setminus X\alpha)h| = q$ for any bijection $h : X \to Y$, we also have $q = n$. □

In [5] Vol. 2, section 10.8, Clifford and Preston give an excellent account of Malcev's well-known results on the lattice of congruences of $T(X)$: namely, it is generated by congruences of three types. In [47], Šutov determined all congruences of $BL(p,p)$ (as we describe below) and he showed that every congruence ρ of $BL(p,p)$ is *induced* by a Malcev congruence in the sense that ρ is the restriction of a Malcev congruence to $BL(p,p)$.

For every infinite cardinal $q \leq p$, and given $\alpha, \beta \in BL(p,q)$, write

$$D(\alpha,\beta) = \{x \in X : x\alpha \neq x\beta\}$$

and, for every cardinal ξ such that $\aleph_0 \leq \xi \leq p'$ (where p' denotes the successor of p), let

$$\sigma_\xi = \{(\alpha,\beta) \in BL(p,q) \times BL(p,q) : |D(\alpha,\beta)| < \xi\}.$$

Theorem 2.1.12. For every infinite cardinal $\xi \leq p'$, σ_ξ is a congruence on $BL(p,p)$. Conversely, every congruence on $BL(p,p)$ different from the identity relation is of this form.

Let S be a semigroup and suppose ρ is a congruence on S. It is well-known that S/ρ is a semigroup with respect to the operation defined by

$$(a\rho)(b\rho) = (ab)\rho \quad (a, b \in S).$$

We say that ρ is a *regular* [*group*, *band*] *congruence on* S if S/ρ is a regular semigroup [respectively: a group, a band]. In [34], Mielke showed that a Baer-Levi semigroup of type (p, q) has a non-trivial regular congruence if and only if $p > q$ and, in this case, each such congruence is a group congruence. He also proved that a Baer-Levi semigroup has no non-trivial band congruences. Some years later, in the main result of [27] (which we quote below), Lindsey and Madison provided more detailed information about the lattice $\mathrm{Cong}(BL(p,q))$ of all congruences on $BL(p,q)$. Let ι and ω denote the identity and the universal relations on $BL(p,q)$, respectively, and, as usual, let $[\rho, \pi] = \{\sigma \in \mathrm{Cong}(BL(p,q)) : \rho \subseteq \sigma \subseteq \pi\}$ for every $\rho, \pi \in \mathrm{Cong}(BL(p,q))$.

Theorem 2.1.13. Let X be an infinite set with cardinal p and let q be an infinite cardinal such that $q \leq p$. Then,

(a) for every infinite cardinal $\xi \leq p'$, σ_ξ is a congruence on $BL(p,q)$;

(b) the relation $\sigma_{\aleph_0}$ is the unique minimum non-trivial congruence on $BL(p,q)$ and $\sigma_{q'}$ is its minimum group congruence;

(c) $[\iota, \sigma_{q'}]$ is a chain consisting of ι and every σ_ξ with $\aleph_0 \leq \xi \leq q'$, where $\sigma_\xi \subseteq \sigma_\chi$ if and only if $\xi \leq \chi$;

(d) $\mathrm{Cong}(BL(p,q)) = [\iota, \sigma_{q'}] \cup [\sigma_{q'}, \omega]$.

As observed by Hotzel in [14] p. 153, it "appears more difficult" to describe the lattice of all subsemigroups of $BL(p,q)$. In [25], Levi and Wood considered the upper part of this lattice and determined a class of maximal subsemigroups of $BL(p,q)$. To do this, for each non-empty subset A of X with $|X \setminus A| \geq q$, they defined the set

$$M_A = \{\alpha \in BL(p,q) : A \not\subseteq \mathrm{ran}\ \alpha \ \text{ or } \ (A\alpha \subseteq A \ \text{ or } \ |\mathrm{ran}\ \alpha \setminus A| < q)\}$$

and in [25] Theorem 1, they proved the following.

Theorem 2.1.14. For each non-empty subset A of X with $|X \setminus A| \geq q$, M_A is a maximal subsemigroup of $BL(p,q)$.

Hotzel generalised this result in [14] Theorem 2.5, which we quote below. To do this, he first considered the following subsets of $BL(p,q)$: for each non-empty subset A of X and each cardinal $s \geq 1$, let

$$L_s^A = \{\alpha \in BL(p,q) : |A \setminus \operatorname{ran} \alpha| \geq s\};$$
$$D_s^A = \{\alpha \in BL(p,q) : |\operatorname{ran} \alpha \setminus A| < s\};$$
$$T_s^A = \{\alpha \in BL(p,q) : |(A\alpha \setminus A)\alpha^{-1}| < s\}.$$

Theorem 2.1.15. Let A be a non-empty subset of X. Then,

(a) if $|X \setminus A| \geq q$ and s is a cardinal such that $1 \leq s = s^2 \leq \min\{|A|, |X \setminus A|, q\}$, then the set $L_s^A \cup D_q^A \cup T_s^A$ is a maximal subsemigroup of $BL(p,q)$;

(b) if $\aleph_0 \leq r = |X \setminus A| < q$, then the set T_r^A is a maximal subsemigroup of $BL(p,q)$.

As noted by Hotzel in [14] p. 163, if $s = 1$ and $|X \setminus A| \geq q$ then we obtain the result by Levi and Wood. Throughout the rest of [14], the author determines other families of maximal subsemigroups of $BL(p,q)$ (see, for example, [14] Theorems 3.1, 3.26 and 4.7). His examples and their complexity suggest we may never be able to describe all the maximal subsemigroups of $BL(p,q)$. However, all maximal subsemigroups of $BL(p,q)$ determined by Hotzel either contain the set $\{\alpha \in BL(p,q) : s(\alpha) < q'\}$, where $s(\alpha) = |\{x \in X : x\alpha \neq x\}|$, or a non-empty left ideal of $BL(p,q)$: whether all maximal subsemigroups of $BL(p,q)$ satisfy one of these conditions remains an open problem. As observed by Hotzel in [14] p. 163, the number of maximal subsemigroups of $BL(p,q)$ is greater than the number of subsets of X. In fact, in [14] p. 186, Hotzel proves that $BL(p,q)$ has 2^{2^p} maximal subsemigroups.

2.2. Baer-Levi semigroups defined on vector spaces

Let V be a vector space over a field F with dimension $m \geq \aleph_0$. For each cardinal n such that $\aleph_0 \leq n \leq m$, we write

$$GS(m,n) = \{\alpha \in T(V) : n(\alpha) = 0,\ d(\alpha) = n\}$$

and, by analogy with the set case, we call this the *linear Baer-Levi semigroup of type* (m,n) *on the vector space* V. In this section, we show that $GS(m,n)$ is a semigroup

with the same properties as $BL(p,q)$. The following result is analogous to [5] Vol. 2, Theorem 8.2 (and to [26] Proposition 4.1 for the case $p = q$).

Theorem 2.2.1. If $\dim V = m \geq n \geq \aleph_0$ then $GS(m,n)$ is a right simple, right cancellative semigroup without idempotents.

Proof. Assume $\alpha, \beta \in GS(m,n)$ and let ran $\alpha = \langle e_i \rangle$ and $V = \langle e_i, e_j \rangle$, so $|J| = n$. Then $\{e_i\beta\} \cup \{e_j\beta\}$ is independent and generates ran β, and ran $\alpha\beta = \langle e_i\beta \rangle$. Hence $d(\alpha\beta) = n+n = n$, and clearly if α, β are injective then $\alpha\beta$ is also, so $\alpha\beta \in GS(m,n)$. Since elements of $GS(m,n)$ are injective, the semigroup is right cancellative; also, if $\varepsilon \in GS(m,n)$ is idempotent then $(u\varepsilon)\varepsilon = (u)\varepsilon$ for all $u \in V$ implies $\varepsilon = \mathrm{id}_V$, a contradiction. Suppose $\alpha, \beta \in GS(m,n)$ and write $V = \langle e_k \rangle$ and

$$\alpha = \begin{pmatrix} e_k \\ a_k \end{pmatrix}, \quad \beta = \begin{pmatrix} e_k \\ b_k \end{pmatrix}.$$

Now if $V = \langle a_k, a_\ell \rangle = \langle b_k, b_\ell, b_r \rangle$ where $|L| = |R| = n$ and we define

$$\mu = \begin{pmatrix} a_k & a_\ell \\ b_k & b_\ell \end{pmatrix}$$

then $\mu \in GS(m,n)$ and $\beta = \alpha\mu$, and we have shown $GS(m,n)$ is right simple. □

Before proceeding any further, it is now important to decide whether any of the semigroups $GS(m,n)$ are isomorphic to any of the semigroups $BL(p,q)$ for appropriate cardinals m, n and p, q (this was not considered in [26]). This question can be answered in one of two ways: by showing that the cardinals of $BL(p,q)$ and $GS(m,n)$ are different; or by finding some algebraic property of $BL(p,q)$ that is not preserved under an isomorphism between it and $GS(m,n)$. In view of the first approach, we now establish some pertinent results. Something like the following appears in [13] Corollary 1.5.13 and Exercise 1.5.36, but for completeness we include a proof.

Lemma 2.2.2. If $|X| = p \geq q$ and $p \geq \aleph_0$ then the number of subsets of X with cardinal q equals p^q. In fact, this is also the number of injective mappings from a set of cardinal q into a set of cardinal p.

Proof. Let $|A| = q, |B| = p$ and note that for each $Y \subseteq B$ with cardinal q, there is an injective map $A \to B$ with range Y. Hence the number k of $Y \subseteq B$ with cardinal

q is at most the number ℓ of injective maps $A \to B$, and clearly $\ell \leq |B^A| = p^q$. Now each $\alpha : A \to B$ is a subset of $A \times B$ and $|\alpha| = q$. Hence $|B^A|$ is at most the number r of subsets of $A \times B$ with cardinal q. But $q \times p = p$, so $r = k$. Hence $k = p^q$. Thus we have $p^q = k \leq \ell \leq p^q$, and the result follows. □

We can now determine the cardinal of $BL(p,q)$. But first we need the order of $G(X)$ where $|X| = p \geq \aleph_0$. To find this, write $X = A \dot{\cup} B$ where $|A| = |B| = p$ and note that for each $Y \subseteq A$, there exists $\pi \in G(X)$ which fixes Y pointwise and shifts all elements of $(A \setminus Y) \cup B$. Hence $|G(X)| \geq 2^{|A|} = 2^p$ and of course $|G(X)| \leq |T(X)| = 2^p$.

Theorem 2.2.3. If $|X| = p \geq q \geq \aleph_0$ then $|BL(p,q)| = 2^p$.

Proof. Suppose $q < p$. For each $Y \subseteq X$ with cardinal q, we have $|X \setminus Y| = p$ and so there exists a bijection $\alpha : X \to X \setminus Y$, hence $\alpha \in BL(p,q)$. In fact, the set of all such α is in one-to-one correspondence with $G(X \setminus Y)$. Therefore, since in this case $p + q = p$, we have:

$$|BL(p,q)| = \sum \{|G(X \setminus Y)| : Y \subseteq X, |Y| = q\} = 2^p.p^q = p^p.p^q = p^p = 2^p.$$

To find the cardinal k of $BL(p,p)$ when $p > \aleph_0$, write $X = Y \dot{\cup} Z$ where $|Y| = |Z| = p$ and fix $\beta \in BL(Z,p,p)$. Then for $\aleph_0 \leq q < p$ and each $\alpha \in BL(Y,p,q)$, we have $\alpha \cup \beta \in BL(X,p,p)$, so $k \geq |BL(Y,p,q)| = 2^p$ and it follows that $k = 2^p$.

Finally for $p = \aleph_0$ we note that for each $Y \subseteq X$ such that $|Y| = |X \setminus Y| = \aleph_0$, there exists $\alpha \in BL(p,p)$ such that $\operatorname{ran} \alpha = Y$, hence in this case $|BL(p,p)|$ is at least the number k of such subsets Y of X. To calculate k, note that $\{Y \subseteq X : |Y| = \aleph_0\}$ equals

$$\bigcup_n \{Y \subseteq X : |Y| = \aleph_0, |X \setminus Y| = n < \aleph_0\} \cup \{Y \subseteq X : |Y| = |X \setminus Y| = \aleph_0\}$$
$$= \bigcup_n \{X \setminus A : |A| = n < \aleph_0\} \cup \{Y \subseteq X : |Y| = |X \setminus Y| = \aleph_0\}$$

and, taking cardinals, we find by Lemma 2.2.2 that

$$2^{\aleph_0} = \aleph_0^{\aleph_0} = \sum_{n < \aleph_0} \aleph_0^n + k = \aleph_0 + k.$$

Hence k must equal $2^{\aleph_0}$. □

To obtain analogous results for $GS(m,n)$, we first recall [16] Vol. II, p. 245: if V is a vector space over a field F and $\dim V = m \geq \aleph_0$ then $|V| = m \times |F|$. Now let A be a basis for V. Since each $\alpha \in T(V)$ determines a unique map from A into V, and conversely any map from A into V can be extended by linearity to a unique $\alpha \in T(V)$, we have $|T(V)| = |V|^m$. In fact, since $m^m = 2^m$, we can deduce that

$$|T(V)| = \begin{cases} 2^m & \text{if } |F| \leq m, \\ |F|^m & \text{if } |F| > m. \end{cases}$$

Lemma 2.2.4. If V is a vector space with $\dim V = m \geq n$ and $m \geq \aleph_0$, then the number of subspaces of V with dimension n equals $|V|^n$. In fact, this is also the number of injective linear mappings from a vector space of dimension n into another with dimension m over the same field.

Proof. Let k be the number of subspaces of V with dimension n. Now, if a subspace U has dimension n then there is a basis $A \subseteq U$ with $|A| = n$, so k is at most the number $|V|^n$ of subsets of V with cardinal n. Now let U be any vector space with dimension n. Note that each linear $\alpha : U \to V$ can be regarded as a subspace of the vector space $U \times V$. In fact, if $A = \{a_i\}$ is a basis for U then $\{(a_i, a_i\alpha)\}$ is a basis for $\alpha \subseteq U \times V$, hence $\dim \alpha = n$. Therefore the number of linear $U \to V$ is at most the number ℓ of subspaces of $U \times V$ with dimension n. But $\dim(U \times V) = n + m = m$ (since if $\{u_i\}$ is a basis for U and $\{v_j\}$ a basis for V then $\{(u_i, 0)\} \cup \{(0, v_j)\}$ is a basis for $U \times V$). Thus, $U \times V$ and V have the same dimension, hence they are isomorphic, so $\ell = k$. Also, if A is a basis for U then any map $A \to V$ can be uniquely extended to a linear $U \to V$; and any linear $U \to V$ induces a unique map $A \to V$. That is, the number of linear $U \to V$ equals $|V^A| = |V|^n$ and it follows that $k = |V|^n$.

Finally, let U be a vector space with dimension n and V a vector space with dimension m over the same field. To find r, the number of injective linear $U \to V$, we follow the corresponding argument in the proof of Lemma 2.2.2. That is, for each injective linear $U \to V$, there is an injective linear $U \to U \times V$ (for example, $U \to \{0\} \times V$); and conversely, since $n \times m = m$ and thus $U \times V$ is isomorphic to V, for each injective linear $U \to U \times V$, there is an injective linear $U \to V$. Now if $\alpha : U \to V$ is any linear map, let $\alpha' : U \to U \times V, u \to (u, u\alpha)$, and note that α' is linear and injective. Hence the number $|V|^n$ of linear $U \to V$ is at most the number of injective linear

$U \to U \times V$, and we have seen this equals r. It follows that $r = |V|^n$ as required. □

Theorem 2.2.5. If $\dim V = m \geq n \geq \aleph_0$, then $|GS(m,n)| = |V|^m$.

Proof. Suppose $\{v_i\} \dot{\cup} \{v_j\}$ is a basis for V with $|I| = m$ and $|J| = n$, and let $W = \langle v_i \rangle$. For each basis $A = \{a_i\}$ for V and each $\alpha \in G(A)$, there exists an invertible linear $\alpha' : V \to V$ and an injective linear $\beta : V \to V, a_i \to v_i$, and clearly $\alpha'\beta \in GS(m,n)$. In other words,

$$|GS(m,n)| \geq \sum \{|G(A)| : A \text{ is a basis for } V\}.$$

But if $|F| \geq 3$ then, for all $k_i \in F^* = F \setminus \{0\}$, $\{k_i a_i\}$ is a basis for V, hence in this case the number of bases for V is at least $|F^*|^m = |F|^m$. Thus

$$|GS(m,n)| \geq 2^m.|F|^m = (m.|F|)^m = |V|^m,$$

and equality follows.

Suppose now that $|F| = 2$. Let $\{e_i\}$ be a basis for V, so $|I| = m$. For each fixed $j \in I$, $\{e_j + e_i\}$ is a basis for V and so the number of bases for V is at least m. Hence

$$|GS(m,n)| \geq \sum \{|G(A)| : A \text{ is a basis for} V\} \geq m.2^m = (m.2)^m = |V|^m,$$

and then we also have equality in case $|F| = 2$. □

From Theorems 2.2.3 and 2.2.5 we deduce that $BL(p,q)$ is not isomorphic to $GS(m,n)$ when $|F| > 2^p$ and $m \geq p$. For, in this case

$$|GS(m,n)| = |V|^m \geq |V|^p = |F|^p \geq |F| > 2^p = |BL(p,q)|.$$

To see that there are fields of any infinite order, we prove the following result (for which I was unable to find a detailed reference).

Lemma 2.2.6. For each $k \geq \aleph_0$, there is a field F such that $|F| = k$.

Proof. We begin by closely following [15] Exercise III.5.4. Namely, let X be a non-empty set with cardinal $k \geq \aleph_0$, let $\mathbb{N}$ denote the set of non-negative integers, and

suppose Φ is the set of all maps $\varphi : X \to \mathbb{N}$ such that $\varphi(x) \neq 0$ for at most a finite number of $x \in X$. Then Φ is an abelian monoid under the operation '$\cdot$' defined by

$$(\varphi \cdot \psi)(x) = \varphi(x) + \psi(x).$$

We write $\varphi \cdot \psi = \varphi\psi$ when it is convenient to do so. For each $x \in X$ and $i \in \mathbb{N}$, we define $x^i \in \Phi$ by

$$x^i(y) = \begin{cases} i & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

If $\varphi \in \Phi$ and $x_1, \ldots, x_n$ are the only $y \in X$ such that $\varphi(y) \neq 0$, it can be shown that

$$\varphi = x_1^{i_1} \cdot x_2^{i_2} \cdots x_n^{i_n}$$

where $i_j = \varphi(x_j)$ for $j = 1, \ldots, n$. If $\mathbb{Q}$ is the field of rational numbers, we let $\mathbb{Q}[X]$ denote the set of all functions $f : \Phi \to \mathbb{Q}$ such that $f(\varphi) \neq 0$ for at most a finite number of $\varphi \in \Phi$. Then $\mathbb{Q}[X]$ is a commutative ring with identity under the operations:

$$(f + g)(\varphi) = f(\varphi) + g(\varphi),$$
$$(fg)(\varphi) = \sum f(\alpha)g(\beta),$$

where the summation is over all pairs (α, β) such that $\alpha\beta = \varphi$. If $\varphi = x_1^{i_1} \cdot x_2^{i_2} \cdots x_n^{i_n} \in \Phi$ and $r \in \mathbb{Q}$, we let $r\varphi$ denote the function $f : \Phi \to \mathbb{Q}$ defined by

$$f(\psi) = \begin{cases} r & \text{if } \psi = \varphi, \\ 0 & \text{if } \psi \neq \varphi. \end{cases}$$

Then every non-zero $f \in \mathbb{Q}[X]$ can be written as

$$f = \sum_{i=0}^{m} r_i x_1^{s_{i1}} x_2^{s_{i2}} \cdots x_n^{s_{in}} \tag{2.2.1}$$

where $r_i \in \mathbb{Q}$, $x_j \in X$ and $m, s_{ij} \in \mathbb{N}$ are all uniquely determined by f.

Now, as in [15] Theorem III.5.3, $\mathbb{Q}[X]$ is an integral domain, so we can form a field of 'rational functions' (compare [15] p. 233, Example) thus:

$$\mathbb{Q}(X) = \{f/g : f, g \in \mathbb{Q}[X], g \neq 0\}.$$

We assert that $|\mathbb{Q}(X)| = k$. To see this, first note that each polynomial $x1 \in \Phi \subseteq \mathbb{Q}[X]$ equals

$$x1/1 \in \mathbb{Q}(X),$$

hence $|\mathbb{Q}(X)| \geq k$. On the other hand, using the map $f/g \mapsto (f,g)$, we have:

$$|\mathbb{Q}(X)| \leq |\mathbb{Q}[X] \times \mathbb{Q}[X]| = |\mathbb{Q}[X]|.$$

Now, by uniqueness, the number of polynomials in $\mathbb{Q}[X]$ with the form $rx_1^{s_1}x_2^{s_2}\cdots x_n^{s_n}$ is exactly

$$|\mathbb{Q}| \times k^{s_1} \times \cdots \times k^{s_n} = k.$$

Thus, to count all $f \in \mathbb{Q}[X]$ expressed as in (2.2.1) is equivalent to counting the number of subsets with cardinal $m < \aleph_0$ in a set with cardinal k, and by Lemma 2.2.2 this number equals $k^m = k$. It then follows that $|\mathbb{Q}(X)| = k$ as asserted. □

Of course, this discussion leaves open the question of whether $BL(p,q)$ and $GS(m,n)$ are isomorphic when the condition "$|F| > 2^p$ and $m \geq p$" does not hold. For, although it is not obvious, the cardinals of $BL(p,q)$ and $GS(m,n)$ can be equal. We consider this possibility in the next section.

2.3. Isomorphisms between Baer-Levi semigroups

In this section we aim to use algebraic conditions on $BL(p,q)$ to decide whether it is ever isomorphic to $GS(m,n)$. To do this, we first recall that in the proof of Theorem 2.1.11 we showed that if $\phi : BL(X,p,q) \to BL(Y,m,n)$ is an isomorphism then it gives rise, in a natural way, to an order-preserving isomorphism between the family $\mathrm{Ran}(BL(X,p,q))$ consisting of the ranges of all $\alpha \in BL(X,p,q)$ and the family $\mathrm{Ran}(BL(Y,m,n))$ consisting of the ranges of all $\beta \in BL(Y,m,n)$. We now use a similar argument to show that $BL(X,p,q)$ is never isomorphic to $GS(V,m,m)$. For this, we need a result for $GS(m,n)$ which is analogous to Lemma 2.1.10 (its proof uses the well-known characterisation of Green's $\mathcal{L}$–relation on $T(V)$: see [5] Vol. 1, p. 57, Exercise 6).

Lemma 2.3.1. If $\alpha, \beta \in GS(m,n)$ then the following are equivalent.

(a) $\mathrm{ran}\ \beta \subseteq \mathrm{ran}\ \alpha$,

(b) for each $\lambda, \mu \in GS(m,n)^1$, $\alpha\lambda = \alpha\mu$ implies $\beta\lambda = \beta\mu$,

(c) for each $\lambda \in GS(m,n)$, $\alpha\lambda = \alpha$ implies $\beta\lambda = \beta$.

Proof. Let $\alpha, \beta \in GS(m,n)$ be such that ran $\beta \subseteq$ ran α. Since $\alpha, \beta \in T(V)$, there is some $\beta_1 \in T(V)$ such that $\beta = \beta_1\alpha$. Let $\lambda, \mu \in GS(m,n)$[1]. Then, $\alpha\lambda = \alpha\mu$ implies $\beta\lambda = (\beta_1\alpha)\lambda = \beta_1(\alpha\lambda) = \beta_1(\alpha\mu) = (\beta_1\alpha)\mu = \beta\mu$. Therefore (a) implies (b). Clearly (b) implies (c). Now assume (c) holds and suppose $\{e_i\}$ is a basis for V. It follows that ran $\alpha = \langle e_i\alpha\rangle$ where $\{e_i\alpha\}$ is linearly independent since α is one-to-one, and $V = \langle e_i\alpha, e_j\rangle$ for some linearly independent subset $\{e_j\}$ of V, with $|J| = d(\alpha) = n$. Write $\{e_j\} = \{u_j\} \dot\cup \{v_j\}$ and define $\lambda, \mu \in T(V)$ as follows:

$$\lambda = \begin{pmatrix} e_i\alpha & e_j \\ e_i\alpha & u_j \end{pmatrix}, \quad \mu = \begin{pmatrix} e_i\alpha & e_j \\ e_i\alpha & v_j \end{pmatrix}.$$

Then $\lambda, \mu \in GS(m,n)$ and $\alpha\lambda = \alpha = \alpha\mu$. Hence $\beta\lambda = \beta = \beta\mu$, so ran $\beta \subseteq$ ran $\lambda = \langle e_i\alpha, u_j\rangle$ and ran $\beta \subseteq$ ran $\mu = \langle e_i\alpha, v_j\rangle$. Now, if $w \in$ ran β then $w = \sum x_i(e_i\alpha) + \sum y_j u_j$ and $w = \sum w_i(e_i\alpha) + \sum z_j v_j$ for some scalars x_i, y_j and w_i, z_j; hence, by linear independence, $y_j = z_j = 0$ for each j. Thus, ran $\beta \subseteq \langle e_i\alpha\rangle =$ ran α, as required for (a). □

Next we need [37] Lemma 6 which we quote below for convenience: as observed by Lima [26] p. 433, this result enhances an essential difference between sets and vector spaces. For, if $X = A \dot\cup B$ where $|A| = |B| = p$ and $A \cap B = \emptyset$, then there is no $C \subseteq X$ such that $|C| = p$ and $C \cap A = \emptyset = C \cap B$.

Lemma 2.3.2. If $\dim V = m \geq \aleph_0$ and U_1, U_2 are subspaces of V with codimension m in V then there is a subspace W of V such that $\dim W = m$ and $W \cap U_1 = \{0\} = W \cap U_2$.

Theorem 2.3.3. The semigroups $BL(X,p,q)$ and $GS(V,m,m)$ are not isomorphic for any (infinite) cardinals p, q and m, with $q \leq p$.

Proof. Suppose ϕ is an isomorphism from $BL(X,p,q)$ onto $GS(V,m,m)$. Then, from Lemmas 2.1.10 and 2.3.1 we have

$$\text{ran } \alpha \subseteq \text{ran } \beta \quad \text{if and only if} \quad \text{ran } (\alpha\phi) \subseteq \text{ran } (\beta\phi). \tag{2.3.1}$$

Let $\mathcal{B}(X,p,q)$ denote the family of all $A \subseteq X$ such that $|A| = p$ and $|X \setminus A| = q$ and let $\mathcal{G}(V,m,m)$ denote the family of all subspaces U of V such that $\dim U = m$

and codim $U = m$. We observe that ϕ gives rise in a natural way to a mapping φ from $\mathcal{B}(X,p,q)$ into $\mathcal{G}(V,m,m)$: for each $A \in \mathcal{B}(X,p,q)$, let $A\varphi = \operatorname{ran}(\alpha\phi)$ for some $\alpha \in BL(X,p,q)$ such that $\operatorname{ran}\alpha = A$. From (2.3.1), we readily deduce that φ is a well-defined order-preserving isomorphism of $\mathcal{B}(X,p,q)$ onto $\mathcal{G}(V,m,m)$.

Let $A_1, A_2 \in \mathcal{B}(X,p,q)$ and write $X = A_1 \dot\cup B_1 = A_2 \dot\cup B_2$ where $|A_i| = p$ and $|B_i| = q$ for $i = 1,2$. Then $A_1\varphi, A_2\varphi$ are elements of $\mathcal{G}(V,m,m)$, and hence $\operatorname{codim}(A_1\varphi) = \dim V = \operatorname{codim}(A_2\varphi)$. By Lemma 2.3.2, there is a subspace W of V such that $\dim W = m$ and $W \cap A_1\varphi = \{0\} = W \cap A_2\varphi$. Let $\{w_i\}$ be a basis for W and $\{a_i\}$ a basis for $A_1\varphi$. Since $W \cap A_1\varphi = \{0\}$, it follows that $\{w_i\} \dot\cup \{a_i\}$ is linearly independent. Hence, it can be expanded to a basis $\{w_i\} \dot\cup \{a_i\} \dot\cup \{v_k\}$ for V, and so $\operatorname{codim} W = |I| + |K| = m$. Thus, $W \in \mathcal{G}(V,m,m)$ and, since φ is onto, there is a subset C of X in $\mathcal{B}(X,p,q)$ such that $W = C\varphi$. We have $C = C \cap X = (C \cap A_1) \dot\cup (C \cap B_1)$. Since $|C| = p$ and $|C \cap B_1| \leq q$, it follows that $|C \cap A_1| = p$ when $q < p$. Moreover, $X = (C \cap A_1) \dot\cup (C \cap B_1) \dot\cup (X \setminus C)$ and so $|X \setminus (C \cap A_1)| = q$. Therefore, $C \cap A_1 \in \mathcal{B}(X,p,q)$ if $q < p$. Since $C \cap A_1 \subseteq C$ and $C \cap A_1 \subseteq A_1$ and φ preserves order, we have $(C \cap A_1)\varphi \subseteq W \cap A_1\varphi = \{0\}$, which contradicts the fact that $(C \cap A_1)\varphi$ belongs to $\mathcal{G}(V,m,m)$. On the other hand, if $q = p$ then either $|C \cap A_1| = p$ or $|C \cap B_1| = p$. Without loss of generality, suppose $|C \cap A_1| = p$ and write $C \cap A_1 = Y \dot\cup Z$ where $|Y| = p = |Z|$. Then $C = Y \dot\cup Z \dot\cup (C \cap B_1)$ and so $|X \setminus Y| \geq |Z \dot\cup (C \cap B_1)| = p$. Therefore, $Y \in \mathcal{B}(X,p,p)$. Since $Y \subseteq C$, $Y \subseteq A_1$ and φ preserves order, we have $Y\varphi \subseteq W \cap A_1\varphi = \{0\}$, which contradicts the fact that $Y\varphi \in \mathcal{G}(V,m,m)$. □

To obtain useful algebraic conditions on $BL(p,q)$ when $q < p$, we first observe that it contains a copy of $BL(q,q)$: namely, if $Y \subseteq X$ has cardinal q, we let

$$B(Y) = \{\alpha \in BL(p,q) : Y\alpha \subseteq Y,\ \alpha \mid (X \setminus Y) = \mathrm{id}_{X\setminus Y}\}$$

which is clearly non-empty and isomorphic to $BL(Y,q,q)$. In chapter 1, section 1.2, we defined the shift of $\alpha \in T(X)$ to be

$$S(\alpha) = \{x \in X : x\alpha \neq x\}, \quad s(\alpha) = |S(\alpha)|.$$

Now, for each $\alpha \in BL(p,q)$ write

$$F(\alpha) = X \setminus S(\alpha) = \{x \in X : x\alpha = x\}.$$

Note that, for every $\alpha, \beta \in BL(p,q)$, $S(\alpha\beta) \subseteq S(\alpha) \cup S(\beta)$, so $s(\alpha\beta) \leq s(\alpha) + s(\beta)$ always. Clearly, $\lambda\alpha = \lambda$ in $BL(p,q)$ if and only if $\operatorname{ran} \lambda \subseteq F(\alpha)$. Also, if $\alpha \in BL(p,q)$, then $s(\alpha) = q$ if and only if $\lambda\alpha = \lambda$ for some $\lambda \in BL(p,q)$. For, we know $X \setminus \operatorname{ran} \alpha \subseteq S(\alpha)$, so $s(\alpha) \geq q$ always. If $\lambda\alpha = \lambda$ for some $\lambda \in BL(p,q)$ then $\operatorname{ran} \lambda \subseteq F(\alpha)$, so $S(\alpha) \subseteq X \setminus \operatorname{ran} \lambda$, and hence $s(\alpha) \leq q$. Conversely, if $s(\alpha) = q < p$ then $|X| = |F(\alpha)|$, and any bijection $\lambda : X \to F(\alpha)$ satisfies $\lambda\alpha = \lambda$ and belongs to $BL(p,q)$. Thus, we have an algebraic characterisation for the elements of the semigroup

$$\Lambda(q) = \{\alpha \in BL(p,q) : s(\alpha) = q\}. \tag{2.3.2}$$

Next we define an equivalence $\sim$ on $\Lambda(q)$ by

$$\alpha \sim \beta \quad \text{if and only if} \quad S(\alpha) = S(\beta).$$

Surprisingly, this has an algebraic characterisation which is similar to Lemma 2.1.10(c). Here it is also worth recalling [5] Vol. 2, Lemma 8.3: namely, the equation $xy = y$ cannot occur in any right simple, right cancellative semigroup without idempotents.

Lemma 2.3.4. If $\alpha, \beta \in \Lambda(q)$ then the following are equivalent.

(a) $S(\beta) \subseteq S(\alpha)$,

(b) for each $\lambda \in BL(p,q)$, $\lambda\alpha = \lambda$ implies $\lambda\beta = \lambda$.

Proof. Suppose $S(\beta) \subseteq S(\alpha)$. Let $\lambda \in BL(p,q)$ be such that $\lambda\alpha = \lambda$. Then $\operatorname{ran} \lambda \subseteq F(\alpha)$ and, since $S(\beta) \subseteq S(\alpha)$, it follows that $\operatorname{ran} \lambda \subseteq F(\beta)$. Therefore, since $x\lambda \in \operatorname{ran} \lambda$ for each x in X, we have $x(\lambda\beta) = (x\lambda)\beta = x\lambda$, and hence $\lambda\beta = \lambda$. Conversely, assume that (b) holds. If $F(\alpha) = \{e_i\}$ and $S(\alpha) = \{x_j\}$, where $|J| = q < p = |I|$ (since $\alpha \in \Lambda(q)$), then write $\{e_i\} = \{f_i\} \dot\cup \{f_j\}$ and

$$\alpha = \begin{pmatrix} f_i & f_j & x_j \\ f_i & f_j & x_j\alpha \end{pmatrix}.$$

Define $\lambda \in T(X)$ by

$$\lambda = \begin{pmatrix} e_i & x_j \\ f_i & f_j \end{pmatrix}.$$

Then $\lambda\alpha = \lambda$ and $\lambda \in BL(p,q)$ since $d(\lambda) = q = s(\alpha)$. Hence $\lambda\beta = \lambda$ and $F(\alpha) = \operatorname{ran} \lambda \subseteq F(\beta)$. Thus $S(\beta) \subseteq S(\alpha)$. □

If we fix some $\beta \in \Lambda(q)$ and put $S(\beta) = Y$ then $F(\beta) = X \setminus Y$ and we have

$$B(Y) = \{\alpha \in \Lambda(q) : S(\alpha) \subseteq Y\},$$

and this is the set of all $\alpha \in BL(p,q)$ such that $\mu\alpha = \mu$ for some $\mu \in BL(p,q)$ and, for each $\lambda \in BL(p,q)$, $\lambda\beta = \lambda$ implies $\lambda\alpha = \lambda$. In other words, we have an algebraic description of each $BL(q,q)$ inside $BL(p,q)$ when $q < p$.

The aim now is to use this description to show that $BL(p,q)$ cannot be isomorphic to any $GS(m,n)$ when $p > q$. However, for this we need to identify a subset of $GS(m,n)$ which will correspond to some $B(Y)$ in $BL(p,q)$ under an isomorphism.

We start by recalling that in chapter 1, section 1.3 we defined

$$\mathrm{Fix}\,(\alpha) = \{u \in V : u\alpha = u\},$$

for each $\alpha \in T(V)$, and we let $s(\alpha) = \mathrm{codim}\,\mathrm{Fix}\,(\alpha)$. It can be shown that $s(\alpha\beta) \leq s(\alpha) + s(\beta)$: see [37] Lemma 5. Hence, by analogy with $\Lambda(q)$ in $BL(p,q)$, if $m > n$ then there exists a subsemigroup of $GS(m,n)$ defined by

$$\Sigma(n) = \{\alpha \in GS(m,n) : s(\alpha) = n\}.$$

Furthermore, we can characterise $\Sigma(n)$ algebraically as follows: given $\alpha \in GS(m,n)$,

$$s(\alpha) = n \quad \text{if and only if} \quad \lambda\alpha = \lambda \text{ for some } \lambda \in GS(m,n). \tag{2.3.3}$$

For, $\mathrm{Fix}\,(\alpha) \subseteq \mathrm{ran}\,\alpha$ implies $n = d(\alpha) \leq s(\alpha)$. If $\lambda\alpha = \lambda$ for some $\lambda \in GS(m,n)$ then $\mathrm{ran}\,\lambda \subseteq \mathrm{Fix}\,(\alpha)$ and this implies $s(\alpha) \leq d(\lambda) = n$; conversely, if $s(\alpha) = n < m$ then $\dim V = \dim \mathrm{Fix}\,(\alpha)$ and any linear bijection $\lambda : V \to \mathrm{Fix}\,(\alpha)$ satisfies $\lambda\alpha = \lambda$ and belongs to $GS(m,n)$.

Next we define an equivalence $\approx$ on $\Sigma(n)$ by

$$\alpha \approx \beta \quad \text{if and only if} \quad \mathrm{Fix}\,(\alpha) = \mathrm{Fix}\,(\beta).$$

Its algebraic characterization is analogous to that of the equivalence $\sim$ defined on the subsemigroup $\Lambda(q)$ of $BL(p,q)$.

Lemma 2.3.5. If $\alpha, \beta \in \Sigma(n)$ then the following conditions are equivalent.

(a) $\mathrm{Fix}\,(\alpha) \subseteq \mathrm{Fix}\,(\beta)$,

(b) for each $\lambda \in GS(m,n)$, $\lambda\alpha = \lambda$ implies $\lambda\beta = \lambda$.

Proof. Assume $\mathrm{Fix}\,(\alpha) \subseteq \mathrm{Fix}\,(\beta)$ and let $\lambda \in GS(m,n)$ be such that $\lambda\alpha = \lambda$. Then $\mathrm{ran}\,\lambda \subseteq \mathrm{Fix}\,(\alpha)$ and so $\mathrm{ran}\,\lambda \subseteq \mathrm{Fix}\,(\beta)$. Therefore, $\lambda\beta = \lambda$. Conversely, suppose $\{e_i\} = \{f_i\}\dot{\cup}\{f_j\}$ is a basis for $\mathrm{Fix}\,(\alpha)$, where $|I| = m > n = |J|$ since $\alpha \in \Sigma(n)$. Expand $\{e_i\}$ to a basis $\{e_i, v_j\}$ for V and note that

$$\alpha = \begin{pmatrix} f_i & f_j & v_j \\ f_i & f_j & v_j\alpha \end{pmatrix}.$$

Define $\lambda \in T(V)$ by

$$\lambda = \begin{pmatrix} e_i & v_j \\ f_i & f_j \end{pmatrix}.$$

Then $\lambda\alpha = \lambda$ and $\lambda \in GS(m,n)$, since $d(\lambda) = n = s(\alpha)$. Hence $\lambda\beta = \lambda$ and so $\mathrm{Fix}\,(\alpha) = \mathrm{ran}\,\lambda \subseteq \mathrm{Fix}\,(\beta)$. □

One candidate for a linear version of $B(Y)$, the copy of $BL(Y,q,q)$ in $BL(p,q)$, can be defined as follows. If U is a subspace of V with dimension m and codimension n and if W is a complement of U in V, then we let

$$G(U,W) = \{\alpha \in GS(m,n) : W\alpha \subseteq W,\ U \subseteq \mathrm{Fix}\,(\alpha)\}$$

which is clearly non-empty and isomorphic to $GS(W,n,n)$. Unfortunately, as observed in chapter 1, section 1.3, whereas the complement of a subset Y in X is unique, this is not true for a complement of a subspace U in V. Therefore, we now fix some $\beta \in \Sigma(n)$ and put $\mathrm{Fix}\,(\beta) = U$ and $V = U \oplus W$, so we have

$$G(U,W) \subsetneq G(U) = \{\alpha \in \Sigma(n) : U \subseteq \mathrm{Fix}\,(\alpha)\}.$$

Note that $G(U)$ is the set of all $\alpha \in GS(m,n)$ such that $\mu\alpha = \mu$ for some μ in $GS(m,n)$ and, for each $\lambda \in GS(m,n)$, $\lambda\beta = \lambda$ implies $\lambda\alpha = \lambda$: that is, $G(U)$ has the same characteristics as $B(Y)$ in $BL(p,q)$. Note also that the above containment is 'proper'. For, if $\{u_i\}$ is a basis for U and $\{v_j\}$ a basis for W then $V = \langle u_i, v_j\rangle$. Write

$\{u_i\} = \{a_i\} \dot{\cup} \{a_j\}$ (possible since $|J| = n \leq m = |I|$ by the choice of U and W) and also write $\{a_j + v_j\} = \{b_j\} \dot{\cup} \{c_j\}$. Then $\{u_i\} \dot{\cup} \{a_j + v_j\}$ is a basis for V and

$$\alpha = \begin{pmatrix} u_i & v_j \\ u_i & b_j \end{pmatrix}$$

is an element of $G(U)$ (note that $v_j\alpha \neq v_j$ for each j) and it does not belong to $G(U, W)$ since $W\alpha \cap W = \{0\}$.

To proceed further, we require two technical results whose purpose will become apparent in the proof of Theorem 2.3.8. The first one is due to Sullivan (personal communication).

Lemma 2.3.6. For each vector space W with dimension $n \geq \aleph_0$, there exists $\alpha \in GS(W, n, n)$ which fixes exactly one element of W, namely 0.

Proof. Consider a basis for W of the form:

$$\{w_{1k}\} \cup \{w_{2k}\} \cup \ldots .$$

That is, $W = \langle w_{ik} \rangle$ where $|I| = \aleph_0$ and $|K| = n$. Define $\alpha \in T(W)$ by

$$\alpha = \begin{pmatrix} w_{1k} & \ldots & w_{ik} & \ldots \\ w_{2k} & \ldots & w_{i+1,k} & \ldots \end{pmatrix}.$$

Then $d(\alpha) = n$, so $\alpha \in GS(W, n, n)$. Now each $v \in W$ can be written as

$$v = \sum_k x_{i_1,k} w_{i_1,k} + \ldots + \sum_k x_{i_r,k} w_{i_r,k} \tag{2.3.4}$$

where the $x_{i_j,k}$ are scalars, each sum is over a finite (and possibly different) index set and we can assume $i_1 < i_2 < \ldots < i_r$. Therefore, if $v\alpha = v$, we have:

$$\begin{aligned} &\sum_k x_{i_1,k} w_{i_1,k} + \sum_k x_{i_2,k} w_{i_2,k} + \ldots + \sum_k x_{i_r,k} w_{i_r,k} \\ &\quad = \sum_k x_{i_1,k} w_{i_1+1,k} + \sum_k x_{i_2,k} w_{i_2+1,k} + \ldots + \sum_k x_{i_r,k} w_{i_r+1,k}. \end{aligned} \tag{2.3.5}$$

Since all the $w_{i_j,k}$ are linearly independent, and $w_{i_1,k}$ does not appear on the right of this equation, we deduce that $x_{i_1,k} = 0$ for all k. Then (2.3.5) reduces to

$$\sum_k x_{i_2,k} w_{i_2,k} + \ldots + \sum_k x_{i_r,k} w_{i_r,k} = \sum_k x_{i_2,k} w_{i_2+1,k} + \ldots + \sum_k x_{i_r,k} w_{i_r+1,k}. \tag{2.3.6}$$

Again, $w_{i_2,k}$ appears nowhere on the right of this new equation, so $x_{i_2,k} = 0$ for all k. Similarly, all coefficients in (2.3.4) equal 0, hence $v = 0$ as required. □

Lemma 2.3.7. Let V be a vector space of dimension m and U a subspace of V with dimension m and codimension n. If W_1, W_2 are subspaces of V with codimension n which contain U and satisfy $\dim(W_1/U) = n = \dim(W_2/U)$, then there exists a subspace L of V with codimension n in V which properly contains U such that $L \cap W_1 = U = L \cap W_2$.

Proof. Let W_1, W_2 be subspaces of V such that $U \subseteq W_1$, $U \subseteq W_2$, codim $W_1 = n =$ codim W_2 and $\dim(W_1/U) = n = \dim(W_2/U)$. Recall that $\dim(V/U)$ equals the codimension of U in V and that there is a natural (linear) isomorphism between V/W_i and $(V/U)/(W_i/U)$ for $i = 1, 2$. Hence, W_i/U has codimension n in V/U. By Lemma 2.3.2, there exists a subspace L/U of V/U such that $\dim(L/U) = n$ and $L/U \cap W_1/U = \{U\} = L/U \cap W_2/U$. Since $\dim(L/U) = n$, U is properly contained in L. Moreover, since $L/U \cap W_1/U = \{U\}$,

$$n = \dim(W_1/U) \leq \operatorname{codim}(L/U) \leq \dim(V/U) = n,$$

and so codim $L = n$. From $L/U \cap W_1/U = \{U\} = L/U \cap W_2/U$, we may conclude that $L \cap W_1 = U = L \cap W_2$. □

Theorem 2.3.8. The semigroups $BL(X, p, q)$ and $GS(V, m, n)$ are not isomorphic for any infinite cardinals p, q, m, n with $q < p$ and $n < m$.

Proof. Suppose ϕ is an isomorphism from $BL(X, p, q)$ onto $GS(V, m, n)$. Let $Y \subseteq X$ be such that $|Y| = q$ and let $\beta \in BL(p, q)$ be such that $S(\beta) = Y$. Then, $\beta\phi \in GS(m, n)$. Moreover, $s(\beta\phi) = n$, since $s(\beta) = q$ and so there exist $\mu \in BL(p, q)$ and $\mu\phi \in GS(m, n)$ such that $\mu\beta = \mu$ and $(\mu\phi)(\beta\phi) = \mu\phi$. Hence, $\dim \operatorname{Fix}(\beta\phi) = m$. Let $U = \operatorname{Fix}(\beta\phi)$ and $V = U \oplus W$. Let $\mathcal{B}$ be the family of all subsets of Y with cardinal q and let $\mathcal{G}$ be the family of all subspaces of V with codimension n which contain U. Consider $\varphi : \mathcal{B} \to \mathcal{G}$ defined as follows: given $B \in \mathcal{B}$, let $B\varphi = \operatorname{Fix}(\alpha\phi)$, where $\alpha \in B(Y)$ is such that $S(\alpha) = B$. We assert that φ is an order-reversing isomorphism from $\mathcal{B}$ onto $\mathcal{G}$.

Let $B = \{b_j\} \dot\cup \{c_j\} \dot\cup \{d_j\} \in \mathcal{B}$, with $|J| = q$ and write $\{d_j\} = \{e_j\} \dot\cup \{f_j\}$. Write $X = \{x_i\} \dot\cup B$ and define $\alpha \in T(X)$ by

$$\alpha = \begin{pmatrix} x_i & b_j & c_j & d_j \\ x_i & e_j & b_j & c_j \end{pmatrix}.$$

Then $c(\alpha) = 0$, $d(\alpha) = q$ and $S(\alpha) = B$. Hence $\alpha \in \Lambda(q)$ and, by the characterisations discussed at (2.3.2) and (2.3.3), we have $\alpha\phi \in \Sigma(n)$. Also, since $S(\alpha) \subseteq Y$, Lemmas 2.3.4 and 2.3.5 imply $U \subseteq \mathrm{Fix}\,(\alpha\phi)$. Therefore, $\mathrm{Fix}\,(\alpha\phi) \in \mathcal{G}$. If $B_1, B_2 \in \mathcal{B}$ and $\alpha_1, \alpha_2 \in B(Y)$ are such that $S(\alpha_1) = B_1$ and $S(\alpha_2) = B_2$, then

$$\begin{aligned} B_1 \subseteq B_2 \quad &\Leftrightarrow \quad S(\alpha_1) \subseteq S(\alpha_2) \\ &\Leftrightarrow \quad \lambda\alpha_2 = \lambda \quad \text{implies} \quad \lambda\alpha_1 = \lambda \quad \text{for all } \lambda \text{ in } BL(p,q) \\ &\Leftrightarrow \quad \mu(\alpha_2\phi) = \mu \quad \text{implies} \quad \mu(\alpha_1\phi) = \mu \quad \text{for all } \mu \text{ in } GS(m,n) \\ &\Leftrightarrow \quad \mathrm{Fix}\,(\alpha_2\phi) \subseteq \mathrm{Fix}\,(\alpha_1\phi) \\ &\Leftrightarrow \quad B_2\varphi \subseteq B_1\varphi. \end{aligned}$$

Thus, φ is a well-defined one-to-one mapping which inverts order. To show that φ is onto, we will use Lemma 2.3.6. Let $G = \langle e_i \rangle \in \mathcal{G}$. Then $\mathrm{codim}\, G = n$ and $U \subseteq G$. Write $V = G \oplus H$, with $H = \langle f_j \rangle$, and define $\varepsilon \in T(V)$ by

$$\varepsilon = \begin{pmatrix} e_i & f_j \\ e_i & f_j\alpha \end{pmatrix},$$

where $\alpha \in GS(H, n, n)$ fixes exactly one element of H, namely 0. Now, $\varepsilon \in GS(V, m, n)$ and $\mathrm{Fix}\,(\varepsilon) = G$. For, if $v = \sum a_i e_i + \sum b_j f_j$, then $v\varepsilon = v$ if and only if α fixes the element $\sum b_j f_j \in H$. But the latter happens if and only if $\sum b_j f_j = 0$ in which case $b_j = 0$ for each j; that is, $v \in G$. Since ε is actually in $\Sigma(n)$, there exists $\delta \in \Lambda(q)$ such that $\varepsilon = \delta\phi$. Let $B = S(\delta)$. Since $\mathrm{Fix}\,(\beta\phi) = U \subseteq G = \mathrm{Fix}\,(\delta\phi)$, we conclude as before that $S(\delta) \subseteq S(\beta) = Y$. That is, $B \in \mathcal{B}$ and $B\varphi = G$.

We now show that, for subspaces $W_1 = B_1\varphi$, $W_2 = B_2\varphi$ of V in $\mathcal{G}$ with $W_1 \cap W_2 = U$, we have $B_1 \cup B_2 = Y$. Since φ inverts order, $(B_1 \cup B_2)\varphi$ is a subset of $B_1\varphi \cap B_2\varphi = W_1 \cap W_2 = U = Y\varphi$ (the last equation holds since Y is the greatest element of $\mathcal{B}$ and U is the least element of $\mathcal{G}$). Hence, $Y \subseteq B_1 \cup B_2$ and so $B_1 \cup B_2 = Y$.

Next, we use the above results to produce a contradiction. Let $B_1, B_2 \in \mathcal{B}$ be such that $B_1 \dot\cup B_2 = Y$. Then, $B_1\varphi = W_1 = \langle u_i, v_k \rangle$ and $B_2\varphi = W_2 = \langle u_i, w_\ell \rangle$, where

$U = \langle u_i \rangle$. Since codim $W_1 = n =$ codim W_2, we can choose bases $\{a_j\} \dot\cup \{b_j\}$ and $\{c_j\} \dot\cup \{d_j\}$ for complements of W_1 and W_2, respectively, where $|J| = n$. Then

$$V = \langle u_i, v_k, a_j, b_j \rangle = \langle u_i, w_\ell, c_j, d_j \rangle.$$

Let $W_1' = \langle u_i, v_k, a_j \rangle$ and $W_2' = \langle u_i, w_\ell, c_j \rangle$. Then $W_1', W_2' \in \mathcal{G}$ and $\dim(W_1'/U) = n = \dim(W_2'/U)$. By Lemma 2.3.7, there exists an element $L \neq U$ in $\mathcal{G}$ such that $L \cap W_1' = U = L \cap W_2'$. Since $W_1 \subseteq W_1'$ and $W_2 \subseteq W_2'$, we have $L \cap W_1 = U = L \cap W_2$. Also, since φ is onto, there exists $B \in \mathcal{B}$ such that $B\varphi = L$. Therefore, $B\varphi \cap B_1\varphi = U = B\varphi \cap B_2\varphi$, which implies that $B \cup B_1 = Y = B \cup B_2$. Thus, $B_1, B_2 \subseteq B$ and $Y = B$. Hence $U = L$, a contradiction. □

Next we show that $BL(p,p)$ and $GS(m,n)$, with $n < m$, are not isomorphic. We recall that $BL(X,p,p)$ is embeddable in $BL(Y,r,p)$, with $X \subsetneq Y$ and $p < r$ (by a remark after Theorem 2.3.3), and consider the semigroup

$$S = \{\alpha \in BL(Y,r,p) : S(\alpha) \subseteq X\}.$$

For each $\alpha \in S$, $s(\alpha) = p$ since $D(\alpha) = Y \setminus \text{ran } \alpha \subseteq S(\alpha) \subseteq X$. Let

$$T = \{\alpha \in S : |X \cap F(\alpha)| = p\}$$

which is easily seen to be non-empty. If $\alpha \in T$, write $X = \{x_j\} = \{s_j\} \dot\cup \{t_j\}$, where $S(\alpha) = \{s_j\}$ and $X \cap F(\alpha) = \{t_j\}$. Write $Y = \{y_i\} \dot\cup \{x_j\}$ and $\{t_j\} = \{u_j\} \dot\cup \{v_j\}$, with $\{v_j\} = \{a_j\} \dot\cup \{b_j\}$. Define λ in $T(Y)$ by

$$\lambda = \begin{pmatrix} y_i & u_j & v_j & s_j \\ y_i & a_j & u_j & b_j \end{pmatrix}.$$

Then $\lambda \in S$ and $\lambda\alpha = \lambda$. On the other hand, let $\alpha \in S$ be such that $\lambda\alpha = \lambda$ for some $\lambda \in S$. Since $\lambda \in S$, we have $S(\lambda) \subseteq X$. Hence $Y \setminus X \subseteq F(\lambda)$. We also have ran $\lambda \subseteq F(\alpha)$ since $\lambda\alpha = \lambda$. Hence $X\lambda \subseteq X \cap F(\alpha)$ and so $|X \cap F(\alpha)| = p$. Thus, we have an algebraic characterisation for the elements of the set T.

However, T is not a semigroup. To see this, let $X = A \dot\cup B \dot\cup C$, each with cardinal p, and let $B = B_1 \dot\cup B_2$, $C = C_1 \dot\cup C_2$, also each with cardinal p. Suppose $\alpha \in S$ fixes both $Y \setminus X$ and A pointwise, and maps B onto C and C onto B_1. Also, let

$\beta \in S$ fix both $Y \setminus X$ and B pointwise, and map A onto C_1 and C onto A. Then $F(\alpha\beta) = Y \setminus X$ and $|X \cap F(\alpha\beta)| = 0$. Hence $\alpha, \beta \in T$ but $\alpha\beta \notin T$.

Theorem 2.3.9. The semigroups $BL(X, p, p)$ and $GS(V, m, n)$ are not isomorphic for any infinite cardinals p, m, n with $n < m$.

Proof. Suppose $BL(X, p, p)$ is isomorphic to $GS(V, m, n)$. Let Y be a set with cardinal $r > p$ such that $Y \supseteq X$. Then, $BL(X, p, p)$ is isomorphic to a subset of $BL(Y, r, p)$ – namely, $S = \{\alpha \in BL(Y, r, p) : S(\alpha) \subseteq X\}$ – and there is an isomorphism ϕ from S onto $GS(V, m, n)$. Let $T = \{\alpha \in S : |X \cap F(\alpha)| = p\}$. Clearly ϕ induces a one-to-one mapping from T onto $\Sigma(n)$. For, $\alpha \in T$ if and only if $\lambda\alpha = \lambda$ for some $\lambda \in S$, which in turn is equivalent to saying: $\mu(\alpha\phi) = \mu$ for some $\mu \in GS(V, m, n)$ (even though T is not a semigroup). But $\Sigma(n)$ is a subsemigroup of $GS(V, m, n)$ and ϕ is an isomorphism, hence $\Sigma(n)\phi^{-1} = T$ must be a subsemigroup of S, contradicting our earlier remark. □

Since we have now shown that $BL(p, q)$ and $GS(m, n)$ are never isomorphic, it is worth observing the following result (recall Teissier's result in Theorem 2.1.1).

Theorem 2.3.10. Any right simple, right cancellative semigroup S without idempotents can be embedded in some $GS(m, m)$.

Proof. Let $|S| = m$ and write $S^1 = \{a_i\}$, with $|I| = m$. Note that S is infinite, since S has no idempotents. Let F be any field and let F_i be a copy of F for each $i \in I$. As in [15] p. 182, Remark (c), we let V be the vector space $\sum F_i$ over F whose basis can be identified in a natural way with $\{a_i\}$: that is, $\sum F_i$ is the set of all $(r_i)_{i \in I}$ where $r_i \in F_i$ and at most finitely many r_i are non-zero. Since S is right cancellative, the extended right regular representation of S is a faithful representation of S as a semigroup of one-to-one mappings of S^1 into itself. Let $x \in S$. Then x is represented by $\rho_x : S^1 \to S^1, a_i \mapsto a_i x$, which is a one-to-one mapping of the basis $\{a_i\}$ into itself. Hence ρ_x can be extended by linearity to a one-to-one linear map $V \to V$. Moreover, since S is infinite, [5] Vol. 2, Lemma 8.4 implies that

$$|S^1| = |S| = |S \setminus Sx| = |S^1 \setminus (x \cup Sx)| = |S^1 \setminus S^1\rho_x|.$$

Therefore, codim $\rho_x = |S| = m$ and hence $\rho_x \in GS(V, m, m)$. The faithfulness of the extended right regular representation implies that S is embedded in $GS(V, m, m)$. □

2.4. Left ideals of $GS(m, n)$

In this section we transfer results on the left ideals of $BL(p, q)$ to the linear Baer-Levi semigroup on V. By analogy with the work by Šutov [46] and Sullivan [41], it can be shown that the left ideals of $GS(m, n)$ are precisely the subsets L of $GS(m, n)$ which satisfy the condition:

$$[\alpha \in L,\ \beta \in GS(m,n),\ \text{ran}\ \beta \subseteq \text{ran}\ \alpha,\ \dim(\text{ran}\ \alpha/\text{ran}\ \beta) = n] \text{ implies } \beta \in L.$$

But, as for the set case, to obtain more information about the left ideals of $GS(m, n)$ we proceed as follows.

If Y is a non-empty subset of $GS(m, n)$, we let $L_Y^+ = Y \cup L_Y$, where

$$L_Y = \{\beta \in GS(m,n) : \text{ran}\ \beta \subseteq \text{ran}\ \alpha,\ \dim(\text{ran}\ \alpha/\ \text{ran}\ \beta) = n \ \text{ for some } \ \alpha \in Y\}.$$

To show L_Y is non-empty, choose any $\alpha \in Y$. Suppose $\{e_i\}$ is a basis for V and write $e_i\alpha = a_i$ for each i. Since α is one-to-one, $\{a_i\}$ is linearly independent and so it can be expanded to a basis $\{a_i\} \dot\cup \{b_j\}$ for V. Note that $|J| = d(\alpha) = n \leq m$, hence we can write $\{a_i\} = \{c_i\} \dot\cup \{d_j\}$. Now define $\beta \in T(V)$ by

$$\beta = \begin{pmatrix} e_i \\ c_i \end{pmatrix}.$$

This is in $GS(m, n)$ since β is one-to-one and $d(\beta) = \dim\langle d_j, b_j\rangle = n$. Clearly we have ran $\beta \subseteq$ ran α and $\dim(\text{ran}\ \alpha/\ \text{ran}\ \beta) = \dim\langle d_j\rangle = n$, hence $\beta \in L_Y$ and so L_Y is non-empty.

Theorem 2.4.1. If Y is a non-empty subset of $GS(m, n)$, then L_Y^+ is a left ideal of $GS(m, n)$. Conversely, if I is a left ideal of $GS(m, n)$, then $I = L_I^+$.

Proof. Suppose Y is a non-empty subset of $GS(m, n)$ and let $\alpha \in L_Y^+$ and $\beta \in GS(m, n)$. Then $\beta\alpha \in GS(m, n)$ and ran $(\beta\alpha) \subseteq$ ran α. Suppose $\{e_i\}$ is a basis for V. Since β is one-to-one, $\{e_i\beta\}$ is a basis for ran β, which can be expanded to another basis $\{e_i\beta, e_j\}$ for V, with $|J| = d(\beta) = n$. Then ran $\alpha = \langle e_i\beta\alpha, e_j\alpha\rangle$. On the other

hand, $\operatorname{ran}(\beta\alpha) = \langle e_i\beta\alpha\rangle$ and so $\dim(\operatorname{ran}\alpha/\operatorname{ran}(\beta\alpha)) = \dim\langle e_j\alpha\rangle = n$. If $\alpha \in Y$, then $\beta\alpha \in L_Y$. If not, then $\alpha \in L_Y$ and so $\operatorname{ran}\alpha \subseteq \operatorname{ran}\gamma$ and $\dim(\operatorname{ran}\gamma/\operatorname{ran}\alpha) = n$ for some $\gamma \in Y$. Thus $\operatorname{ran}(\beta\alpha) \subseteq \operatorname{ran}\alpha \subseteq \operatorname{ran}\gamma$ and $n = \dim(\operatorname{ran}\gamma/\operatorname{ran}\alpha) \leq \dim(\operatorname{ran}\gamma/\operatorname{ran}(\beta\alpha)) \leq d(\beta\alpha) = n$. Therefore $\beta\alpha \in L_Y$. In other words, we have shown that L_Y^+ is a left ideal of $GS(m,n)$.

Suppose I is a left ideal of $GS(m,n)$. We assert that $I = L_I^+$. Let $\beta \in L_I$. Then there exists $\alpha \in I$ such that $\operatorname{ran}\beta \subseteq \operatorname{ran}\alpha$ and $\dim(\operatorname{ran}\alpha/\operatorname{ran}\beta) = n$. If $\{e_i\}$ is a basis for V then $\operatorname{ran}\beta = \langle e_i\beta\rangle$ and, since $\operatorname{ran}\beta \subseteq \operatorname{ran}\alpha$, $\operatorname{ran}\alpha = \langle e_i\beta, e_j\rangle$ for some linearly independent set $\{e_i\beta, e_j\}$. Moreover, $|J| = n$ since $\dim(\operatorname{ran}\alpha/\operatorname{ran}\beta) = n$. Since α is one-to-one and $e_i\beta, e_j \in \operatorname{ran}\alpha$, we can choose unique f_i and f_j in V such that $f_i\alpha = e_i\beta$ and $f_j\alpha = e_j$. Then $\{f_i\}\dot\cup\{f_j\}$ is a basis for V (since α is one-to-one). Thus, we have

$$\alpha = \begin{pmatrix} f_i & f_j \\ e_i\beta & e_j \end{pmatrix}, \quad \beta = \begin{pmatrix} e_i \\ e_i\beta \end{pmatrix}.$$

Define $\gamma \in T(V)$ by

$$\gamma = \begin{pmatrix} e_i \\ f_i \end{pmatrix}.$$

Then $\gamma \in GS(m,n)$ and $\beta = \gamma\alpha$. Since I is a left ideal, it follows that $\beta \in I$. Therefore, $L_I \subseteq I$ and so $L_I^+ = I$. □

Remark 2.4.2. The left ideals of $GS(m,n)$ do not form a chain under $\subseteq$. For, suppose $\{e_i\}$ is a basis for V, let $\alpha \in GS(m,n)$ and write $e_i\alpha = a_i$ for each i. We can expand $\{a_i\}$ to a basis $\{a_i\}\dot\cup\{b_j\}$ for V, with $|J| = n$. Let $|K| < n$ and write $\{e_i\} = \{f_i\}\dot\cup\{f_k\}$ and $\{b_j\} = \{c_k\}\dot\cup\{d_j\}$. Define β in $T(V)$ by

$$\beta = \begin{pmatrix} f_i & f_k \\ a_i & c_k \end{pmatrix}.$$

Then $\alpha \notin L_{\{\beta\}}^+$ and $\beta \notin L_{\{\alpha\}}^+$. Thus $L_{\{\alpha\}}^+ \not\subseteq L_{\{\beta\}}^+$ and $L_{\{\beta\}}^+ \not\subseteq L_{\{\alpha\}}^+$.

The next result determines when one left ideal of $GS(m,n)$ is contained in another.

Theorem 2.4.3. Let A, B be non-empty subsets of $GS(m,n)$. Then $L_A^+ \subseteq L_B^+$ if and only if $A \setminus B \subseteq L_B$.

Proof. If $L_A^+ \subseteq L_B^+$, then $A \subseteq B \cup L_B$ and so $A \setminus B \subseteq L_B$. Now assume the latter occurs and let $\alpha \in L_A^+$. Then $\alpha \in A$ or $\alpha \in L_A$. If $\alpha \in A \cap B$, then $\alpha \in B$. If $\alpha \in A \setminus B$, then $\alpha \in L_B$. On the other hand, if $\alpha \in L_A$, then there exists $\beta \in A$ such that $\operatorname{ran} \alpha \subseteq \operatorname{ran} \beta$ and $\dim(\operatorname{ran} \beta / \operatorname{ran} \alpha) = n$. If $\beta \in B$, then $\alpha \in L_B$. If not, then $\beta \in A \setminus B \subseteq L_B$ and so there exists $\gamma \in B$ such that $\operatorname{ran} \beta \subseteq \operatorname{ran} \gamma$ and $\dim(\operatorname{ran} \gamma / \operatorname{ran} \beta) = n$. Therefore $\operatorname{ran} \alpha \subseteq \operatorname{ran} \gamma$ and $n \geq \dim(\operatorname{ran} \gamma / \operatorname{ran} \alpha) \geq \dim(\operatorname{ran} \beta / \operatorname{ran} \alpha) = n$, hence $\alpha \in L_B$. Thus we have shown that $\alpha \in L_B^+$ and the result follows. □

Hence $A \subseteq B$ implies $L_A^+ \subseteq L_B^+$, but not conversely. For, suppose $\{e_i\}$ is a basis for V and write $\{e_i\} = \{a_i\} \dot\cup \{b_j\}$ and $\{a_i\} = \{c_i\} \dot\cup \{c_j\}$, with $|J| = n$. Define $\alpha, \beta \in T(V)$ by

$$\alpha = \begin{pmatrix} e_i \\ a_i \end{pmatrix}, \qquad \beta = \begin{pmatrix} e_i \\ c_i \end{pmatrix}.$$

Since α, β are one-to-one and $d(\alpha) = \dim\langle b_j \rangle = n = \dim\langle b_j, c_j \rangle = d(\beta)$, α and β are elements of $GS(m,n)$. If $A = \{\beta\}$ and $B = \{\alpha\}$ then $L_A^+ \subseteq L_B^+$ but $A \not\subseteq B$.

Corollary 2.4.4. Let A, B be non-empty subsets of $GS(m,n)$. Then $L_A^+ \cup L_B^+ = L_{A \cup B}^+$.

Proof. Since $A, B \subseteq A \cup B$, we have $L_A^+ \cup L_B^+ \subseteq L_{A \cup B}^+$. Let $\gamma \in L_{A \cup B}^+$. Then $\gamma \in A \cup B$, and so $\gamma \in A$ or $\gamma \in B$, or $\gamma \in L_{A \cup B}$. If the latter happens, then there exists $\alpha \in A \cup B$ such that $\operatorname{ran} \gamma \subseteq \operatorname{ran} \alpha$ and $\dim(\operatorname{ran} \alpha / \operatorname{ran} \gamma) = n$. Hence $\gamma \in L_A \cup L_B$. Therefore $\gamma \in L_A^+ \cup L_B^+$ and the result follows. □

A similar result does not hold for the intersection of two non-empty subsets of $GS(m,n)$. That is, there are non-empty subsets A, B of $GS(m,n)$ whose intersection is also non-empty but $L_{A \cap B}^+ \subsetneq L_A^+ \cap L_B^+$. To see this, suppose $\{e_i\}$ is a basis for V and write $\{e_i\} = \{a_i\} \dot\cup \{b_j\} \dot\cup \{c_j\} \dot\cup \{d_j\}$, with $|J| = n$. Since $n \leq m$, we can also write $\{a_i\} \dot\cup \{b_j\} = \{u_i\}$, $\{a_i\} \dot\cup \{b_j\} \dot\cup \{c_j\} = \{v_i\}$ and $\{a_i\} \dot\cup \{d_j\} = \{w_i\}$. Now define $\alpha, \beta, \gamma \in T(V)$ by

$$\alpha = \begin{pmatrix} e_i \\ u_i \end{pmatrix}, \quad \beta = \begin{pmatrix} e_i \\ v_i \end{pmatrix}, \quad \gamma = \begin{pmatrix} e_i \\ w_i \end{pmatrix}.$$

It is easy to see that $\alpha, \beta, \gamma \in GS(m,n)$ and $\operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$, $\dim(\operatorname{ran}\beta/\operatorname{ran}\alpha) = n$ and $\operatorname{ran}\alpha \not\subseteq \operatorname{ran}\gamma$. Let $A = \{\alpha, \gamma\}$ and $B = \{\beta, \gamma\}$. Then $A \cap B = \{\gamma\}$. Since $\alpha \in A$ and $\alpha \in L_B$, it follows that $\alpha \in L_A^+ \cap L_B^+$. On the other hand, $\alpha \neq \gamma$ and $\alpha \notin L_{\{\gamma\}}$, hence $\alpha \notin L_{A\cap B}^+$.

In addition, the correspondence $A \mapsto L_A^+$ is not one-to-one. For example, if $C = \{\alpha, \beta\}$ and $D = \{\beta\}$ where α, β are the linear transformations defined in the last paragraph, then $L_C^+ = L_D^+$. To see this, let $\delta \in GS(m,n)$ be such that $\operatorname{ran}\delta \subseteq \operatorname{ran}\alpha$ and $\dim(\operatorname{ran}\alpha/\operatorname{ran}\delta) = n$. Then $\operatorname{ran}\delta \subseteq \operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$ and

$$n = \dim(\operatorname{ran}\alpha/\operatorname{ran}\delta) \leq \dim(\operatorname{ran}\beta/\operatorname{ran}\delta) \leq d(\delta) = n.$$

That is, if $\delta \in L_C^+$ then $\delta = \beta$ or ($\operatorname{ran}\delta \subseteq \operatorname{ran}\beta$ and $\dim(\operatorname{ran}\beta/\operatorname{ran}\delta) = n$) (by the definition of α and β, this covers the possibility that $\delta = \alpha$). Hence $\delta \in L_D^+$, and clearly $L_D^+ \subseteq L_C^+$, so we have equality as stated.

As observed in a remark before Corollary 2.1.7, any right simple semigroup without idempotents is not a principal left ideal of itself. In particular, $GS(m,n)$ is not a principal left ideal of itself. To decide when other left ideals of $GS(m,n)$ are principal, we first observe that the principal left ideal generated by $\alpha \in GS(m,n)$ is $L_{\{\alpha\}}^+$. For, clearly $GS(m,n)^1\alpha \subseteq L_{\{\alpha\}}^+$ since $\alpha \in L_{\{\alpha\}}^+$ and $L_{\{\alpha\}}^+$ is a left ideal of $GS(m,n)$. Conversely, the argument in the second paragraph of the proof of Theorem 2.4.1 shows that if $\alpha \in A \subseteq GS(m,n)$ and $\beta \in L_A$ then $\beta = \gamma\alpha$ for some $\gamma \in GS(m,n)$. In other words, $L_{\{\alpha\}} \subseteq GS(m,n)\alpha$ and it follows that $L_{\{\alpha\}}^+ = GS(m,n)^1\alpha$.

Corollary 2.4.5. Let A be a non-empty subset of $GS(m,n)$ and $\alpha \in GS(m,n)$. Then $L_A^+ = L_{\{\alpha\}}^+$ if and only if $\alpha \in L_A^+$ and $A \setminus \{\alpha\} \subseteq L_{\{\alpha\}}$.

In effect, the following result determines when left ideals are proper.

Theorem 2.4.6. Let A be a non-empty subset of $GS(m,n)$. Then $L_A^+ = GS(m,n)$ if and only if for each $\alpha \in GS(m,n)$ there exists $\lambda \in A$ such that $\operatorname{ran}\alpha \subseteq \operatorname{ran}\lambda$.

Proof. Suppose the latter condition holds for a non-empty $A \subseteq GS(m,n)$. Let $\{e_i\}$ be a basis for V, suppose $\beta \in GS(m,n)$ and write $e_i\beta = b_i$ for each i. We can expand $\{b_i\}$ to a basis for V, say $\{b_i\} \dot\cup \{b_j\}$. Write $\{b_j\} = \{c_j\} \dot\cup \{d_j\}$ and let

$\{c_i\} = \{b_i\} \dot\cup \{c_j\}$. Define

$$\gamma = \begin{pmatrix} e_i \\ c_i \end{pmatrix}.$$

Then $\gamma \in GS(m,n)$ and so there exists $\lambda \in A$ such that $\operatorname{ran}\gamma \subseteq \operatorname{ran}\lambda$. Hence $\operatorname{ran}\beta \subseteq \operatorname{ran}\gamma \subseteq \operatorname{ran}\lambda$ and $n \geq \dim(\operatorname{ran}\lambda/\operatorname{ran}\beta) \geq \dim(\operatorname{ran}\gamma/\operatorname{ran}\beta) = n$. Therefore $\beta \in L_A \subseteq L_A^+$. Thus $GS(m,n) \subseteq L_A^+$ and equality follows. Conversely, if there exists $\alpha \in GS(m,n)$ such that $\operatorname{ran}\alpha \not\subseteq \operatorname{ran}\lambda$ for all $\lambda \in A$, then clearly $\alpha \notin L_A^+$ and hence L_A^+ is a proper subset of $GS(m,n)$. □

To see that A may not equal $GS(m,n)$ in the above result, fix $\alpha \in GS(m,n) = G$ say, and write $\beta = \gamma\alpha$ for some fixed $\gamma \in G$. Put $A = G \setminus \{\beta\}$ and recall (see before Lemma 2.3.4) that $\alpha \neq \gamma\alpha$ in G, so $\alpha \in A$. Clearly $G = GA \cup A$. Also, if $\mu \in G$ then either $\mu \in A$ or $\mu = \gamma\alpha$ with $\alpha \in A$, and in each case $\operatorname{ran}\mu \subseteq \operatorname{ran}\lambda$ for some $\lambda \in A$. Hence, by the Theorem, $L_A^+ = G$ where $A \subsetneq G$.

It is easy to see that $GS(m,n)$ has no minimal left ideals, since it is right simple and has no idempotents (see a remark before Theorem 2.1.9). However, it is not as easy to see that $GS(m,n)$ has no maximal left ideals.

Theorem 2.4.7. The semigroup $GS(m,n)$ has no maximal (proper) left ideals.

Proof. From Theorem 2.4.6, L_A^+ is a proper left ideal if and only if there exists some α in $GS(m,n)$ such that $\operatorname{ran}\alpha \not\subseteq \operatorname{ran}\lambda$ for all $\lambda \in A$.

Let L_Y^+ be a proper left ideal of $GS(m,n)$. Then there exists $\alpha \in GS(m,n)$ such that $\operatorname{ran}\alpha \not\subseteq \operatorname{ran}\lambda$ for all $\lambda \in Y$. Let $Z = Y \cup \{\alpha\}$. Then $L_Y^+ \subseteq L_Z^+$. Obviously $\alpha \notin L_Y^+$ and so $L_Y^+ \subsetneq L_Z^+$. We assert that $L_Z^+ \subsetneq GS(m,n)$.

Write $e_i\alpha = a_i$ where $\{e_i\}$ is a basis for V, and expand $\{a_i\}$ to a basis for V, say $\{a_i\} \dot\cup \{a_j\}$ with $|J| = n \leq m = |I|$. Write $\{a_j\} = \{b_j\} \dot\cup \{c_j\}$ and let $\{b_i\} = \{a_i\} \dot\cup \{b_j\}$ (this is possible since m and n are both infinite). Define $\beta \in T(V)$ by

$$\beta = \begin{pmatrix} e_i \\ b_i \end{pmatrix}.$$

Clearly, $\beta \in GS(m,n)$ and $\operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$, so $\beta \notin Y$. Since $\alpha \neq \beta$, we have $\beta \notin Z$. Suppose $\beta \in L_Z$. Then $\operatorname{ran}\beta \subseteq \operatorname{ran}\gamma$ and $\dim(\operatorname{ran}\gamma/\operatorname{ran}\beta) = n$ for some $\gamma \in Z$. If $\gamma = \alpha$, then $\operatorname{ran}\beta \subseteq \operatorname{ran}\alpha$, a contradiction. Then $\gamma \in Y$, but

ran $\alpha \subseteq$ ran $\beta \subseteq$ ran γ, which contradicts our condition on α and Y. Therefore, $\beta \notin L_Z^+$ and hence $L_Z^+ \subsetneq GS(m,n)$. In other words, given any proper left ideal L, we can find a strictly larger proper left ideal that contains L. Hence there are no maximal left ideals of $GS(m,n)$. □

2.5. Maximal subsemigroups of $GS(m,n)$

In this section, we show that any non-zero subspace U of V with codimension at least n gives rise to a maximal subsemigroup of $GS(m,n)$: here, our work closely follows that in [25] (see Theorem 2.1.14).

Let U be a non-zero subspace of V with codim $U \geq n$ and define

$$M_U = \{\alpha \in GS(m,n) : U \not\subseteq \text{ran } \alpha \quad \text{or} \quad (U\alpha \subseteq U \quad \text{or} \quad \dim(\text{ran } \alpha/U) < n)\}.$$

Theorem 2.5.1. For each non-zero subspace U of V with codim $U \geq n$, M_U is a maximal subsemigroup of $GS(m,n)$.

Proof. We first show that M_U is a subsemigroup of $GS(m,n)$. Let $\alpha, \beta \in M_U$. Since $\alpha, \beta \in GS(m,n)$, it follows that $\alpha\beta \in GS(m,n)$. If $U \not\subseteq \text{ran}\,(\alpha\beta)$ then $\alpha\beta \in M_U$. If $U \subseteq \text{ran}\,(\alpha\beta)$ then $U \subseteq \text{ran } \beta$. Hence $U\beta \subseteq U$ or $\dim(\text{ran } \beta/U) < n$. If the latter holds then $\dim(\text{ran}\,(\alpha\beta)/U) \leq \dim(\text{ran } \beta/U) < n$ and so $\alpha\beta \in M_U$. If $U\beta \subseteq U$ then $U\beta \subseteq \text{ran}\,(\alpha\beta)$ and so $U \subseteq \text{ran } \alpha$. Thus, $U\alpha \subseteq U$ or $\dim(\text{ran } \alpha/U) < n$ since $\alpha \in M_U$. Suppose $U\alpha \subseteq U$. Then $U\alpha\beta \subseteq U\beta \subseteq U$ and therefore $\alpha\beta \in M_U$. If $\dim(\text{ran } \alpha/U) < n$, write $U = \langle u_i \rangle$ and so $U\beta = \langle u_i\beta \rangle$. Hence $U = \langle u_i\beta, u_j \rangle$ for some linearly independent set $\{u_i\beta\} \dot\cup \{u_j\}$, and likewise ran $\alpha = \langle u_i, w_r \rangle$ and ran $(\alpha\beta) = \langle u_i\beta, w_r\beta \rangle$. On the other hand, since $U = \langle u_i\beta, u_j \rangle \subseteq \text{ran}\,(\alpha\beta)$, we have ran $(\alpha\beta) = \langle u_i\beta, u_j, w_s \rangle$. Hence $|R| = |J| + |S|$. Thus,

$$\dim(\text{ran}\,(\alpha\beta)/U) = |S| \leq |R| < n.$$

Therefore, $\alpha\beta \in M_U$ and M_U is a subsemigroup of $GS(m,n)$.

In order to prove the maximality of M_U, we show that a subsemigroup M of $GS(m,n)$ properly containing M_U necessarily is $GS(m,n)$ itself. Let M be a subsemigroup of $GS(m,n)$ satisfying these conditions. Let $\gamma \in M \setminus M_U$ and $\alpha \in GS(m,n) \setminus M_U$.

Since $\gamma, \alpha \notin M_U$, we know that $U \subseteq \text{ran } \gamma$, $U\gamma \not\subseteq U$, $\dim(\text{ran } \gamma/U) \geq n$ and $U \subseteq \text{ran } \alpha$, $U\alpha \not\subseteq U$, $\dim(\text{ran } \alpha/U) \geq n$. If $U\alpha^{-1} = \langle a_i \rangle$ and $U\gamma^{-1} = \langle b_j \rangle$, then $U = \langle a_i\alpha \rangle = \langle b_j\gamma \rangle$ and $\{a_i\alpha\}, \{b_j\gamma\}$ are bases for U (since α and γ are one-to-one). Therefore $|I| = |J|$ and we can write $U\gamma^{-1} = \langle b_i \rangle$ and $U = \langle a_i\alpha \rangle = \langle b_i\gamma \rangle$. Since $U\alpha^{-1}$ is a subspace of V, we can expand $\{a_i\}$ to a basis for V, say $\{a_i\} \dot\cup \{e_k\}$. Then $\text{ran } \alpha = \langle a_i\alpha, e_k\alpha \rangle$ where $\{a_i\alpha, e_k\alpha\}$ is linearly independent. Hence $\text{codim}(U\alpha^{-1}) = |K| = \dim(\text{ran } \alpha/U)$. Since $\text{ran } \alpha = \langle a_i\alpha, e_k\alpha \rangle$ and $\text{ran } \alpha \subseteq V$, we can expand $\{a_i\alpha\} \dot\cup \{e_k\alpha\}$ to a basis for V, say $\{a_i\alpha, e_k\alpha, e_\ell\}$ with $|L| = n$ and so $\text{codim } U = |K| + n = |K|$.

Analogously we can expand $\{b_i\}$ to a basis for V, say $\{b_i, f_r\}$, and $\text{ran } \gamma$ is spanned by the linearly independent set $\{b_i\gamma, f_r\gamma\}$. Hence

$$\text{codim}(U\gamma^{-1}) = |R| = \dim(\text{ran } \gamma/U) \geq n.$$

Now extend $\{b_i\gamma, f_r\gamma\}$ to a basis for V, say $\{b_i\gamma, f_r\gamma, f_s\}$, where $|S| = d(\gamma) = n$. Since $|L| = n$, this means we can write $\{f_\ell\}$ instead of $\{f_s\}$. Moreover $\text{codim } U = |R| = |K|$. Therefore, we can also write $\{f_k\}$ and $\{f_k\gamma\}$ instead of $\{f_r\}$ and $\{f_r\gamma\}$, respectively.

Since $U\gamma \not\subseteq U$, there exists $u \in U$ such that $u\gamma \notin U$. It follows that $\{b_i, u\}$ and $\{b_i\gamma, u\gamma\}$ are linearly independent. We can expand these sets to bases for V and for $\text{ran } \gamma$, respectively, say $\{b_i, u, h_k\}$ and $\{b_i\gamma, u\gamma, g_k\}$ (note that $|K| = \text{codim}(U\gamma^{-1}) = \dim\langle u, h_k \rangle$ and $|K| = \dim(\text{ran } \gamma/U) = \dim\langle u\gamma, g_k \rangle$). We can also expand $\{b_i\gamma, u\gamma, g_k\}$ to a basis $\{b_i\gamma, u\gamma, g_k, g_t\}$ for V, where $|T| = d(\gamma) = n = |L|$. Write $\{g_\ell\}$ instead of $\{g_t\}$ and let $W = \langle u\gamma, g_k, g_\ell \rangle$. Then W is a complement of U in V. We have $\langle u \rangle \subseteq U \cap W\gamma^{-1}$. Also $\langle u \rangle \subseteq \langle u, h_k \rangle$, which is a complement of $U\gamma^{-1}$ in V. Since $|K| = \dim(\text{ran } \gamma/U) \geq n = |L|$, we may write $\{h_k\} = \{c_k\} \dot\cup \{d_\ell\}$. Now define β in $T(V)$ by

$$\beta = \begin{pmatrix} a_i & e_k \\ b_i & c_k \end{pmatrix}.$$

Since $u \in U$ and $u \notin \text{ran } \beta$, it follows that $U \not\subseteq \text{ran } \beta$ and so $\beta \in M_U$. Write $\{u\} \cup \{d_\ell\} = \{c_\ell\}$ and $c_\ell\gamma = z_\ell$ for each ℓ. Then

$$\gamma = \begin{pmatrix} b_i & c_k & c_\ell \\ b_i\gamma & c_k\gamma & z_\ell \end{pmatrix}.$$

Let $\langle w_\ell \rangle$ be a complement of ran γ in V. As in the second paragraph above, let $\{e_\ell\}$ be a basis for a complement of ran α in V and write $\{e_\ell\} = \{x_\ell\} \dot\cup \{y_\ell\}$. Now write $\{z_\ell\} \dot\cup \{w_\ell\} = \{v_\ell\}$ and define $\delta \in T(V)$ by

$$\delta = \begin{pmatrix} b_i\gamma & c_k\gamma & v_\ell \\ a_i\alpha & e_k\alpha & x_\ell \end{pmatrix}.$$

Since $U = \langle a_i\alpha \rangle \subseteq$ ran δ and $U\delta = \langle b_i\gamma \rangle \delta = \langle a_i\alpha \rangle = U$, it follows that $\delta \in M_U$. Since $\beta\gamma\delta = \alpha$, we have $\alpha \in M_U.M.M_U \subseteq M$. Therefore, $M = GS(m,n)$ and hence M_U is maximal. □

As already noted at the end of section 2.1, the complexity of Hotzel's examples in [14] suggests that we may never be able to describe all the maximal subsemigroups of $BL(p,q)$, and this may be true also for $GS(m,n)$.

Chapter 3

Injective total linear transformations with infinite defect

According to Steinfeld [39] p. 99, the notion of a 'bi-ideal' in a semigroup was first introduced in 1952 by Good and Hughes and, a few years later, Steinfeld defined the concept of a 'quasi-ideal' in a semigroup. Since every quasi-ideal of a semigroup S is a bi-ideal of S, it is natural to ask if the sets of quasi-ideals and bi-ideals of S coincide. In [39] p. 74, Steinfeld states that Lajos showed in 1961 that regular semigroups belong to **BQ**, the class of all semigroups whose sets of bi-ideals and quasi-ideals coincide. And, according to Namnak and Kemprasit [36] p. 238, Kapp proved in 1969 that every right [left] simple semigroup and every right [left] 0-simple semigroup belong to **BQ**. In 2002, Kemprasit gave examples of semigroups of transformations on infinite sets which are in **BQ** but are not of the above kinds. On the other hand, one year before, she gave examples of semigroups of transformations on infinite sets which have infinitely many bi-ideals which are not quasi-ideals. In the same year, Kemprasit and Namnak considered the semigroup of all injective linear transformations of an infinite-dimensional vector space V into itself, whose ranges have infinite codimension, and they showed it is in **BQ** if and only if $\dim V = \aleph_0$ and, in this case, it is right simple.

In this chapter, we consider a natural generalisation of their semigroup, which we define as follows. Let V be a vector space with infinite dimension p. For each cardinal q such that $\aleph_0 \leq q \leq p$, let $KN(p,q)$ denote the semigroup of all injective linear transformations $\alpha : V \to V$ for which the codimension of the range of α is at least q (we note that 'KN' stands for Kemprasit and Namnak). In section 3.1, we start by determining the ideals of $M(V)$, the semigroup of all injective linear transformations of V into itself. The analogue of this problem for the set case was considered by Šutov in 1961: he described all left and two-sided ideals of $M(X)$, the semigroup of all injective transformations of a set X into itself, and he showed each right ideal of $M(X)$ is two-sided. Next we prove that the prime ideals of $M(V)$ are exactly the semigroups $KN(p,q)$ with $\aleph_0 \leq q \leq p$: this provides one reason for our interest in such semigroups. Then we show that $KN(p,q)$ is right cancellative and

we generalise Kemprasit and Namnak's results: we prove that $KN(p,q)$ is in **BQ** if and only if $p = q$ and $KN(p,q)$ is right simple only when $p = q$.

In section 3.2, we investigate the ideal structure and Green's relations on $KN(p,q)$ for $p > q$ (clearly, if $p = q$ then $KN(p,q)$ is the linear Baer-Levi semigroup $GS(p,p)$, which was studied in the previous chapter). We show that the ideals of $KN(p,q)$, when $p > q$, are precisely the semigroups $KN(p,\xi)$ provided that $q < \xi < p$, or ξ equals q or p, but not both (hence they form a chain under $\subseteq$) and that each right ideal of $KN(p,q)$ is a two-sided ideal. In section 3.3, we determine the left ideals of $KN(p,q)$ for $p > q$ and we show that this semigroup has no minimal or maximal (proper) left ideals. We also determine the principal left ideals of $KN(p,q)$ and we see that there are no maximal principal left ideals in $KN(p,q)$.

As noted by Clifford and Preston [5] Vol. 2, p. 82, Teissier showed in 1953 that any right simple, right cancellative semigroup S without idempotents can be embedded in some Baer-Levi semigroup defined on a set. By chapter 2, Theorem 2.3.10, we know that S can also be embedded in some linear Baer-Levi semigroup $GS(p,p)$ with $p = |S|$. In section 3.4, we prove that any right cancellative semigroup S without idempotents whose right ideals form a chain and which has no maximal principal left ideals can be embedded in some $KN(p,q)$, where $p = |S|$ and where, among all the cardinals of the complements of principal left ideals in S, q is the least of them: this extends previous work on Baer-Levi semigroups in a natural way, since the two types of Baer-Levi semigroup (one defined on sets, the other on vector spaces) have no proper right ideals and no maximal principal left ideals.

In section 3.5, we describe the quasi-ideals and the bi-ideals of $KN(p,q)$ for $p > q$ and we show that the principal quasi-ideals are exactly the principal left ideals of $KN(p,q)$. Also, we show that the principal quasi-ideal and the principal bi-ideal generated by $\{\alpha\}$ in $KN(p,q)$ coincide if and only if $d(\alpha) = q$ and, using this, we conclude that there are infinitely many bi-ideals of $KN(p,q)$ which are neither quasi-ideals nor ideals of $KN(p,q)$. Finally, in section 3.6 we determine all maximal right simple subsemigroups of $KN(p,q)$ for $p > q$: they are precisely the semigroups $GS(p,\xi)$ where $q \leq \xi \leq p$ (with the same proviso on ξ as before). All the work in this chapter has appeared in [29].

3.1. Basic properties

Let V be a vector space over a field F with infinite dimension p. For each cardinal q such that $\aleph_0 \leq q \leq p$, we write

$$KN(p,q) = \{\alpha \in T(V) : n(\alpha) = 0,\ d(\alpha) \geq q\}.$$

Clearly $KN(p,q)$ is a subsemigroup of $T(V)$: for, if $\alpha, \beta \in KN(p,q)$ then $\alpha\beta$ is one-to-one; and, since $\text{ran}\,(\alpha\beta) \subseteq \text{ran}\ \beta$, we have

$$d(\alpha\beta) = \dim(V/\,\text{ran}\,(\alpha\beta)) \geq \dim(V/\,\text{ran}\ \beta) \geq q,$$

and so $\alpha\beta \in KN(p,q)$. We show below that these semigroups are exactly the prime ideals of $M(V)$, the semigroup of all injective linear transformations from V to V. In fact, we will show that the ideals of $M(V)$ are precisely the sets $\{\alpha \in T(V) : n(\alpha) = 0,\ d(\alpha) \geq r\}$ where $0 \leq r \leq p$. Šutov ([46] Theorem 6) proved a similar result for the ideals of the semigroup of all injective transformations of an infinite set into itself.

Since elements of $KN(p,q)$ are injective, the semigroup is right cancellative. Also, if $\varepsilon \in KN(p,q)$ is idempotent then $(u\varepsilon)\varepsilon = u\varepsilon$ for all $u \in V$ implies $\varepsilon = \text{id}_V$, a contradiction. Thus $KN(p,q)$ has no idempotents.

Given $\alpha, \beta \in T(V)$, it is clear that $d(\alpha\beta) \geq d(\beta)$, since $\text{ran}\,(\alpha\beta) \subseteq \text{ran}\ \beta$, but it is not always true that $d(\alpha\beta) \geq d(\alpha)$. Likewise, we know that $n(\alpha\beta) \geq n(\alpha)$ (since $\ker\alpha \subseteq \ker(\alpha\beta)$) but $n(\alpha\beta) \geq n(\beta)$ does not necessarily hold. Part (b) of the following result shows that the latter is true in $E(V)$, the semigroup of all surjective linear transformations of V. In part (a), we show that $d(\alpha\beta) \geq d(\alpha)$ for all α, β in $M(V)$: in [23] Lemma 2.2, Levi proved a similar result for injective transformations of an infinite set into itself.

Lemma 3.1.1. Let $\alpha, \beta \in T(V)$.

(a) If $n(\alpha) = 0 = n(\beta)$, then $d(\alpha\beta) = d(\alpha) + d(\beta)$;

(b) If $d(\alpha) = 0 = d(\beta)$, then $n(\alpha\beta) = n(\alpha) + n(\beta)$.

Proof. To see that (a) holds, suppose $\alpha, \beta \in M(V)$ and let $\{e_i\}$ be a basis for V. Write $e_i\alpha = a_i$ for each i. Since α is one-to-one, the set $\{a_i\}$ is linearly independent and so it

can be expanded to a basis $\{a_i\} \dot\cup \{a_j\}$ for V with $|J| = d(\alpha)$. Then, $\{a_i\beta\} \dot\cup \{a_j\beta\}$ is a basis for ran β and we can expand it to a basis for V, say $\{a_i\beta\} \dot\cup \{a_j\beta\} \dot\cup \{b_k\}$ with $|K| = d(\beta)$. Clearly, $\{a_i\beta\}$ is a basis for ran $(\alpha\beta)$ and hence $d(\alpha\beta) = \dim\langle a_j\beta, b_k\rangle = |J| + |K| = d(\alpha) + d(\beta)$.

Now suppose $\alpha, \beta \in E(V)$ and let $\{e_j\}$ and $\{f_k\}$ be bases for $\ker\alpha$ and $\ker\beta$, respectively, where $|J| = n(\alpha)$ and $|K| = n(\beta)$. Since α is onto, there exist a_k in V such that $a_k\alpha = f_k$ for each k. We assert that $\{e_j\} \dot\cup \{a_k\}$ is a basis for $\ker(\alpha\beta)$. Since $a_k(\alpha\beta) = (a_k\alpha)\beta = f_k\beta = 0$, we have $\{e_j\} \dot\cup \{a_k\} \subseteq \ker(\alpha\beta)$. Also, if $\sum x_j e_j + \sum y_k a_k = 0$ for some scalars x_j and y_k, then $(\sum x_j e_j + \sum y_k a_k)\alpha = 0$, and so $\sum y_k f_k = 0$. But $\{f_k\}$ is linearly independent, hence $y_k = 0$ for every k. Thus, $\sum x_j e_j = 0$ and, since $\{e_j\}$ is linearly independent, it follows that $x_j = 0$ for all j. Therefore, $\{e_j\} \dot\cup \{a_k\}$ is linearly independent. Now, if $u \in \ker(\alpha\beta)$ then $u\alpha \in \ker\beta$ and hence $u\alpha = \sum x_k f_k$ for some scalars x_k. Therefore, $u\alpha = (\sum x_k a_k)\alpha$ and so $u - \sum x_k a_k \in \ker\alpha$. Thus, there exist scalars y_j such that $u - \sum x_k a_k = \sum y_j e_j$. Hence, $u = \sum y_j e_j + \sum x_k a_k$ and $\{e_j\} \dot\cup \{a_k\}$ is a basis for $\ker(\alpha\beta)$. In particular, $n(\alpha\beta) = n(\alpha) + n(\beta)$, as required. □

Theorem 3.1.2. If $\dim V = p \geq q \geq \aleph_0$ then $KN(p, q)$ is an ideal of $M(V)$. In fact, the ideals of $M(V)$ are precisely the sets

$$M_\xi = \{\alpha \in T(V) : n(\alpha) = 0,\ d(\alpha) \geq \xi\}$$

where $0 \leq \xi \leq p$.

Proof. Let ξ be a cardinal such that $0 \leq \xi \leq p$. To see that M_ξ is non-empty, let $\{e_i\}$ be a basis for V and write $\{e_i\}$ as $\{a_i\} \dot\cup \{b_k\}$ with $|K| = \xi$. Define γ in $T(V)$ by

$$\gamma = \begin{pmatrix} e_i \\ a_i \end{pmatrix}.$$

Since γ is one-to-one and $d(\gamma) = \dim\langle b_k\rangle = \xi$, it follows that $\gamma \in M_\xi$ and so this set is non-empty. Now suppose $\alpha \in M_\xi$ and $\lambda, \mu \in M(V)$. Then $\lambda\alpha\mu \in M(V)$ and, from Lemma 3.1.1(a), $d(\lambda\alpha\mu) \geq d(\lambda\alpha) \geq d(\alpha) \geq \xi$. Therefore $\lambda\alpha\mu \in M_\xi$ and so M_ξ is an ideal of $M(V)$ (note that the latter semigroup contains id_V).

Conversely, suppose I is an ideal of $M(V)$ and choose $\delta \in I$ with minimal defect ξ. Note that this is possible since the cardinals are well-ordered. We then have

$d(\alpha) \geq \xi$ for every $\alpha \in I$ and so $I \subseteq M_\xi$. Now let $\beta \in M_\xi$. Then $d(\beta) \geq \xi = d(\delta)$. Suppose $\{e_i\}$ is a basis for V and write $e_i\beta = b_i$ and $e_i\delta = d_i$ for each i. Expand $\{b_i\}$ and $\{d_i\}$ to bases for V, say $\{b_i\} \dot\cup \{b_j\}$ and $\{d_i\} \dot\cup \{d_k\}$, respectively, with $|J| = d(\beta) \geq d(\delta) = |K|$. We can write $\{b_j\}$ as $\{u_k\} \dot\cup \{v_\ell\}$ where $|J| = |K| + |L|$, and define μ in $T(V)$ by

$$\mu = \begin{pmatrix} d_i & d_k \\ b_i & u_k \end{pmatrix}.$$

Then μ is one-to-one and so $\mu \in M(V)$. Also we have $\beta = \delta\mu$. Since I is an ideal of $M(V)$ and $\delta \in I$, it follows that $\beta \in I$ and this completes the proof. □

A proper ideal I of a semigroup S is *prime* if $ab \in I$ implies $a \in I$ or $b \in I$ (compare [5] Vol. 1, p. 40). The next result provides one reason for our interest in the semigroups $KN(p,q)$ (another is discussed in section 3.4).

Theorem 3.1.3. If $\dim V = p \geq \aleph_0$, then the prime ideals of $M(V)$ are exactly the semigroups $KN(p,q)$ with $\aleph_0 \leq q \leq p$.

Proof. From Lemma 3.1.1(a) and Theorem 3.1.2, it is easy to see that $KN(p,q)$ is a prime ideal of $M(V)$ for every infinite cardinal $q \leq p$. If $r = 0$ then $M_r = M(V)$. Now let r be a non-zero finite cardinal and suppose $\{e_i\}$ is a basis for V. Write $\{e_i\}$ as $\{a_i\} \dot\cup \{b_j\} \dot\cup \{c\}$ with $|J| = r - 1$. Also write $\{a_i\} \dot\cup \{b_j\}$ as $\{f_i\}$ and $\{a_i\} \dot\cup \{c\}$ as $\{g_i\}$. Define $\alpha, \beta \in M(V)$ by

$$\alpha = \begin{pmatrix} e_i \\ g_i \end{pmatrix}, \qquad \beta = \begin{pmatrix} e_i \\ f_i \end{pmatrix}.$$

Then, $d(\alpha) = \dim\langle b_j \rangle = r - 1$ and $d(\beta) = \dim\langle c \rangle = 1$, and so $\alpha, \beta \notin M_r$. From Lemma 3.1.1(a), $d(\alpha\beta) = d(\alpha) + d(\beta) = (r-1) + 1 = r$ and hence $\alpha\beta \in M_r$. Therefore, M_r is not prime for every cardinal r such that $0 \leq r < \aleph_0$ and the result follows. □

By chapter 2, Theorem 2.2.5, we know that $|GS(p,q)| = |V|^p$. In fact, $KN(p,q)$ has the same order since it equals $\bigcup_{p \geq \xi \geq q} GS(p,\xi)$. For, this equality implies

$$|KN(p,q)| = \sum_{p \geq \xi \geq q} |GS(p,\xi)| = \sum_{p \geq \xi \geq q} |V|^p \leq p^p |V|^p = 2^p |V|^p = (2|V|)^p = |V|^p.$$

Since $|V|^p \leq \sum_{p \geq \xi \geq q} |V|^p = |KN(p,q)|$, it follows that $|KN(p,q)| = |V|^p$.

A subsemigroup Q of a semigroup S is called a *quasi-ideal* of S if $SQ \cap QS \subseteq Q$. A subsemigroup B of S is a *bi-ideal* of S if $BSB \subseteq B$. Note that every right and every left ideal of S is a quasi-ideal, and every quasi-ideal Q of a semigroup S is a bi-ideal of S since $QSQ \subseteq SQ \cap QS$. Given a non-empty subset X of S, the quasi-ideal and the bi-ideal generated by X will be denoted respectively by $(X)_Q$ and $(X)_B$. If $X = \{x_1, x_2, \ldots, x_n\}$ then we write $(x_1, x_2, \ldots, x_n)_Q$ and $(x_1, x_2, \ldots, x_n)_B$ instead of $(\{x_1, x_2, \ldots, x_n\})_Q$ and $(\{x_1, x_2, \ldots, x_n\})_B$, respectively. By [5] Vol. 1, pp. 84-85, Exercises 15 and 17, if X is a non-empty subset of a semigroup S, then

$$(X)_Q = S^1X \cap XS^1 = (SX \cap XS) \cup X, \quad \text{and}$$
$$(X)_B = (XS^1X) \cup X = XSX \cup X \cup X^2.$$

We now decide when the semigroup $KN(p,q)$ is in the class **BQ** of all semigroups whose sets of bi-ideals and quasi-ideals coincide. For this, we closely follow the arguments in [19] Theorem 3.5.

Theorem 3.1.4. $KN(p,q) \in \mathbf{BQ}$ if and only if $\dim V = q$.

Proof. If $\dim V = q$, then $KN(p,q) = GS(p,p)$ and so $KN(p,q)$ is right simple (by chapter 2, Theorem 2.2.1). Thus, by [19] Proposition 1.3, $KN(p,q) \in \mathbf{BQ}$.

Conversely, assume $q < p = \dim V$ and let $\{b_i\}$ be a basis for V. Write $\{b_i\} = \{c_i\} \dot{\cup} \{e_i\} = \{d_j\} \dot{\cup} \{f_i\}$ and $\{c_i\} = \{a_i\} \dot{\cup} \{g_i\}$, with $|I| = p$ and $|J| = q$ (this is possible since p and q are infinite). Define $\alpha, \beta \in T(V)$ by

$$\alpha = \begin{pmatrix} b_i \\ e_i \end{pmatrix}, \quad \beta = \begin{pmatrix} b_i \\ f_i \end{pmatrix}.$$

Since α, β are one-to-one, $d(\alpha) = p > q$ and $d(\beta) = q$, it follows that $\alpha, \beta \in KN(p,q)$. Now define γ in $T(V)$ by

$$\gamma = \begin{pmatrix} c_i & e_i \\ a_i & f_i\alpha \end{pmatrix}.$$

Since $\{a_i\} \cap \{f_i\alpha\} \subseteq \{c_i\} \cap \{e_i\} = \emptyset$, γ is one-to-one. Moreover, $d(\gamma) \geq \dim\langle g_i \rangle = p$ and so $d(\gamma) = p > q$. Therefore, $\gamma \in KN(p,q)$. Clearly, $\beta\alpha = \alpha\gamma$, so $\beta\alpha \in KN(p,q)\alpha \cap \alpha KN(p,q) \subseteq (\alpha)_Q$.

Suppose $\beta\alpha \in (\alpha)_B$. Then $\beta\alpha \in \alpha KN(p,q)\alpha \cup \{\alpha, \alpha^2\}$. If $\beta\alpha = \alpha$, then $\beta = \mathrm{id}_V$, a contradiction since $\beta \in KN(p,q)$. If $\beta\alpha = \alpha^2$, then $\beta = \alpha$ and so $q = d(\beta) = d(\alpha) = p$, which contradicts our assumption. Therefore $\beta\alpha \in \alpha KN(p,q)\alpha$, hence there exists λ in $KN(p,q)$ such that $\beta = \alpha\lambda$. Thus, using the fact that λ is injective, we obtain

$$q = d(\beta) = d(\alpha\lambda) = \dim(V/(\text{ran }\alpha)\lambda) \geq \dim(V\lambda/(\text{ran }\alpha)\lambda) = d(\alpha) = p,$$

which contradicts $q < p$. Therefore, $\beta\alpha \notin (\alpha)_B$. By [19] Proposition 1.4, $KN(p,q) \notin \mathbf{BQ}$. □

From the result above, we determine when the semigroup $KN(p,q)$ is right simple.

Lemma 3.1.5. The semigroup $KN(p,q)$ is right simple if and only if $\dim V = q$.

Proof. If $\dim V = q$, then $KN(p,q) = GS(p,p)$ and so it is right simple. Conversely, suppose $\dim V > q$. Then, $KN(p,q) \notin \mathbf{BQ}$. By [19] Proposition 1.3, $KN(p,q)$ is not right simple. □

If $p = q$, then $KN(p,q)$ is the semigroup $GS(p,q)$, which was studied in chapter 2. Therefore, *we assume $p > q$ throughout the rest of this chapter.*

3.2. Green's relations and ideals in $KN(p,q)$

It is easy to see that Green's $\mathcal{L}$ relation on $KN(p,q)$ equals the identity relation. For, if α and β are two distinct elements of $KN(p,q)$ such that $\alpha = \lambda\beta$ and $\beta = \mu\alpha$ for some $\lambda, \mu \in KN(p,q)$, then $\alpha = (\lambda\mu)\alpha$. Therefore, for every $u \in V$, $u\alpha = u(\lambda\mu)\alpha$ and, since α is one-to-one, we have $u = u(\lambda\mu)$. Hence, $\lambda\mu = \mathrm{id}_V$, a contradiction. As $KN(p,q)$ is not right simple if $p > q$, it is of interest to characterise the $\mathcal{R}$ relation on it.

Lemma 3.2.1. Suppose $p > q$. Given $\alpha, \beta \in KN(p,q)$, there exists $\mu \in KN(p,q)$ such that $\beta = \alpha\mu$ if and only if $d(\alpha) \leq d(\beta)$. Hence, $\alpha \mathcal{R} \beta$ if and only if $d(\alpha) = d(\beta)$. Consequently, in this case, $\mathcal{R} = \mathcal{D} = \mathcal{J}$ on $KN(p,q)$.

Proof. Suppose $\beta = \alpha\mu$ for some $\mu \in KN(p,q)$. From Lemma 3.1.1(a), $d(\beta) = d(\alpha) + d(\mu) \geq d(\alpha)$. Conversely, if $d(\alpha) \leq d(\beta)$ and if $\{e_i\}$ is a basis for V, write

$e_i\alpha = a_i$ and $e_i\beta = b_i$. We can expand $\{a_i\}$ and $\{b_i\}$ to bases for V, say $\{a_i\}\dot\cup\{a_j\}$ and $\{b_i\}\dot\cup\{b_k\}$, respectively, with $|J| = d(\alpha) \leq d(\beta) = |K|$. Since, by the definition of $KN(p,q)$, the defects are infinite, we can write $\{b_k\} = \{c_k\}\dot\cup\{c_j\}$ and define $\mu \in T(V)$ by

$$\mu = \begin{pmatrix} a_i & a_j \\ b_i & c_j \end{pmatrix}.$$

Then μ is one-to-one and $d(\mu) = \dim\langle c_k\rangle = |K| \geq q$, so $\mu \in KN(p,q)$. Also, we have $\beta = \alpha\mu$.

Since $\mathcal{L}$ equals the identity relation, we know $\mathcal{R} = \mathcal{D} \subseteq \mathcal{J}$. Moreover, if $\alpha, \beta \in KN(p,q)$ and $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in KN(p,q)^1$ then, by Lemma 3.1.1(a), we have $d(\alpha) \leq d(\lambda\alpha) \leq d(\beta)$ and so $\mathcal{R} = \mathcal{J}$ in $KN(p,q)$. □

We note that the $\mathcal{R}$–classes on $KN(p,q)$ are exactly the subsemigroups $GS(p,\xi)$ with $p \geq \xi \geq q$ (and $p \neq q$ by supposition). As usual, we may regard $KN(p,q)/\mathcal{R}$ as a partially ordered set: given $\alpha, \beta \in KN(p,q)$, if R_α denotes the $\mathcal{R}$–class containing α, then

$$R_\alpha \leq R_\beta \quad \text{if and only if} \quad \alpha KN(p,q)^1 \subseteq \beta KN(p,q)^1.$$

From the Lemma above, $R_\alpha \leq R_\beta$ if and only if $d(\beta) \leq d(\alpha)$. Hence, $KN(p,q)/\mathcal{R}$ is a chain with $GS(p,q)$ and $GS(p,p)$ as its maximum and minimum element, respectively.

As observed in chapter 1, the ideals of $T(V)$ are well-known (compare [16] Vol. 2, section IX.9): they are precisely the sets $\{\alpha \in T(V) : r(\alpha) < \xi\}$ where $1 \leq \xi \leq p'$ and p' denotes the successor of p. We now determine the ideals of $KN(p,q)$.

Theorem 3.2.2. If $p > q$ then the ideals of $KN(p,q)$ are precisely the semigroups $KN(p,\xi)$ where $q \leq \xi \leq p$. In addition, each right ideal of $KN(p,q)$ is a two-sided ideal.

Proof. Let ξ be a cardinal such that $q \leq \xi \leq p$. To simplify notation, we write I_ξ instead of $KN(p,\xi)$. Clearly, $I_\xi = KN(p,q) \cap M_\xi$, hence I_ξ is an ideal of $KN(p,q)$ by Theorem 3.1.2.

Conversely, suppose I is an ideal of $KN(p,q)$ and choose $\gamma \in I$ with minimal defect ξ (possible since the cardinals are well-ordered). Then $d(\alpha) \geq \xi$ for every $\alpha \in I$ and so $I \subseteq I_\xi$. Let $\beta \in I_\xi$. Then $d(\beta) \geq \xi = d(\gamma)$ and so there exists $\mu \in KN(p,q)^1$ such

that $\beta = \gamma\mu$. Since $\gamma \in I$ and I is an ideal of $KN(p,q)$, it follows that $\beta \in I$ and hence $I = I_\xi$.

Given a right ideal R of $KN(p,q)$, choose $\varepsilon \in R$ with minimum defect. Let $\alpha \in R$ and $\lambda \in KN(p,q)$. We have $d(\lambda\alpha) \geq d(\alpha) \geq d(\varepsilon)$ and from Lemma 3.2.1 there exists $\gamma \in KN(p,q)$ such that $\lambda\alpha = \varepsilon\gamma$. Since $\varepsilon \in R$ and R is a right ideal of $KN(p,q)$, it follows that $\lambda\alpha \in R$. Hence R is an ideal of $KN(p,q)$. □

Clearly, the principal ideal generated by $\alpha \in KN(p,q)$ is $KN(p,d(\alpha))$. Moreover, we know that, for every cardinal ξ such that $q \leq \xi \leq p$, there exists some $\alpha \in KN(p,q)$ such that $d(\alpha) = \xi$. Hence, all ideals of $KN(p,q)$ are principal and form a chain under $\subseteq$, with the smallest being $KN(p,p) = GS(p,p)$ and the largest being $KN(p,q)$.

An ideal I of a semigroup S is said to be *reflective* if $ab \in I$ implies $ba \in I$. In [41], Sullivan studied the ideals in certain transformation semigroups defined on a set, and he characterised when they are prime or reflective. From Lemma 3.1.1(a), it is easy to see that every ideal of $KN(p,q)$ is prime and reflective.

As in [37] Theorem 2, we now consider $KN(p,\xi)/KN(p,\xi')$: that is, the Rees quotient semigroup where $q \leq \xi < p$. We may regard $KN(p,\xi)/KN(p,\xi')$ as the set $GS(p,\xi)$ of all $\alpha \in KN(p,q)$ with defect ξ together with a zero element. In fact, since $GS(p,\xi)$ is a semigroup under composition, and the product of non-zero $\alpha, \beta \in KN(p,\xi)/KN(p,\xi')$ equals the composition of α and β provided $\alpha\beta \notin KN(p,\xi')$, we conclude that $KN(p,\xi)/KN(p,\xi')$ is simply the semigroup $GS(p,\xi)$ with 0 adjoined. Since $GS(p,\xi)$ is right simple, it follows that $KN(p,\xi)/KN(p,\xi')$ is right 0–simple.

3.3. Left ideals of $KN(p,q)$

Recall that the right ideals of $KN(p,q)$ are precisely its ideals. The characterisation of the left ideals of $KN(p,q)$ is much harder but is similar to the characterisation of the left ideals of $GS(m,n)$ in chapter 2, section 2.4.

If A is a non-empty subset of $KN(p,q)$, we let $L_A^* = A \cup L_A$, where

$$L_A = \{\beta \in KN(p,q) : \operatorname{ran}\beta \subseteq \operatorname{ran}\alpha,\ \dim(\operatorname{ran}\alpha/\operatorname{ran}\beta) \geq q \ \text{ for some } \ \alpha \in A\}.$$

To see that L_A is non-empty, let $\alpha \in A$ and suppose $\{e_i\}$ is a basis for V. Write $e_i\alpha = a_i$ for each i. Since α is injective, $\{a_i\}$ is linearly independent and so it can be

expanded to a basis for V, say $\{a_i\} \dot{\cup} \{a_j\}$ with $|J| = d(\alpha) \geq q$. Since $|I| = p \geq d(\alpha)$, we can write $\{a_i\} = \{b_i\} \dot{\cup} \{b_j\}$ and define $\beta \in T(V)$ by

$$\beta = \begin{pmatrix} e_i \\ b_i \end{pmatrix}.$$

Then, $\beta \in KN(p,q)$ since it is one-to-one and $d(\beta) = \dim\langle b_j, a_j \rangle = d(\alpha) \geq q$. We have $\operatorname{ran} \beta \subseteq \operatorname{ran} \alpha$ and $\dim(\operatorname{ran} \alpha / \operatorname{ran} \beta) = d(\alpha) \geq q$. Thus $\beta \in L_A$ and so L_A is non-empty.

Theorem 3.3.1. If A is a non-empty subset of $KN(p,q)$, then L_A^* is a left ideal of $KN(p,q)$. Conversely, if I is a left ideal of $KN(p,q)$ then $I = L_I^*$. Furthermore, the principal left ideal generated by $\alpha \in KN(p,q)$ is $L_{\{\alpha\}}^*$.

Proof. Suppose A is a non-empty subset of $KN(p,q)$. Let $\alpha \in L_A^*$ and $\beta \in KN(p,q)$. Then $\beta\alpha \in KN(p,q)$ and $\operatorname{ran}(\beta\alpha) \subseteq \operatorname{ran} \alpha$. If $\{e_i\}$ is a basis for V, then $\{e_i\beta\}$ is a basis for $\operatorname{ran} \beta$ and $\{e_i\beta\alpha\}$ is a basis for $\operatorname{ran}(\beta\alpha)$. Expand $\{e_i\beta\}$ to a basis $\{e_i\beta, e_j\}$ for V with $|J| = d(\beta) \geq q$. Then $\{e_i\beta\alpha, e_j\alpha\}$ is a basis for $\operatorname{ran} \alpha$ and so $\dim(\operatorname{ran} \alpha / \operatorname{ran}(\beta\alpha)) = \dim\langle e_j\alpha \rangle = |J| \geq q$. If $\alpha \in A$, then $\beta\alpha \in L_A$. If $\alpha \in L_A$, then $\operatorname{ran} \alpha \subseteq \operatorname{ran} \gamma$ and $\dim(\operatorname{ran} \gamma / \operatorname{ran} \alpha) \geq q$ for some $\gamma \in A$. Therefore $\operatorname{ran}(\beta\alpha) \subseteq \operatorname{ran} \alpha \subseteq \operatorname{ran} \gamma$ and $\dim(\operatorname{ran} \gamma / \operatorname{ran}(\beta\alpha)) \geq \dim(\operatorname{ran} \gamma / \operatorname{ran} \alpha) \geq q$. Thus, $\beta\alpha \in L_A$. Hence L_A^* is a left ideal of $KN(p,q)$.

Suppose now that I is a left ideal of $KN(p,q)$ and let $\beta \in L_I$. Then $\operatorname{ran} \beta \subseteq \operatorname{ran} \alpha$ and $\dim(\operatorname{ran} \alpha / \operatorname{ran} \beta) \geq q$ for some $\alpha \in I$. Let $\{e_i\}$ be a basis for V and write $e_i\beta = b_i$ for each i. Since β is one-to-one, $\{b_i\}$ is a basis for $\operatorname{ran} \beta$ and so it can be expanded to a basis $\{b_i\} \dot{\cup} \{a_k\}$ for $\operatorname{ran} \alpha$ with $|K| = \dim(\operatorname{ran} \alpha / \operatorname{ran} \beta)$. Since α is injective, we can choose unique f_i and f_k in V such that $f_i\alpha = b_i$ and $f_k\alpha = a_k$, and $\{f_i\} \dot{\cup} \{f_k\}$ is a basis for V. We then have

$$\alpha = \begin{pmatrix} f_i & f_k \\ b_i & a_k \end{pmatrix}, \quad \beta = \begin{pmatrix} e_i \\ b_i \end{pmatrix}.$$

Define $\lambda \in T(V)$ by

$$\lambda = \begin{pmatrix} e_i \\ f_i \end{pmatrix}.$$

Since λ is one-to-one and $d(\lambda) = \dim\langle f_k \rangle = |K| \geq q$, we know $\lambda \in KN(p,q)$. We also have $\beta = \lambda\alpha$ with $\lambda \in KN(p,q)$ and $\alpha \in I$. Hence $\beta \in I$, a left ideal of $KN(p,q)$, and so $L_I \subseteq I$. Therefore $L_I^* = I$.

Finally we observe that the principal left ideal generated by $\alpha \in KN(p,q)$ is $L^*_{\{\alpha\}}$. Clearly $KN(p,q)^1\alpha \subseteq L^*_{\{\alpha\}}$ since $\alpha \in L^*_{\{\alpha\}}$ and $L^*_{\{\alpha\}}$ is a left ideal of $KN(p,q)$. Conversely, the argument in the paragraph above shows that if $\alpha \in A \subseteq KN(p,q)$ and $\beta \in L_A$ then $\beta = \lambda\alpha$ for some $\lambda \in KN(p,q)$. Hence $L_{\{\alpha\}} \subseteq KN(p,q)\alpha$ and so the result follows. □

In chapter 2, section 2.4, we closely examined the poset of left ideals in $GS(m,n)$ for $m \geq n \geq \aleph_0$. Similar results can be proved for the poset of left ideals in $KN(p,q)$ for $p > q \geq \aleph_0$. However, here we simply note some key differences between the right and the left ideals of $KN(p,q)$ for $p > q$.

Remark 3.3.2. The left ideals of $KN(p,q)$ do not form a chain under $\subseteq$. To see this, suppose $\{e_i\}$ is a basis for V and let $\alpha \in KN(p,q)$. Write $e_i\alpha = a_i$ for each i and expand $\{a_i\}$ to a basis $\{a_i\} \dot\cup \{a_j\}$ for V, with $|J| = d(\alpha) \geq q$. Let $|K| < q$ and write $\{e_i\} = \{f_i\} \dot\cup \{f_k\}$ and $\{a_j\} = \{b_j\} \dot\cup \{b_k\}$. Define β in $T(V)$ by

$$\beta = \begin{pmatrix} f_i & f_k \\ a_i & b_k \end{pmatrix}.$$

Since β is one-to-one and $d(\beta) = \dim\langle b_j \rangle = d(\alpha) \geq q$, β is in $KN(p,q)$. Moreover, $\operatorname{ran}\alpha \subsetneq \operatorname{ran}\beta$ and $\dim(\operatorname{ran}\beta/\operatorname{ran}\alpha) = |K| < q$. Hence, $\alpha \notin L^*_{\{\beta\}}$ and $\beta \notin L^*_{\{\alpha\}}$. Therefore, $L^*_{\{\alpha\}} \not\subseteq L^*_{\{\beta\}}$ and $L^*_{\{\beta\}} \not\subseteq L^*_{\{\alpha\}}$.

Remark 3.3.3. $KN(p,q)$ has no minimal left ideals. To see this, let L_Y^* be a left ideal of $KN(p,q)$ with $\emptyset \neq Y \subseteq KN(p,q)$ and let $\alpha \in Y$. Then, $L^*_{\{\alpha\}} \subseteq L_Y^*$. Let $\{e_i\}$ be a basis for V and write $e_i\alpha = a_i$ for each i. We can expand $\{a_i\}$ to a basis for V, say $\{a_i\} \dot\cup \{a_j\}$ with $|J| = d(\alpha) \geq q$. Write $\{a_i\}$ as $\{b_i\} \dot\cup \{b_j\}$ and define $\gamma \in T(V)$ by

$$\gamma = \begin{pmatrix} e_i \\ b_i \end{pmatrix}.$$

Then $\gamma \in KN(p,q)$, $\operatorname{ran}\gamma \subseteq \operatorname{ran}\alpha$ and $\dim(\operatorname{ran}\alpha/\operatorname{ran}\gamma) = \dim\langle b_j \rangle = d(\alpha) \geq q$. Hence $\gamma \in L^*_{\{\alpha\}}$ and so $L^*_{\{\gamma\}} \subseteq L^*_{\{\alpha\}}$. Since $\operatorname{ran}\alpha \not\subseteq \operatorname{ran}\gamma$, it follows that $\alpha \notin L^*_{\{\gamma\}}$.

Thus, $L^*_{\{\gamma\}} \subsetneq L^*_{\{\alpha\}} \subseteq L^*_Y$. Hence, given any left ideal, we can find another left ideal properly contained in it and so there are no minimal left ideals of $KN(p,q)$. We recall that, in Theorem 3.3.1, we determined all principal left ideals of $KN(p,q)$; and so the argument above also shows that $KN(p,q)$ has no minimal principal left ideals.

Remark 3.3.4. To see that $KN(p,q)$ has no maximal (proper) left ideals, we let L^*_A be a proper left ideal of $KN(p,q)$, with $\emptyset \neq A \subseteq KN(p,q)$, and we suppose that, for each $\alpha \in KN(p,q)$, there exists $\lambda \in A$ such that $\operatorname{ran}\alpha \subseteq \operatorname{ran}\lambda$. Let $\{e_i\}$ be a basis for V and let $\beta \in KN(p,q)$. Write $e_i\beta = b_i$ for each i and expand $\{b_i\}$ to a basis $\{b_i\}\dot\cup\{b_j\}$ for V, with $|J| = d(\beta) \geq q$. Now write $\{b_j\} = \{c_j\}\dot\cup\{d_j\}$ and let $\{f_i\} = \{b_i\}\dot\cup\{c_j\}$. Define γ in $T(V)$ by

$$\gamma = \begin{pmatrix} e_i \\ f_i \end{pmatrix}.$$

Then $\gamma \in KN(p,q)$ and so there exists $\lambda \in A$ such that $\operatorname{ran}\gamma \subseteq \operatorname{ran}\lambda$. Hence $\operatorname{ran}\beta \subseteq \operatorname{ran}\gamma \subseteq \operatorname{ran}\lambda$ and $\dim(\operatorname{ran}\lambda/\operatorname{ran}\beta) \geq \dim(\operatorname{ran}\gamma/\operatorname{ran}\beta) = |J| \geq q$. Thus $\beta \in L_A$ and this implies $L^*_A = KN(p,q)$ which contradicts our choice of a proper left ideal.

Consequently, there exists $\alpha \in KN(p,q)$ such that $\operatorname{ran}\alpha \not\subseteq \operatorname{ran}\lambda$ for all $\lambda \in A$. Let $B = A \cup \{\alpha\}$. Clearly $L^*_A \subseteq L^*_B$ and $\alpha \notin L^*_A$. Hence, $L^*_A \subsetneq L^*_B$. Now suppose $\{e_i\}$ is a basis for V and write $e_i\alpha = a_i$ for each i. We can expand $\{a_i\}$ to a basis $\{a_i\}\dot\cup\{a_j\}$ for V, with $|J| = d(\alpha) \geq q$. Write $\{a_j\} = \{u_j\}\dot\cup\{v_j\}$ and let $\{b_i\} = \{a_i\}\dot\cup\{u_j\}$. Define $\beta \in T(V)$ by

$$\beta = \begin{pmatrix} e_i \\ b_i \end{pmatrix}.$$

Then $\beta \in KN(p,q)$ and $\operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$, so $\beta \notin A$. Since $\alpha \neq \beta$, $\beta \notin B$. If $\beta \in L_B$, then there exists some $\delta \in B$ such that $\operatorname{ran}\beta \subseteq \operatorname{ran}\delta$ and $\dim(\operatorname{ran}\delta/\operatorname{ran}\beta) \geq q$. Since $\operatorname{ran}\beta \not\subseteq \operatorname{ran}\alpha$, we conclude that $\delta \in A$. Thus $\operatorname{ran}\alpha \subseteq \operatorname{ran}\beta \subseteq \operatorname{ran}\delta$ with $\delta \in A$, which contradicts our assumption on α and A. Therefore, $\beta \notin L_B$ and so $L^*_B \subsetneq KN(p,q)$. Hence L^*_B is a proper left ideal of $KN(p,q)$ that strictly contains L^*_A.

Remark 3.3.5. To see that $KN(p,q)$ has no maximal principal left ideals, let $\alpha \in KN(p,q)$ and suppose $\{e_i\}$ is a basis for V. Write $e_i\alpha = a_i$ for each i. We can expand

$\{a_i\}$ to a basis $\{a_i\} \dot{\cup} \{a_j\}$ for V, with $|J| = d(\alpha) \geq q$. Write $\{e_i\} = \{f_i\} \dot{\cup} \{f_j\}$ and $\{a_j\} = \{b_j\} \dot{\cup} \{c_j\}$. Define β in $T(V)$ by

$$\beta = \begin{pmatrix} f_i & f_j \\ a_i & b_j \end{pmatrix}.$$

Then $\beta \in KN(p,q)$ and $\alpha \in L^*_{\{\beta\}}$. Hence $L^*_{\{\alpha\}} \subseteq L^*_{\{\beta\}}$. Since $\beta \notin L^*_{\{\alpha\}}$, we have $L^*_{\{\alpha\}} \subsetneq L^*_{\{\beta\}}$. Hence, by Theorem 3.3.1, there are no maximal principal left ideals in $KN(p,q)$.

3.4. $KN(p,q)$ as a model for some algebraic semigroups

In [5] Vol. 1, Lemma 1.0, the authors show that any right cancellative semigroup S having no idempotents (except possibly an identity element) can be embedded in $M(X)$, the semigroup of one-to-one transformations on some set X, with $|X| = |S|$; and in [5] Vol. 2, Theorem 8.5, they show that any right simple, right cancellative semigroup S without idempotents can be embedded in a Baer-Levi semigroup of type (p,p), where $p = |S|$. In chapter 2, Theorem 2.3.10, we showed that S can also be embedded in some linear Baer-Levi semigroup $GS(p,p)$.

In this section, we prove that any right cancellative semigroup S without idempotents whose right ideals form a chain and which has no maximal principal left ideals can be embedded in some $KN(p,q)$, where $p = |S|$. This result is due to Sullivan (personal communication). We have already seen that any semigroup $KN(p,q)$ with $p > q \geq \aleph_0$ has such properties (see Theorem 3.2.2, Remarks 3.3.3 and 3.3.5 and the remark before Lemma 3.1.1). Note also that, from [5] Vol. 2, p. 85, Exercise 3, the Baer-Levi semigroups of the two types (defined on sets and on vector spaces) have no maximal principal left ideals and, since they are right simple, they also have no proper right ideals. Hence Sullivan's result extends previous work on Baer-Levi semigroups in a natural way. For the embedding theorem, we need a preliminary Lemma (compare [5] Vol. 2, Lemma 8.3).

Lemma 3.4.1. Let S be a right cancellative semigroup without idempotents whose right ideals form a chain. Then the equation $xy = y$ cannot hold for any two elements $x, y \in S$.

Proof. Suppose $xy = y$ for some distinct $x, y \in S$. Since the right ideals of S form a chain, we have $xS^1 \subseteq yS^1$ or $yS^1 \subseteq xS^1$. If the latter holds, then $y = xb$ for some $b \in S$ and so $xb = y = xy = x(xb) = x^2b$. Since S is right cancellative, this implies $x = x^2$, contrary to the assumption that S has no idempotents. Then, $xS^1 \subseteq yS^1$ and hence there exists a in S such that $x = ya$. Therefore, $y = xy = (ya)y = yay$ and ya is an idempotent, a contradiction. □

Theorem 3.4.2. Let S be a right cancellative semigroup without idempotents whose right ideals form a chain and which has no maximal principal left ideals. Then S can be embedded in some $KN(p, q)$, where $p = |S|$.

Proof. Let $|S| = p$ and write $S^1 = \{a_i\}$, with $|I| = p$. Note that S is infinite, since S has no idempotents. Let F be any field and let F_i be a copy of F for each $i \in I$. As in chapter 2, Theorem 2.3.10, we let V be the vector space $\sum F_i$ over F whose basis can be identified in a natural way with $\{a_i\}$: that is, $\sum F_i$ is the set of all $(r_i)_{i \in I}$ where $r_i \in F_i$ and at most finitely many r_i are non-zero.

Since S is right cancellative, the extended right regular representation of S is a faithful representation of S as a semigroup of one-to-one mappings of S^1 into itself. Let $x \in S$. Then x is represented by $\rho_x : S^1 \to S^1, a_i \mapsto a_i x$, which is a one-to-one mapping of the basis $\{a_i\}$ into itself. Hence ρ_x can be extended by linearity to a one-to-one linear map $V \to V$.

We assert that $S^1 \setminus S^1x$ is infinite. For, since there are no maximal principal left ideals of S, given $a \in S$, there exists $b \in S$ such that $S^1a \subsetneq S^1b \subsetneq S^1$ and so $\emptyset \neq S^1 \setminus S^1b \subsetneq S^1 \setminus S^1a$. Therefore, if $q = \min\{|S^1 \setminus S^1a| : a \in S\}$, then $q \geq \aleph_0$ and codim $\rho_x \geq q$ and hence $\rho_x \in KN(p, q)$. The faithfulness of the extended right regular representation implies that S is embedded in $KN(p, q)$. □

3.5. Quasi-ideals and bi-ideals of $KN(p, q)$

In this section, we describe the quasi-ideals and the bi-ideals of $KN(p, q)$ for $p > q$ and, using this, we determine when the principal quasi-ideal and the principal bi-ideal generated by $\{\alpha\}$ in $KN(p, q)$ coincide.

Recall that every right and every left ideal of a semigroup S is a quasi-ideal, and

every quasi-ideal of S is a bi-ideal (see remark before Theorem 3.1.4). By [5] Vol. 1, p. 85, Exercise 17, a subset Q of a semigroup S is a quasi-ideal of S if and only if it is a non-empty intersection of a right ideal and a left ideal of S. For, if $Q = R \cap L \neq \emptyset$, where R and L are a right and a left ideal of S, respectively, then

$$QS \cap SQ \subseteq RS \cap SL \subseteq R \cap L = Q.$$

Conversely, if Q is a quasi-ideal of S then

$$Q \subseteq (Q \cup QS) \cap (Q \cup SQ) = Q \cup (Q \cap SQ) \cup (QS \cap Q) \cup (QS \cap SQ).$$

But $QS \cap SQ \subseteq Q$, hence $Q = QS^1 \cap S^1Q$ and the assertion follows.

In sections 3.2 and 3.3, we characterised the right and the left ideals of $KN(p,q)$ and hence we may conclude that every quasi-ideal of $KN(p,q)$ is a left ideal (since every right ideal of $KN(p,q)$ is an ideal and the intersection of an ideal and a left ideal is still a left ideal). Since the converse is always true, the quasi-ideals of $KN(p,q)$ are exactly its left ideals.

From what we have just seen, a quasi-ideal Q of $KN(p,q)$, with $p > q$, is of the form $I_\xi \cap L_A^*$ where ξ is a cardinal such that $p \geq \xi \geq q$, I_ξ denotes the semigroup $KN(p,\xi)$ (as in the proof of Theorem 3.2.2) and A is a non-empty subset of $KN(p,q)$. Note that $I_\xi \cap L_A^*$ is always non-empty: for example, choose $\beta \in A$ such that $e_i\beta = a_i$ where $\{e_i\}$ is a basis for V, and write $\{a_i\} = \{b_i\} \dot\cup \{b_j\}$ with $|J| = \xi \geq q$. Define α in $T(V)$ by

$$\alpha = \begin{pmatrix} e_i \\ b_i \end{pmatrix}.$$

Since α is one-to-one and $d(\alpha) = d(\beta) + \xi \geq \xi \geq q$, it follows that $\alpha \in I_\xi$. Moreover $\operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$ and $\dim(\operatorname{ran}\beta / \operatorname{ran}\alpha) = \xi \geq q$. Hence $\alpha \in I_\xi \cap L_A^*$.

Lemma 3.5.1. Suppose $p > q$. If ξ is a cardinal such that $p \geq \xi \geq q$ and A is a non-empty subset of $KN(p,q)$, and if we write $A = C \dot\cup D$ where $C = \{\alpha \in A : d(\alpha) \geq \xi\}$, then $I_\xi \cap L_A^*$ equals

$$L_C^* \cup \{\beta \in KN(p,q) : \operatorname{ran}\beta \subseteq \operatorname{ran}\gamma \text{ and } \dim(\operatorname{ran}\gamma / \operatorname{ran}\beta) \geq \xi \text{ for some } \gamma \in D\}.$$

Proof. Since $A = C \cup D$, we have $L^*_A = L^*_C \cup L^*_D$. Therefore, $I_\xi \cap L^*_A = (I_\xi \cap L^*_C) \cup (I_\xi \cap L^*_D)$. Clearly $C \subseteq I_\xi$, hence $L^*_C \subseteq L^*_{I_\xi} = I_\xi$.

Since $I_\xi \cap D = \emptyset$, it follows that $I_\xi \cap L^*_D = I_\xi \cap L_D$. If $\beta \in I_\xi \cap L_D$, then $d(\beta) \geq \xi$ and ran $\beta \subseteq$ ran γ, where $\dim(\text{ran } \gamma / \text{ran } \beta) \geq q$ for some $\gamma \in D$. Since $d(\gamma) < \xi$ and $d(\beta) \geq \xi$, we have that $\dim(\text{ran } \gamma / \text{ran } \beta) \geq \xi$. Hence, $I_\xi \cap L_D$ is contained in

$$\{\beta \in KN(p,q) : \text{ran } \beta \subseteq \text{ran } \gamma \text{ and } \dim(\text{ran } \gamma / \text{ran } \beta) \geq \xi \text{ for some } \gamma \in D\}.$$

Conversely, if $\beta \in KN(p,q)$ is such that ran $\beta \subseteq$ ran γ and $\dim(\text{ran } \gamma / \text{ran } \beta) \geq \xi$ for some $\gamma \in D$, then $d(\beta) = \dim(V / \text{ran } \beta) \geq \dim(\text{ran } \gamma / \text{ran } \beta) \geq \xi$ and $\beta \in L_D$, since $\xi \geq q$. Therefore, $\beta \in I_\xi \cap L_D$ and the desired equality follows. □

For each cardinal ξ such that $p \geq \xi \geq q$ and each non-empty subset A of $KN(p,q)$, let $C_A = \{\alpha \in A : d(\alpha) \geq \xi\}$ and $D_A = A \setminus C$ and let $Q(\xi, A)$ be the set

$$L^*_{C_A} \cup \{\beta \in KN(p,q) : \text{ran } \beta \subseteq \text{ran } \gamma \text{ and } \dim(\text{ran } \gamma / \text{ran } \beta) \geq \xi \text{ for some } \gamma \in D_A\}.$$

The next result shows that the quasi-ideals of $KN(p,q)$ are precisely the sets $Q(\xi, A)$.

Theorem 3.5.2. Suppose $p > q$. If ξ is a cardinal such that $p \geq \xi \geq q$ and A is a non-empty subset of $KN(p,q)$, then $Q(\xi, A)$ is a quasi-ideal of $KN(p,q)$. Conversely, any quasi-ideal of $KN(p,q)$ equals some $Q(\xi, A)$ with $p \geq \xi \geq q$ and $\emptyset \neq A \subseteq KN(p,q)$.

Proof. Let ξ be a cardinal such that $p \geq \xi \geq q$ and let A be a non-empty subset of $KN(p,q)$. Write $A = C_A \dot\cup D_A$ where $C_A = \{\alpha \in A : d(\alpha) \geq \xi\}$. From Lemma 3.5.1, $Q(\xi, A) = I_\xi \cap L^*_A$ and so $Q(\xi, A)$ is the intersection of a right and a left ideal of $KN(p,q)$. Therefore, it is a quasi-ideal of $KN(p,q)$.

Conversely, if Q is a quasi-ideal of $KN(p,q)$ then it is the intersection of a right ideal R and a left ideal M of $KN(p,q)$. Choose $\alpha \in R$ with minimum defect and write $d(\alpha) = \xi$. Then $R = I_\xi$ and $M = L^*_M$, hence $Q = I_\xi \cap L^*_M = Q(\xi, M)$. □

Remark 3.5.3. Next we show that the principal quasi-ideals are the principal left ideals of $KN(p,q)$. To see this, let $\alpha \in KN(p,q)$. We have already observed (see before Theorem 3.1.4) that $(\alpha)_Q = \alpha KN(p,q)^1 \cap KN(p,q)^1 \alpha$. From Theorem 3.2.2 and the remark after its proof, $\alpha KN(p,q)^1 = I_{d(\alpha)}$; and $KN(p,q)^1 \alpha = L^*_{\{\alpha\}}$ by

Theorem 3.3.1. Hence $(\alpha)_Q = I_{d(\alpha)} \cap L^*_{\{\alpha\}} = L^*_{\{\alpha\}}$, since $\{\alpha\} \subseteq I_{d(\alpha)}$. However, note that no principal left ideal $L^*_{\{\alpha\}}$ of $KN(p,q)$ is an ideal: otherwise, from Theorem 3.2.2, there exists a cardinal ξ such that $p \geq \xi \geq q$ and $L^*_{\{\alpha\}} = I_\xi$. This is a contradiction since I_ξ contains linear transformations whose range is not a subset of ran α. Therefore, no principal quasi-ideal of $KN(p,q)$ is a two-sided ideal.

By [5] Vol. 1, p. 85, Exercise 18, a subset B of a semigroup S is a bi-ideal of S if and only if $RL \subseteq B \subseteq R \cap L$ for some right ideal R and some left ideal L of S. For, if the latter holds then

$$BSB \subseteq (R \cap L)S(R \cap L) \subseteq RS.L \subseteq RL \subseteq B.$$

Conversely, if B is a bi-ideal of S, then it is not difficult to see that $RL \subseteq B \subseteq R \cap L$ where $R = BS^1$ and $L = S^1B$.

Lemma 3.5.4. Suppose $p > q$. Let ξ be a cardinal such that $p \geq \xi \geq q$ and let A be a non-empty subset of $KN(p,q)$. Then $I_\xi L^*_A$ equals

$$\{\beta \in KN(p,q) : \text{ran } \beta \subseteq \text{ran } \alpha \quad \text{and} \quad \dim(\text{ran } \alpha / \text{ran } \beta) \geq \xi \text{ for some } \alpha \in A\}. \tag{3.5.1}$$

Proof. It is not difficult to see that the set $I_\xi L^*_A$ is a subset of the set in (3.5.1). Now let $\beta \in KN(p,q)$ be such that ran $\beta \subseteq$ ran α and $\dim(\text{ran } \alpha / \text{ran } \beta) \geq \xi$ for some $\alpha \in A$. Suppose $\{e_i\}$ is a basis for V and write $e_i\beta = b_i$ for each i. Expand $\{b_i\}$ to a basis for ran α, say $\{b_i\} \dot\cup \{a_j\}$, with $|J| = \dim(\text{ran } \alpha / \text{ran } \beta) \geq \xi$. Since α is one-to-one, there exist unique f_i and f_j in V such that $f_i\alpha = b_i$ and $f_j\alpha = a_j$. Moreover, $\{f_i\} \dot\cup \{f_j\}$ is a basis for V. Define λ in $T(V)$ by

$$\lambda = \begin{pmatrix} e_i \\ f_i \end{pmatrix}.$$

Then λ is one-to-one and $d(\lambda) = |J| \geq \xi$, hence $\lambda \in I_\xi$. Since $\beta = \lambda\alpha$, we have $\beta \in I_\xi A \subseteq I_\xi L^*_A$ and the desired equality follows. □

For each cardinal ξ such that $p \geq \xi \geq q$ and each non-empty subset A of $KN(p,q)$, let $B(\xi, A)$ denote the set defined in (3.5.1). The following result characterises the bi-ideals of $KN(p,q)$.

Theorem 3.5.5. Suppose $p > q$ and ξ is a cardinal such that $p \geq \xi \geq q$. Let A be a non-empty subset of $KN(p,q)$ and write $A = C_A \dot{\cup} D_A$, where $C_A = \{\alpha \in A : d(\alpha) \geq \xi\}$. Then, for each subset M of $C_A \cup K_A$, where K_A equals

$$\{\beta \in KN(p,q) : \text{ran } \beta \subseteq \text{ran } \delta \text{ and } q \leq \dim(\text{ran } \delta / \text{ran } \beta) < \xi \text{ for some } \delta \in C_A\},$$

$B(\xi, A) \cup M$ is a bi-ideal of $KN(p,q)$. Conversely, any bi-ideal of $KN(p,q)$ is of the form $B(\xi, A) \cup M$ for some cardinal ξ, some non-empty subset A of $KN(p,q)$ and some subset M of $C_A \cup K_A$, as defined above.

Proof. Let ξ be a cardinal such that $p \geq \xi \geq q$ and let A be a non-empty subset of $KN(p,q)$. Write $A = C_A \dot{\cup} D_A$ where $C_A = \{\alpha \in A : d(\alpha) \geq \xi\}$ and suppose M is a subset of $C_A \cup K_A$. Then,

$$B(\xi, A) \subseteq B(\xi, A) \cup M \subseteq Q(\xi, A).$$

From Lemmas 3.5.1 and 3.5.4, we have $I_\xi L_A^* \subseteq B(\xi, A) \cup M \subseteq I_\xi \cap L_A^*$, where I_ξ is a right ideal of $KN(p,q)$ and L_A^* is a left ideal. Hence, $B(\xi, A) \cup M$ is a bi-ideal of $KN(p,q)$.

Conversely, let B be a bi-ideal of $KN(p,q) = KN$, say. Then there exist a right ideal R and a left ideal L of KN such that $RL \subseteq B \subseteq R \cap L$. Choose $\alpha \in R$ with minimum defect and write $d(\alpha) = \xi$. Since $L = L_L^*$, we have $RL = I_\xi L_L^* = B(\xi, L)$ and $R \cap L = I_\xi \cap L_L^* = Q(\xi, L) = B(\xi, L) \cup C_L \cup K_L$, where $C_L = \{\alpha \in L : d(\alpha) \geq \xi\}$ and $K_L = \{\beta \in KN : \text{ran } \beta \subseteq \text{ran } \delta \text{ and } q \leq \dim(\text{ran } \delta / \text{ran } \beta) < \xi \text{ for some } \delta \in C_L\}$, and the result follows. □

We now show that the principal bi-ideal $(\alpha)_B$ of $KN(p,q)$ is a principal quasi-ideal of $KN(p,q)$ if and only if $d(\alpha) = q$. To do this, let $\alpha \in KN(p,q)$ and write $d(\alpha) = \xi$. We have already noted in a remark before Theorem 3.1.4 that $(\alpha)_B = \alpha KN(p,q)^1 \alpha \cup \{\alpha\}$. Now we assert that $(\alpha)_B = I_\xi L_{\{\alpha\}}^* \cup \{\alpha\}$. For, if $\beta \in I_\xi$ and $\gamma \in L_{\{\alpha\}}^* = L_{\{\alpha\}} \cup \{\alpha\}$, we know that $d(\beta) \geq \xi = d(\alpha)$, and so Lemma 3.2.1 implies that $\beta = \alpha\mu$ for some $\mu \in KN(p,q)^1$. Also, by Theorem 3.3.1 we know $L_{\{\alpha\}}^*$ is a left ideal, hence $\mu\gamma \in L_{\{\alpha\}}^* = KN(p,q)^1\alpha$ and this implies $\beta\gamma = \alpha\mu\gamma \in \alpha KN(p,q)^1\alpha$. Therefore $I_\xi L_{\{\alpha\}}^* \subseteq (\alpha)_B$ and, since $(\alpha)_B$ contains $\{\alpha\}$, we have $I_\xi L_{\{\alpha\}}^* \cup \{\alpha\} \subseteq (\alpha)_B$. Clearly $\alpha . KN(p,q)^1\alpha \subseteq I_\xi . L_{\{\alpha\}}^*$ and so $(\alpha)_B \subseteq I_\xi L_{\{\alpha\}}^* \cup \{\alpha\}$. Hence, $(\alpha)_B = I_\xi L_{\{\alpha\}}^* \cup \{\alpha\}$.

Since every quasi-ideal of $KN(p,q)$ is a bi-ideal, then $(\alpha)_Q$ is a bi-ideal of $KN(p,q)$ which contains $\{\alpha\}$. Therefore, $(\alpha)_B \subseteq (\alpha)_Q$. Suppose $\xi > q$, let $\{e_i\}$ be a basis for V and write $e_i\alpha = a_i$ for each i. Now write $\{e_i\} = \{f_i\} \dot\cup \{f_r\}$ with $|R| = q$. Also write $\{a_i\} = \{c_i\} \dot\cup \{c_r\} \dot\cup \{d_r\}$ and define λ in $T(V)$ by

$$\lambda = \begin{pmatrix} f_i & f_r \\ c_i & c_r \end{pmatrix}.$$

Then λ is one-to-one, $d(\lambda) = d(\alpha) + q = \xi$, $\operatorname{ran}\lambda \subseteq \operatorname{ran}\alpha$ and $\dim(\operatorname{ran}\alpha/\operatorname{ran}\lambda) = \dim\langle d_r\rangle = q < \xi$. Hence $\lambda \in L^*_{\{\alpha\}}$. But $\lambda \notin (\alpha)_B$ by Lemma 3.5.4, so $(\alpha)_B \subsetneq L^*_{\{\alpha\}} = (\alpha)_Q$. On the other hand, if $\xi = q$ then, since $(\alpha)_B = I_\xi L^*_{\{\alpha\}} \cup \{\alpha\}$, we conclude from Lemma 3.5.4 that $(\alpha)_B = L^*_{\{\alpha\}}$ which equals $(\alpha)_Q$ by Remark 3.5.3.

We have proved the following result.

Theorem 3.5.6. Suppose $p > q$. Given $\alpha \in KN(p,q)$, $(\alpha)_B = (\alpha)_Q$ if and only if $d(\alpha) = q$.

Let $\alpha \in KN(p,q)$ with $\xi = d(\alpha)$ and suppose $\{e_i\}$ is a basis for V. Write $e_i\alpha = a_i$ for each i and expand $\{a_i\}$ to a basis $\{a_i\} \dot\cup \{a_j\}$ for V with $|J| = \xi$. For each $j_0 \in J$, write $\{a_i\} \dot\cup \{a_{j_0}\}$ as $\{b_{i,j_0}\}$ and define $\gamma_{j_0} \in T(V)$ by

$$\gamma_{j_0} = \begin{pmatrix} e_i \\ b_{i,j_0} \end{pmatrix}.$$

Clearly, $d(\gamma_{j_0}) = d(\alpha) = \xi$ and so $\gamma_{j_0} \in KN(p,q)$. From Lemma 3.5.4, $\gamma_{j_0} \notin (\alpha)_B$ and hence $(\alpha)_B \neq (\gamma_{j_0})_B$. Moreover, if $j \neq j'$, then $(\gamma_j)_B \neq (\gamma_{j'})_B$.

From [17] Proposition 1.2, it follows that if $\alpha \in KN(p,q)$ is such that $d(\alpha) > q$, then $(\alpha)_B$ is not a quasi-ideal. Thus, in this case $(\alpha)_B$ is also not an ideal of $KN(p,q)$ (since ideals are quasi-ideals). Therefore, from what we observed in the paragraph above, there are infinitely many bi-ideals of $KN(p,q)$ which are neither quasi-ideals nor ideals of $KN(p,q)$.

3.6. Maximal right simple subsemigroups of $KN(p,q)$

We have already observed in a remark after Theorem 3.1.3 that

$$KN(p,q) = \bigcup_{p \geq \xi \geq q} GS(p,\xi).$$

For every cardinal ξ such that $p \geq \xi \geq q$, $GS(p, \xi)$ is a right simple subsemigroup of $KN(p, q)$. The result that follows shows us that these are precisely the maximal right simple subsemigroups of $KN(p, q)$.

Theorem 3.6.1. Suppose $p > q$. If ξ is a cardinal such that $p \geq \xi \geq q$, then $GS(p, \xi)$ is a maximal right simple subsemigroup of $KN(p, q)$. Conversely, any maximal right simple subsemigroup of $KN(p, q)$ equals some $GS(p, \xi)$ with $p \geq \xi \geq q$.

Proof. Suppose $GS(p, \xi) \subseteq M \subseteq KN(p, q)$ with M a right simple subsemigroup of $KN(p, q)$. Since $p > q$, $KN(p, q)$ is not right simple and so $M \neq KN(p, q)$. Let $\alpha \in M$ and $\gamma \in GS(p, \xi)$. Then both α and γ are elements of M and, since this semigroup is right simple, there exist $\lambda, \mu \in M^1$ such that $\alpha = \gamma\lambda$ and $\gamma = \alpha\mu$. From Lemma 3.2.1, $d(\alpha) = d(\gamma) = \xi$. Thus $\alpha \in GS(p, \xi)$ and so $GS(p, \xi) = M$. Hence, $GS(p, \xi)$ is a maximal right simple subsemigroup of $KN(p, q)$.

Now suppose M is a maximal right simple subsemigroup of $KN(p, q)$ and let $\alpha \in M$. For every $\beta \in M$, there exist $\lambda, \mu \in M^1$ such that $\alpha = \beta\lambda$ and $\beta = \alpha\mu$. Therefore, α and β are $\mathcal{R}$–related in $KN(p, q)$ and so $d(\beta) = d(\alpha)$. Hence, $M \subseteq GS(p, d(\alpha))$. Since $GS(p, d(\alpha))$ is a right simple subsemigroup of $KN(p, q)$, and M is maximal in $KN(p, q)$, we conclude that $M = GS(p, d(\alpha))$. □

Chapter 4

Linear transformation semigroups with restricted cardinalities

Let X be an infinite set with cardinal p. A (total) transformation α of X is said to be *almost one-to-one* if the set $C(\alpha) = \bigcup\{y\alpha^{-1} : |y\alpha^{-1}| \geq 2\}$ is finite. By an *almost onto transformation of* X, we mean a transformation α of X such that $X \setminus X\alpha$ is finite. On the other hand, if $X \setminus X\alpha$ is infinite, we say α is *opposite onto*. In 2001, Kemprasit showed that $AM(X)$, the semigroup of all almost one-to-one transformations of X, and $AE(X)$, the semigroup of all almost onto transformations of X, do not belong to **BQ**, the class of all semigroups whose sets of bi-ideals and quasi-ideals coincide. And, one year later, she showed that $OE(X)$, the semigroup of all opposite onto transformations of X, is in **BQ**.

In this chapter, we examine related semigroups defined as follows. Let V be a vector space over a field F with dimension $p \geq \aleph_0$. For each cardinal q such that $\aleph_0 \leq q \leq p$, we denote by $AM(p,q)$ the set of all linear transformations from V into itself whose kernel has dimension smaller than q and similarly, we denote by $AE(p,q)$ the set of all linear transformations $\alpha : V \to V$ whose range has codimension smaller than q. Likewise, we let $OM(p,q)$ denote the set consisting of all linear transformations $\alpha : V \to V$ whose kernel has dimension at least q, and let $OE(p,q)$ denote the set consisting of all linear transformations from V into itself whose range has codimension at least q. In 2001, Kemprasit and Namnak showed that $OE(p,\aleph_0)$ is in **BQ** and one year later, they showed that $AM(p,\aleph_0)$ and $AE(p,\aleph_0)$ do not belong to **BQ**, but $OM(p,\aleph_0) \in$ **BQ**. In section 4.1, we generalise these results: we show that the sets defined above are subsemigroups of $T(V)$; we also show that $AM(p,q)$ and $AE(p,q)$ do not belong to **BQ**, while $OM(p,q)$ and $OE(p,q)$ are always in this class of semigroups. For each of the four semigroups defined, we characterise its regular elements; and using this, we determine its unique maximal regular subsemigroup. In section 4.2, we characterise the Green's relations and the ideals in $AM(p,q)$ and $AE(p,q)$, and in section 4.3 we do the same for $OM(p,q)$ and $OE(p,q)$.

According to Clifford and Preston [5] Vol. 1, p. 51, in 1955 Miller and Doss studied the

$\mathcal{D}$–structure of the full transformation semigroup $T(X)$ on an arbitrary non-empty set X. In particular, they determined all $\mathcal{D}$–classes and all $\mathcal{H}$–classes of $T(X)$. Also, Miller and Doss characterised the group $\mathcal{H}$–classes of $T(X)$ and showed they are isomorphic to a symmetric group $G(Y)$ for some $Y \subseteq X$. As observed by Clifford and Preston in [5] Vol. 1, p. 51, similar results hold for $T(V)$, where V is an arbitrary vector space over a field F. By analogy with their work, in section 4.4 we describe the $\mathcal{D}$–classes and the $\mathcal{H}$–classes of the four semigroups defined above, and we also determine their group $\mathcal{H}$–classes.

In the previous chapter, we determined all maximal right simple subsemigroups of $KN(p,q)$, the semigroup consisting of every injective element of $OE(p,q)$. In section 4.5, we show that $KN(p,q)$ is a maximal right cancellative subsemigroup of $OE(p,q)$ and we describe all maximal right simple subsemigroups of $AM(p,q)$.

In 1966, Howie characterised the elements of $\mathcal{E}(X)$, the semigroup generated by the non-identity idempotents of $T(X)$, and two decades later Reynolds and Sullivan showed that Howie's characterisation has an analogue for the linear case. Several years later, Sullivan determined the subsemigroup of $T(V)$ generated by all nilpotent linear transformations of V. In section 4.6, we use the work by Reynolds and Sullivan to describe products of idempotents in the four semigroups studied in this chapter; and also the work by Sullivan to describe products of nilpotents in them. All the work in this chapter (except for the results in section 4.4) will appear in [33].

4.1. Basic properties

Let V be a vector space over a field F with dimension $p \geq \aleph_0$. As before, let $M(V)$ denote the subsemigroup of $T(V)$ consisting of all one-to-one linear transformations, and let $E(V)$ denote the subsemigroup of $T(V)$ consisting of all onto linear transformations. For each cardinal q such that $\aleph_0 \leq q \leq p$, we write

$$AM(p,q) = \{\alpha \in T(V) : n(\alpha) < q\}; \tag{S1}$$

$$AE(p,q) = \{\alpha \in T(V) : d(\alpha) < q\}; \tag{S2}$$

$$OM(p,q) = \{\alpha \in T(V) : n(\alpha) \geq q\}; \tag{S3}$$

$$OE(p,q) = \{\alpha \in T(V) : d(\alpha) \geq q\}. \tag{S4}$$

Clearly, $M(V) \subseteq AM(p,q)$, $E(V) \subseteq AE(p,q)$ and $0 \in OM(p,q) \cap OE(p,q)$, where 0 denotes the zero map on V.

For every $\alpha, \beta \in T(V)$, we have $n(\alpha) \leq n(\alpha\beta)$ and $d(\beta) \leq d(\alpha\beta)$, since $\ker \alpha \subseteq \ker(\alpha\beta)$ and $\operatorname{ran}(\alpha\beta) \subseteq \operatorname{ran} \beta$. Thus, it is easy to see that the sets $OM(p,q)$ and $OE(p,q)$, as defined in $(S3)$ and $(S4)$, respectively, are subsemigroups of $T(V)$.

The fact that the sets $AM(p,q)$ and $AE(p,q)$ (as defined in $(S1)$ and $(S2)$, respectively) are semigroups follows from parts (a) and (b), respectively, of the following result, and our assumption that q is infinite. In effect, this result was proved by Namnak and Kemprasit in [35] pp. 217-218, but we include a brief proof for completeness. First, we observe that if $\alpha, \beta \in T(V)$, then

$$(\ker(\alpha\beta))\alpha = \operatorname{ran} \alpha \cap \ker \beta. \tag{4.1.1}$$

For, if $a \in \ker(\alpha\beta)$ then clearly $a\alpha \in \operatorname{ran} \alpha$ and $(a\alpha)\beta = a(\alpha\beta) = 0$. Conversely, if $a \in \operatorname{ran} \alpha \cap \ker \beta$ then $a\beta = 0$ and $a = b\alpha$ for some $b \in V$, hence $b(\alpha\beta) = (b\alpha)\beta = a\beta = 0$. Thus, $a = b\alpha$ with $b \in \ker(\alpha\beta)$.

Lemma 4.1.1. If $\alpha, \beta \in T(V)$ then

(a) $n(\alpha) \leq n(\alpha\beta) \leq n(\alpha) + n(\beta)$, and

(b) $d(\beta) \leq d(\alpha\beta) \leq d(\alpha) + d(\beta)$.

Proof. Let $\alpha, \beta \in T(V)$. If $\ker(\alpha\beta) = \ker \alpha \oplus \langle e_j \rangle$ then, by (4.1.1), $(\ker(\alpha\beta))\alpha = \langle e_j\alpha \rangle \subseteq \ker \beta$, so $|J| \leq n(\beta)$ and hence $n(\alpha\beta) = n(\alpha) + |J| \leq n(\alpha) + n(\beta)$. Now suppose $\operatorname{ran} \beta = \operatorname{ran}(\alpha\beta) \oplus \langle e_i \rangle$, then $d(\alpha\beta) = d(\beta) + |I|$, where $|I| = \dim(\operatorname{ran} \beta / \operatorname{ran}(\alpha\beta))$. Now, if $V = (\operatorname{ran} \alpha + \ker \beta) \oplus U$ then $d(\alpha) \geq \dim U$ and $\operatorname{ran} \beta = \operatorname{ran}(\alpha\beta) \oplus U\beta$ (if $w = v\alpha\beta = u\beta$ then $v\alpha - u \in \ker \beta$, so $u \in \operatorname{ran} \alpha + \ker \beta$ and this implies $u = 0$, so $w = 0$). Hence $\dim(\operatorname{ran} \beta / \operatorname{ran}(\alpha\beta)) = \dim(U\beta) \leq \dim U \leq \dim(V / \operatorname{ran} \alpha) = d(\alpha)$, and the result follows. □

Example 4.1.2. We note that $OM(p,q)$ and $OE(p,q)$ are semigroups even if q is finite, whereas q must be infinite (or 1) for $AM(p,q)$ and $AE(p,q)$ to be semigroups. For, suppose q is finite, $q \neq 1$, and let $\{e_i\} \dot\cup \{u_1, u_2, \ldots, u_q\}$ be a basis for V, with $|I| = p$.

Now define $\alpha, \beta \in T(V)$ by

$$\alpha = \begin{pmatrix} u_1 & u_2 & \dots & u_q & e_i \\ 0 & u_2 & \dots & u_q & e_i \end{pmatrix}, \qquad \beta = \begin{pmatrix} u_1 & u_2 & \dots & u_q & e_i \\ u_1 & 0 & \dots & 0 & e_i \end{pmatrix}.$$

Clearly, $n(\alpha) = d(\alpha) = 1$ and $n(\beta) = d(\beta) = q-1$, and so $\alpha, \beta \in AM(p,q) \cap AE(p,q)$. It is easy to see that $\ker(\alpha\beta) = \langle u_1, u_2, \dots, u_q \rangle$ and $V = \operatorname{ran}(\alpha\beta) \oplus \langle u_1, u_2, \dots, u_q \rangle$. Therefore, $n(\alpha\beta) = d(\alpha\beta) = q$ and hence $\alpha\beta \notin AM(p,q) \cup AE(p,q)$.

In chapter 3, Theorem 3.1.4, we showed that the semigroup $KN(p,q)$ consisting of all injective elements of $OE(p,q)$ belongs to the class **BQ** of all semigroups whose sets of bi-ideals and quasi-ideals coincide if and only if $p = q$. It is known that regular semigroups, right [left] simple semigroups and right [left] 0-simple semigroups are in **BQ** (see [18] Propositions 1.2 and 1.3).

In [35] Theorem 2.2, Namnak and Kemprasit showed that $AM(p, \aleph_0)$ does not belong to **BQ**. Likewise, we now decide whether the semigroup defined in $(S1)$ belongs to **BQ**. For this, we closely follow the argument in [35] Theorem 2.2.

Theorem 4.1.3. For any infinite cardinals $p \geq q$, the semigroup $AM(p,q)$ does not belong to **BQ**.

Proof. Suppose $\{e_i\}$ is a basis for V and write $\{e_i\} = \{f_i\} \dot\cup \{f_j\}$ with $|J| = q$. Now write $\{f_j\} = \{a_j\} \dot\cup \{b_k\}$ with $|K| < q$ and $\{a_j\} = \{g_j\} \dot\cup \{h_j\}$. Put $\{h_j\} \dot\cup \{b_k\} = \{c_j\}$ and define $\alpha, \beta \in T(V)$ by

$$\alpha = \begin{pmatrix} f_i & f_j \\ f_i & g_j \end{pmatrix}, \qquad \beta = \begin{pmatrix} f_i & f_j \\ f_i & a_j \end{pmatrix}.$$

Since $n(\alpha) = 0 = n(\beta)$, we have $\alpha, \beta \in AM(p,q)$. Now define $\gamma \in T(V)$ by

$$\gamma = \begin{pmatrix} f_i & g_j & h_j & b_k \\ f_i & a_j\alpha & h_j & b_k \end{pmatrix}.$$

Since $\{a_j\alpha\} \subseteq \{f_j\alpha\} = \{g_j\}$, it follows that γ is one-to-one and so $\gamma \in AM(p,q)$. Clearly, $\beta\alpha = \alpha\gamma$ and hence $\beta\alpha \in AM(p,q)\alpha \cap \alpha AM(p,q) = (\alpha)_Q$ (the intersection contains α since $AM(p,q)$ contains id_V).

Suppose $\beta\alpha \in (\alpha)_B$. Then, $\beta\alpha \in \alpha AM(p,q)\alpha \cup \{\alpha\}$ (again, note that $AM(p,q)$ contains id_V, so the first set in this union contains α^2). If $\beta\alpha = \alpha$ then, since α is one-to-one, $\beta = \mathrm{id}_V$, a contradiction. Thus, there exists $\lambda \in AM(p,q)$ such that $\beta\alpha = \alpha\lambda\alpha$. Since α is one-to-one, it follows that $\beta = \alpha\lambda$. Hence, $\langle f_i, a_j\rangle = \mathrm{ran}\ \beta = \mathrm{ran}\,(\alpha\lambda) = (\mathrm{ran}\ \alpha)\lambda = \langle f_i, g_j\rangle\lambda$ and so $V = \langle f_i, a_j, b_k\rangle = \langle f_i, g_j\rangle\lambda + \langle b_k\rangle$. For each j, $c_j\lambda \in V$, and so there exist $u_j \in \langle f_i, g_j\rangle$ and $v_j \in \langle b_k\rangle$ such that $c_j\lambda = u_j\lambda + v_j$. Then, $(c_j - u_j)\lambda = v_j \in \langle b_k\rangle$. Since $\{c_j\}\dot\cup\{f_i\}\dot\cup\{g_j\}$ is linearly independent, it follows that $\{c_j - u_j\}$ is also linearly independent and $c_r - u_r \neq c_s - u_s$ if $r \neq s$. Let $C = \langle c_j - u_j\rangle$. Then, $\dim C = q$ and $\mathrm{ran}\,(\lambda|C) \subseteq \langle b_k\rangle$. Hence, $\dim(\mathrm{ran}\,(\lambda|C)) < q$. Since $q = \dim C = \dim(\ker(\lambda|C)) + \dim(\mathrm{ran}\,(\lambda|C))$ by the Rank-Nullity Theorem, it follows that $\dim(\ker(\lambda|C)) = q$. But $\ker(\lambda|C) \subseteq \ker\lambda$ and so $n(\lambda) \geq n(\lambda|C) = q$, which contradicts the fact that $\lambda \in AM(p,q)$. Therefore, $\beta\alpha \notin (\alpha)_B$ and so $(\alpha)_Q \neq (\alpha)_B$. By [35] Corollary 1.5, $AM(p,q) \notin \mathbf{BQ}$. □

From a remark before Theorem 4.1.3, it follows that the semigroup $AM(p,q)$ is neither regular nor right simple or left simple, for any infinite cardinals p, q such that $p \geq q$. Hence, it is worth determining all regular elements in $AM(p,q)$. First, we assert that, given an idempotent ε in $T(V)$, then

$$V = \ker\varepsilon \oplus \mathrm{ran}\ \varepsilon.$$

For, clearly, if $v \in V$ then $v = (v - v\varepsilon) + v\varepsilon$, and $\varepsilon^2 = \varepsilon$ implies $(v - v\varepsilon)\varepsilon = v\varepsilon - v\varepsilon^2 = 0$, hence $v - v\varepsilon \in \ker\varepsilon$. Thus, $V = \ker\varepsilon + \mathrm{ran}\ \varepsilon$. If $a \in \ker\varepsilon \cap \mathrm{ran}\ \varepsilon$, then $a\varepsilon = 0$ and $a = b\varepsilon$ for some $b \in V$. Therefore $a = b\varepsilon = b\varepsilon^2 = (b\varepsilon)\varepsilon = a\varepsilon = 0$, so $V = \ker\varepsilon \oplus \mathrm{ran}\ \varepsilon$. Moreover, $\varepsilon|\ \mathrm{ran}\ \varepsilon = \mathrm{id}_{\mathrm{ran}\ \varepsilon}$: for, if $a = b\varepsilon$ then $a\varepsilon = b\varepsilon^2 = b\varepsilon = a$. Hence, the idempotents in $T(V)$ are exactly the elements of the form

$$\varepsilon = \begin{pmatrix} e_j & e_i \\ 0 & e_i \end{pmatrix}, \tag{4.1.2}$$

where $\{e_j\}\dot\cup\{e_i\}$ is a basis for V. In particular, $n(\varepsilon) = d(\varepsilon)$.

Theorem 4.1.4. Let $\alpha \in AM(p,q)$. Then, α is regular if and only if $\alpha \in AE(p,q)$. Consequently, $AM(p,q) \cap AE(p,q)$ is the largest regular subsemigroup of $AM(p,q)$.

Proof. Suppose $\alpha \in AM(p,q) \cap AE(p,q)$. Let $\{e_j\}$ be a basis for $\ker\alpha$ and expand it to a basis $\{e_j\}\dot\cup\{e_i\}$ for V. Now write $e_i\alpha = a_i$ for each i. Since $\{a_i\}$ is a basis

for ran α, it can be expanded to a basis for V, say $\{a_i\} \dot\cup \{a_k\}$. Define $\beta \in T(V)$ by

$$\beta = \begin{pmatrix} a_i & a_k \\ e_i & 0 \end{pmatrix}.$$

Clearly, $n(\beta) = d(\alpha) < q$ and $d(\beta) = n(\alpha) < q$, hence $\beta \in AM(p,q) \cap AE(p,q)$. Also, $\alpha = \alpha\beta\alpha$ and so α is regular in $AM(p,q)$. Conversely, suppose $\alpha = \alpha\beta\alpha$ for some $\beta \in AM(p,q)$. Then $\beta\alpha$ is an idempotent in $T(V)$. By the remark above, $V = \ker(\beta\alpha) \oplus \operatorname{ran}(\beta\alpha)$ and, since $AM(p,q)$ is closed, it follows that $q > n(\beta\alpha) = d(\beta\alpha) \geq d(\alpha)$. Therefore, $\alpha \in AE(p,q)$ as required.

Finally, given a regular subsemigroup S of $AM(p,q)$, we know it is contained in $AE(p,q)$, and so $S \subseteq AM(p,q) \cap AE(p,q)$. Thus, the latter is the largest regular subsemigroup of $AM(p,q)$. □

Similar results hold for the semigroup $AE(p,q)$, as we now proceed to show. In the proof of our next Theorem, we closely follow the argument in [35] Theorem 2.3, where Namnak and Kemprasit proved that $AE(p,\aleph_0)$ does not belong to **BQ**.

Theorem 4.1.5. For any infinite cardinals $p \geq q$, the semigroup $AE(p,q)$ does not belong to **BQ**.

Proof. Suppose $\{e_i\}$ is a basis for V and write $\{e_i\} = \{f_i\} \dot\cup \{h\}$. Now write $\{f_i\} = \{a_i\} \dot\cup \{b_i\}$ and define $\alpha, \beta \in T(V)$ by

$$\alpha = \begin{pmatrix} a_i & b_i & h \\ f_i & 0 & h \end{pmatrix}, \qquad \beta = \begin{pmatrix} a_i & b_i & h \\ a_i & b_i & 0 \end{pmatrix}.$$

Since $d(\alpha) = 0$ and $d(\beta) = \dim\langle h\rangle = 1 < q$, we have $\alpha, \beta \in AE(p,q)$. Also, $\alpha \neq \beta\alpha = \alpha\beta$ and so $\alpha\beta \in AE(p,q)\alpha \cap \alpha AE(p,q) = (\alpha)_Q$ (note that the intersection contains α, since $AE(p,q)$ contains id_V).

Now suppose $\alpha\beta \in (\alpha)_B = \alpha AE(p,q)\alpha \cup \{\alpha\}$ (again, note that $AE(p,q)$ contains id_V, and so the first set in this union contains α^2). Then, since $\alpha\beta \neq \alpha$, we know $\alpha\beta = \alpha\lambda\alpha$ for some $\lambda \in AE(p,q)$ and the surjectivity of α implies $\beta = \lambda\alpha$. Thus, $(h\lambda)\alpha = h(\lambda\alpha) = h\beta = 0$ and so $h\lambda \in \ker\alpha$. Hence, there exist a natural number n and scalars $x_1, \ldots, x_n$ such that

$$h\lambda = \sum_{r=1}^{n} x_r b_{i_r}. \tag{4.1.3}$$

Put $\{b_i\} \setminus \{b_{i_1}, \ldots, b_{i_n}\} = \{c_i\}$. We assert that $\{c_i + \operatorname{ran} \lambda\}$ is a linearly independent subset of $V/\operatorname{ran} \lambda$. Suppose $\sum y_i(c_i + \operatorname{ran} \lambda) = \operatorname{ran} \lambda$ for some scalars y_i. Then, $\sum y_i c_i \in \operatorname{ran} \lambda$ and so there exists some $u \in V$ such that $\sum y_i c_i = u\lambda$. Since $V = \langle a_i \rangle \oplus \langle b_i \rangle \oplus \langle h \rangle$, there exist scalars r_i and s, and a vector $v \in \langle a_i \rangle$, such that $u = v + \sum r_i b_i + sh$. Hence,

$$\sum y_i c_i = v\lambda + \sum r_i(b_i\lambda) + s(h\lambda). \tag{4.1.4}$$

Thus,

$$\sum y_i(c_i\alpha) = v(\lambda\alpha) + \sum r_i(b_i\lambda\alpha) + s(h\lambda\alpha).$$

Since $\ker \alpha = \langle b_i \rangle$, $\lambda\alpha = \beta$ and $\ker \beta = \langle h \rangle$, it follows that $0 = v\beta + \sum r_i(b_i\beta)$. That is, $v + \sum r_i b_i \in \ker \beta$ and, by our choice of bases, this implies $v = 0$ and $r_i = 0$ for each i. Thus, we can rewrite (4.1.4),

$$\sum y_i c_i = s(h\lambda).$$

From (4.1.3),

$$\sum y_i c_i = \sum_{r=1}^{n} (sx_r) b_{i_r}.$$

Since $\{c_i\} \dot{\cup} \{b_{i_1}, \ldots, b_{i_n}\}$ is linearly independent, it follows that $y_i = 0$ for each i. Hence, $\{c_i + \operatorname{ran} \lambda\}$ is linearly independent, and so $q > d(\lambda) = \dim(V/\operatorname{ran} \lambda) = p$, a contradiction. Therefore, $\alpha\beta \notin (\alpha)_B$ and so $(\alpha)_B \neq (\alpha)_Q$. Hence, by [35] Corollary 1.5, $AE(p,q) \notin \mathbf{BQ}$. □

From the previous Theorem, it follows that $AE(p,q)$ is neither regular nor right simple or left simple, for any infinite cardinals p and q such that $p \geq q$. In the next result, we determine all regular elements in $AE(p,q)$.

Theorem 4.1.6. Let $\alpha \in AE(p,q)$. Then, α is regular if and only if $\alpha \in AM(p,q)$. Consequently, $AM(p,q) \cap AE(p,q)$ is the largest regular subsemigroup of $AE(p,q)$.

Proof. By Theorem 4.1.4, if $\alpha \in AM(p,q) \cap AE(p,q)$ then $\alpha = \alpha\beta\alpha$ and $\beta = \beta\alpha\beta$ for some $\beta \in AM(p,q)$, and hence $\beta \in AE(p,q)$ (by Theorem 4.1.4 again). That is, every $\alpha \in AM(p,q) \cap AE(p,q)$ is a regular element of $AE(p,q)$. Conversely, suppose $\alpha \in AE(p,q)$ and $\alpha = \alpha\beta\alpha$ for some $\beta \in AE(p,q)$. Then $\alpha\beta$ is an idempotent

in $T(V)$, hence (by a remark before Theorem 4.1.4) $V = \ker(\alpha\beta) \oplus \operatorname{ran}(\alpha\beta)$ and, since $AE(p,q)$ is closed, it follows that $q > d(\alpha\beta) = n(\alpha\beta) \geq n(\alpha)$. Therefore, $\alpha \in AM(p,q)$ as required. Finally, it is easy to see that every regular subsemigroup of $AE(p,q)$ is contained in $AM(p,q) \cap AE(p,q)$, hence the latter is the largest regular subsemigroup of $AE(p,q)$. □

We have already remarked that regular semigroups, right [left] simple semigroups and right [left] 0-simple semigroups are in **BQ**. Sullivan (personal communication) extended this remark with the following result and its dual: it can be used to simplify some of the arguments in [18], [19] and [35].

Lemma 4.1.7. If S is a regular semigroup, then any right ideal R of S belongs to **BQ**.

Proof. Suppose S is a regular semigroup and let R be a right ideal of S. Let X be a non-empty subset of R. We know that $(X)_B \subseteq (X)_Q$ always. We assert that $(X)_Q \subseteq (X)_B$. Let $a \in RX \cap XR$. Then, there exist $b, c \in R$ and $s, t \in X$ such that $a = bs = tc$. Since S is regular, $s = sxs$ for some $x \in S$. Since R is a right ideal of S, $cx \in R$. Therefore, $a = bs = b(sxs) = (bs)xs = (tc)xs = t(cx)s \in XRX$. Hence, $RX \cap XR \subseteq XRX$ and so $(X)_Q = R^1X \cap XR^1 \subseteq XRX \cup X \cup X^2 = (X)_B$. Thus, $(X)_B = (X)_Q$ for every non-empty subset X of R and so $R \in \mathbf{BQ}$. □

By Lemma 4.1.1, $OM(p,q)$ and $OE(p,q)$ are a right and a left ideal of $T(V)$, respectively. Moreover, by [5] Vol. 1, p. 57, Exercise 6, the semigroup $T(V)$ is regular. Hence, by the above result and its dual, $OM(p,q)$ and $OE(p,q)$ are always in **BQ**. In passing, we observe that in [18] Theorem 2.1, Kemprasit showed that $OE(p, \aleph_0)$ is in **BQ**; in the same year, Namnak and Kemprasit proved in [35] Theorem 3.4 that $OM(p, \aleph_0)$ is also in **BQ**.

In [36], Namnak and Kemprasit considered the semigroup $OM(p, \aleph_0) \cap OE(p, \aleph_0)$, and they showed that this is a regular subsemigroup of $T(V)$ and hence belongs to **BQ**. The next result extends their work by determining all the regular elements of $OM(p,q)$.

Theorem 4.1.8. Let $\alpha \in OM(p,q)$. Then, α is regular if and only if $\alpha \in OE(p,q)$. Consequently, $OM(p,q) \cap OE(p,q)$ is the largest regular subsemigroup of $OM(p,q)$.

Proof. Suppose $\alpha \in OM(p,q) \cap OE(p,q)$ and let $\{e_j\}$ be a basis for $\ker \alpha$ with $|J| = n(\alpha) \geq q$. Expand $\{e_j\}$ to a basis $\{e_j\} \dot{\cup} \{e_i\}$ for V and write $e_i\alpha = a_i$ for each i. Then, $\{a_i\}$ is a basis for $\operatorname{ran} \alpha$ and it can be expanded to a basis for V, say $\{a_i\} \dot{\cup} \{a_k\}$, where $|K| = d(\alpha) \geq q$. Define $\beta \in T(V)$ by

$$\beta = \begin{pmatrix} a_i & a_k \\ e_i & 0 \end{pmatrix}.$$

Then, $n(\beta) = \dim\langle a_k \rangle = d(\alpha) \geq q$ and $d(\beta) = \dim\langle e_j \rangle = n(\alpha) \geq q$, and hence $\beta \in OM(p,q) \cap OE(p,q)$. Also, $\alpha\beta\alpha = \alpha$ and so α is regular in $OM(p,q)$.

Conversely, suppose $\alpha \in OM(p,q)$ and $\alpha = \alpha\beta\alpha$ for some $\beta \in OM(p,q)$. Then $d(\alpha) = d(\alpha(\beta\alpha)) \geq d(\beta\alpha)$ and, like before, the fact that $\beta\alpha$ is idempotent implies $d(\beta\alpha) = n(\beta\alpha) \geq n(\beta) \geq q$. Hence $\alpha \in OE(p,q)$ as required. Also, if S is a regular subsemigroup of $OM(p,q)$, then $S \subseteq OE(p,q)$. Thus, $S \subseteq OM(p,q) \cap OE(p,q)$ and the latter is the largest regular subsemigroup of $OM(p,q)$. □

Lemma 4.1.9. The semigroup $OM(p,q)$ is neither left 0-simple nor right 0-simple.

Proof. For each cardinal r such that $1 < r \leq p$, let $T_r = \{\alpha \in T(V) : r(\alpha) < r\}$. As observed in chapter 1, section 1.3, T_r is a non-zero ideal of $T(V)$. Given $\alpha \in T_r$, $r(\alpha) < r \leq p$ and, since $p = \dim V = n(\alpha) + r(\alpha)$, it follows that $n(\alpha) = p \geq q$. Therefore, T_r is a subset of $OM(p,q)$. It is easy to see that T_r is properly contained in $OM(p,q)$. For, suppose $\{e_i\}$ is a basis for V and write $\{e_i\} = \{f_i\} \dot{\cup} \{g_i\}$. Now define $\alpha \in T(V)$ by

$$\alpha = \begin{pmatrix} f_i & g_i \\ 0 & e_i \end{pmatrix}.$$

Clearly, $\alpha \in OM(p,q)$ and, since $\operatorname{ran} \alpha = V$, it follows that $\alpha \notin T_r$. Therefore, T_r is a non-zero proper ideal of $OM(p,q)$, and the result follows. □

In passing, we note that $OM(p,q)$ is never left [right] 0-simple even if q is finite (recall Example 4.1.2). For, in this case, the sets T_r with $1 < r \leq p$ (as defined in the proof of Lemma 4.1.9) still are non-zero proper ideals of $OM(p,q)$.

We now determine all regular elements of $OE(p,q)$.

Theorem 4.1.10. Let α in $OE(p,q)$. Then, α is regular if and only if $\alpha \in OM(p,q)$. Consequently, $OM(p,q) \cap OE(p,q)$ is the largest regular subsemigroup of $OE(p,q)$.

Proof. By Theorem 4.1.8, every $\alpha \in OM(p,q) \cap OE(p,q)$ is regular in $OE(p,q)$ (using an argument similar to that in the proof of Theorem 4.1.6). Conversely, suppose $\alpha \in OE(p,q)$ and $\alpha = \alpha\beta\alpha$ for some $\beta \in OE(p,q)$. Then, $n(\alpha) = n((\alpha\beta)\alpha) \geq n(\alpha\beta)$. Also $\alpha\beta$ is an idempotent in $T(V)$, hence $V = \ker(\alpha\beta) \oplus \operatorname{ran}(\alpha\beta)$ and, since $OE(p,q)$ is closed, it follows that $n(\alpha) \geq n(\alpha\beta) = d(\alpha\beta) \geq d(\beta) \geq q$. So, $\alpha \in OM(p,q)$ as required. Clearly, every regular subsemigroup of $OE(p,q)$ is contained in $OM(p,q) \cap OE(p,q)$, hence this semigroup is the largest regular subsemigroup of $OE(p,q)$. □

Lemma 4.1.11. The semigroup $OE(p,q)$ is neither left 0-simple nor right 0-simple.

Proof. For each cardinal r such that $1 < r \leq p$, the set $T_r = \{\alpha \in T(V) : r(\alpha) < r\}$ is a non-zero ideal of $T(V)$. Given $\alpha \in T_r$, we have $r(\alpha) < r \leq p$. Since $p = \dim V = r(\alpha) + d(\alpha)$, it follows that $d(\alpha) = p \geq q$. Thus, $\alpha \in OE(p,q)$ and so T_r is a non-zero subset of $OE(p,q)$. In fact, it is a proper subset of $OE(p,q)$. For, let $\{e_i\}$ be a basis for V and write $\{e_i\} = \{u_i\} \dot{\cup} \{v_i\}$. Now define $\alpha \in T(V)$ by

$$\alpha = \begin{pmatrix} e_i \\ u_i \end{pmatrix}.$$

Since $d(\alpha) = \dim\langle v_i \rangle = p$ and $r(\alpha) = \dim\langle u_i \rangle = p$, we have $\alpha \in OE(p,q) \setminus T_r$. Hence, T_r is a non-zero proper ideal of $OE(p,q)$, and so the latter is neither left 0-simple nor right 0-simple. □

By analogy with the 'opposite one-to-one' case, we note that the semigroup $OE(p,q)$ is never right [left] 0-simple even for finite q (recall Example 4.1.2). This is because, for every r such that $1 < r \leq p$, the set $T_r = \{\alpha \in T(V) : r(\alpha) < r\}$ is still a non-zero proper ideal of $OE(p,q)$.

4.2. Green's relations and ideals in $AM(p,q)$ and $AE(p,q)$

In this section, we describe Green's relations on each of the semigroups in $(S1)$–$(S2)$ and, using this information, we investigate their ideal structures. First, we characterise the $\mathcal{L}$ relation on $AM(p,q)$ and the $\mathcal{R}$ relation on $AE(p,q)$.

Lemma 4.2.1. Let $\alpha, \beta \in AM(p,q)$. Then $\alpha \mathcal{L} \beta$ if and only if $\operatorname{ran}\alpha = \operatorname{ran}\beta$.

Proof. Let α, β be distinct elements of $AM(p,q)$ and suppose $\alpha \mathcal{L} \beta$. Then there exist λ, μ in $AM(p,q)$ such that $\beta = \lambda\alpha$ and $\alpha = \mu\beta$. Therefore, ran $\alpha =$ ran $(\mu\beta) \subseteq$ ran β and ran $\beta =$ ran $(\lambda\alpha) \subseteq$ ran α, so ran $\alpha =$ ran β.

Conversely, suppose ran $\alpha =$ ran β and let $\{e_j\}$ be a basis for ker β. Expand $\{e_j\}$ to a basis $\{e_j\} \dot{\cup} \{e_i\}$ for V and write $e_i\beta = b_i$ for each i. Then, $\{b_i\}$ is a basis for ran $\beta =$ ran α. For every i, choose $f_i \in b_i\alpha^{-1}$. Clearly, $\{f_i\}$ is linearly independent. Now define $\lambda \in T(V)$ by

$$\lambda = \begin{pmatrix} e_j & e_i \\ 0 & f_i \end{pmatrix}.$$

Since ker $\lambda =$ ker β, it follows that $\lambda \in AM(p,q)$. Also, $\beta = \lambda\alpha$. Similarly, we conclude that there exists $\mu \in AM(p,q)$ such that $\alpha = \mu\beta$, and so $\alpha \mathcal{L} \beta$. □

Lemma 4.2.2. Let $\alpha, \beta \in AE(p,q)$. Then $\alpha \mathcal{R} \beta$ if and only if ker $\alpha =$ ker β.

Proof. Suppose $\alpha, \beta \in AE(p,q)$ are distinct and $\mathcal{R}$–related. Then, there exist $\lambda, \mu \in AE(p,q)$ such that $\alpha = \beta\lambda$ and $\beta = \alpha\mu$. Thus, ker $\alpha \subseteq$ ker $(\alpha\mu) =$ ker β and ker $\beta \subseteq$ ker $(\beta\lambda) =$ ker α. Hence, ker $\alpha =$ ker β.

Conversely, suppose ker $\alpha =$ ker β and let $\{e_j\}$ be a basis for this subspace. Expand $\{e_j\}$ to a basis $\{e_j\} \dot{\cup} \{e_i\}$ for V and, for each i, write $e_i\alpha = a_i$ and $e_i\beta = b_i$. Clearly, $\{a_i\}$ and $\{b_i\}$ are bases for ran α and ran β, respectively. Now expand $\{b_i\}$ to a basis for V, say $\{b_i\} \dot{\cup} \{b_\ell\}$, and define $\lambda \in T(V)$ by

$$\lambda = \begin{pmatrix} b_\ell & b_i \\ 0 & a_i \end{pmatrix}.$$

Since $d(\lambda) = d(\alpha)$, it follows that $\lambda \in AE(p,q)$. Also, $\alpha = \beta\lambda$. Similarly, expand $\{a_i\}$ to a basis $\{a_i\} \dot{\cup} \{a_k\}$ for V and define $\mu \in T(V)$ by

$$\mu = \begin{pmatrix} a_k & a_i \\ 0 & b_i \end{pmatrix}.$$

Clearly, $d(\mu) = d(\beta)$ and so $\mu \in AE(p,q)$. Also, $\beta = \alpha\mu$. Hence, $\alpha \mathcal{R} \beta$. □

We proceed to characterise the $\mathcal{R}$ relation on $AM(p,q)$. For this, we need two preliminary Lemmas.

Lemma 4.2.3. If $\alpha, \beta, \lambda \in T(V)$ satisfy $\alpha = \beta\lambda$ then

$$d(\beta) \leq n(\lambda) + \dim(\text{ran } \lambda / \text{ran } \alpha).$$

In fact, if we also have $\ker \alpha = \ker \beta$, then $d(\beta) = n(\lambda) + \dim(\text{ran } \lambda / \text{ran } \alpha)$.

Proof. Since $\alpha = \beta\lambda$ implies $\ker \beta \subseteq \ker \alpha$, we can write $\ker \beta = \langle e_r \rangle$, $\ker \alpha = \langle e_r, e_s \rangle$ and $V = \langle e_r \rangle \oplus \langle e_s \rangle \oplus \langle e_j \rangle$. Write $e_j\alpha = a_j$, $e_s\beta = b_s$ and $e_j\beta = b_j$, and note that $a_j = e_j\alpha = (e_j\beta)\lambda = b_j\lambda$ for each j. In addition, $\{a_j\}$ and $\{b_s, b_j\}$ are bases for ran α and ran β, respectively. Now, if $\sum x_j b_j \in \ker \lambda$ for some scalars x_j, then $\sum x_j a_j = 0$ and so $x_j = 0$ for each j: that is, $\langle b_j \rangle \cap \ker \lambda = \{0\}$. Hence we can write $V = \langle b_j \rangle \oplus \ker \lambda \oplus \langle e_k \rangle$ for some linearly independent subset $\{e_k\}$ of V, and we assert that ran $\lambda =$ ran $\alpha \oplus \langle e_k\lambda \rangle$. For, if $\sum x_j a_j = \sum y_k(e_k\lambda)$ for some scalars x_j and y_k then $\sum x_j b_j - \sum y_k e_k \in \ker \lambda$ and, by our choice of bases, this implies $x_j = 0 = y_k$ for all j and k. Clearly, $\{e_k\lambda\}$ is linearly independent. Since $\langle b_j \rangle \subseteq$ ran β, we have

$$d(\beta) \leq \text{codim } \langle b_j \rangle = n(\lambda) + |K| = n(\lambda) + \dim(\text{ran } \lambda / \text{ran } \alpha).$$

Finally, if we also have $\ker \alpha = \ker \beta$ then, with the previous notation, ran $\beta = \langle b_j \rangle$ and $V = \langle b_j \rangle \oplus \ker \lambda \oplus \langle e_k \rangle$ and so $d(\beta) = n(\lambda) + |K|$. □

Lemma 4.2.4. If $\alpha, \beta \in AM(p,q)$ and $\alpha \mathcal{R} \beta$, then $\alpha \in AE(p,q)$ if and only if $\beta \in AE(p,q)$.

Proof. Suppose the conditions hold and $\alpha \in AE(p,q)$. By Theorem 4.1.4, α is a regular element of $AM(p,q)$, and so D_α, the $\mathcal{D}$–class of α in $AM(p,q)$, is regular (by [5] Vol. 1, Theorem 2.11). Now let R_α denote the $\mathcal{R}$–class of α in $AM(p,q)$. Since $\beta \in R_\alpha \subseteq D_\alpha$, this implies β is a regular element of $AM(p,q)$ and so $\beta \in AE(p,q)$ by Theorem 4.1.4. Similarly, if $\beta \in AE(p,q)$ then $\alpha \in AE(p,q)$. □

Lemma 4.2.5. Let $\alpha \in AM(p,q)$ and denote the $\mathcal{R}$–class of $AM(p,q)$ containing α by R_α. Then,

(a) $\alpha \in AE(p,q)$ implies $R_\alpha = \{\beta \in AM(p,q) : \beta \in AE(p,q) \text{ and } \ker \beta = \ker \alpha\}$;

(b) $\alpha \notin AE(p,q)$ implies $R_\alpha = \{\beta \in AM(p,q) : \ker \beta = \ker \alpha \text{ and } d(\beta) = d(\alpha)\}$.

Proof. First suppose $\alpha \in AE(p,q)$. If $\beta \in AM(p,q)$ is such that $\alpha \mathcal{R} \beta$ then, since $\text{id}_V \in AM(p,q)$, there exist $\lambda, \mu \in AM(p,q)$ such that $\alpha = \beta\lambda$ and $\beta = \alpha\mu$.

Therefore, $\ker \alpha \subseteq \ker(\alpha\mu) = \ker \beta$ and $\ker \beta \subseteq \ker(\beta\lambda) = \ker \alpha$, and so $\ker \alpha = \ker \beta$. Also, we know $\beta \in AE(p,q)$, from Lemma 4.2.4.

Conversely, suppose $\beta \in AM(p,q) \cap AE(p,q)$ and $\ker \beta = \ker \alpha$. Since $AM(p,q) \cap AE(p,q)$ is a regular subsemigroup of $AE(p,q)$, Hall's Theorem ([11], Proposition II.4.5) implies that the $\mathcal{R}$ relation on $AM(p,q) \cap AE(p,q)$ is the restriction of the $\mathcal{R}$ relation on $AE(p,q)$ to $AM(p,q) \cap AE(p,q)$. In other words, since $\alpha, \beta \in AM(p,q) \cap AE(p,q)$ and $\ker \alpha = \ker \beta$, we deduce from Lemma 4.2.2 that $\alpha \mathcal{R} \beta$ in $AM(p,q) \cap AE(p,q)$ and hence $\alpha \mathcal{R} \beta$ in $AM(p,q)$. That is, $\beta \in R_\alpha$ as required, and (a) holds.

Now, suppose $\alpha \notin AE(p,q)$ and $\alpha \mathcal{R} \beta$ in $AM(p,q)$. Then $\beta \notin AE(p,q)$ (by Lemma 4.2.4) and $\alpha = \beta\lambda$, $\beta = \alpha\mu$ for some $\lambda, \mu \in AM(p,q)$. As we already know, the latter implies $\ker \alpha = \ker \beta$. Moreover, since $\alpha = \beta\lambda$, $n(\lambda) < q$ and $d(\beta) \geq q$, by Lemma 4.2.3 we have $d(\beta) \leq \dim(\operatorname{ran} \lambda / \operatorname{ran} \alpha) \leq \dim(V / \operatorname{ran} \alpha) = d(\alpha)$. Similarly, since $\beta = \alpha\mu$, $n(\mu) < q$ and $d(\alpha) \geq q$, we deduce that $d(\alpha) \leq d(\beta)$ and equality follows.

Conversely, suppose $\beta \in AM(p,q)$ is such that $\ker \beta = \ker \alpha$ and $d(\beta) = d(\alpha)$. Let $\{e_j\}$ be a basis for $\ker \alpha = \ker \beta$, with $|J| = n(\alpha) = n(\beta)$, and expand it to a basis $\{e_j\} \dot\cup \{e_i\}$ for V. Now write $e_i\alpha = a_i$ and $e_i\beta = b_i$ for each i. Then, $\{a_i\}$ is a basis for $\operatorname{ran} \alpha$ and it can be expanded to a basis for V, say $\{a_i\} \dot\cup \{a_k\}$, where $|K| = d(\alpha) \geq q$. Similarly, $\{b_i\}$ is a basis for $\operatorname{ran} \beta$ and we can expand it to a basis $\{b_i\} \dot\cup \{b_k\}$ for V (note that $d(\beta) = d(\alpha) = |K|$). Since $|K| \geq q \geq \aleph_0$, we can write $\{a_k\}$ as $\{u_k\} \dot\cup \{u_r\}$ and $\{b_k\}$ as $\{v_k\} \dot\cup \{v_r\}$, where $|R| < q$. Now define $\lambda, \mu \in T(V)$ by

$$\lambda = \begin{pmatrix} b_i & v_k & v_r \\ a_i & u_k & 0 \end{pmatrix}, \quad \mu = \begin{pmatrix} a_i & u_k & u_r \\ b_i & v_k & 0 \end{pmatrix}.$$

Since $n(\lambda) = \dim\langle v_r \rangle < q$ and $n(\mu) = \dim\langle u_r \rangle < q$, we have $\lambda, \mu \in AM(p,q)$. Also, $\alpha = \beta\lambda$ and $\beta = \alpha\mu$. Hence, $\alpha \mathcal{R} \beta$ and (b) holds. □

The next two results are crucial for the characterisation of the $\mathcal{L}$ relation on $AE(p,q)$.

Lemma 4.2.6. If $\alpha, \beta, \lambda \in T(V)$ satisfy $\alpha = \lambda\beta$, then

$$n(\beta) \leq d(\lambda) + \dim(\ker \alpha / \ker \lambda).$$

In fact, if $\operatorname{ran} \alpha = \operatorname{ran} \beta$ then $n(\beta) = d(\lambda) + \dim(\ker \alpha / \ker \lambda)$.

Proof. Since $\alpha = \lambda\beta$, we can write $\ker \lambda = \langle e_j \rangle$, $\ker \alpha = \langle e_j \rangle \oplus \langle e_i \rangle$ and $V = \langle e_j \rangle \oplus \langle e_i \rangle \oplus \langle f_k \rangle$. Write $f_k\alpha = a_k$ and $f_k\lambda = u_k$ for each k, and note that $\{a_k\}$ is a basis for $\operatorname{ran} \alpha$. In addition, $a_k = f_k\alpha = u_k\beta$. Clearly, the set $\{e_i\lambda\} \dot\cup \{u_k\}$ is linearly independent, hence $\operatorname{ran} \lambda = \langle e_i\lambda \rangle \oplus \langle u_k \rangle$. Moreover, if $(\sum x_k u_k)\beta = 0$ for some scalars x_k, then $\sum x_k(u_k\beta) = 0$, hence $\sum x_k a_k = 0$ and this implies $x_k = 0$ for each k. Thus $\ker \beta \cap \langle u_k \rangle = \{0\}$. Therefore,

$$n(\beta) \leq \operatorname{codim} \langle u_k \rangle = d(\lambda) + |I| = d(\lambda) + \dim(\ker \alpha / \ker \lambda).$$

Now suppose $\operatorname{ran} \beta = \operatorname{ran} \alpha = \langle a_k \rangle$. If $v \in V$, then there exist scalars y_k such that $v\beta = \sum y_k a_k$ and so $v\beta = (\sum y_k u_k)\beta$. Hence, $v - \sum y_k u_k \in \ker \beta$ and thus $v \in \ker \beta \oplus \langle u_k \rangle$. Therefore, $V = \ker \beta \oplus \langle u_k \rangle$ and, in this case, $n(\beta) = \operatorname{codim} \langle u_k \rangle = d(\lambda) + \dim(\ker \alpha / \ker \lambda)$. □

Lemma 4.2.7. If $\alpha, \beta \in AE(p,q)$ and $\alpha \mathcal{L} \beta$, then $\alpha \in AM(p,q)$ if and only if $\beta \in AM(p,q)$.

Proof. Suppose the conditions hold and let $\alpha \in AM(p,q)$. By Theorem 4.1.6, α is a regular element of $AE(p,q)$ and thus D_α, the $\mathcal{D}$–class of α in $AE(p,q)$, is regular (by [5] Vol. 1, Theorem 2.11). If L_α denotes the $\mathcal{L}$–class of α in $AE(p,q)$, then $\beta \in L_\alpha \subseteq D_\alpha$ implies β is a regular element of $AE(p,q)$ and so $\beta \in AM(p,q)$ by Theorem 4.1.6. Similarly, if $\beta \in AM(p,q)$ then $\alpha \in AM(p,q)$. □

Lemma 4.2.8. Let $\alpha \in AE(p,q)$ and denote the $\mathcal{L}$–class of $AE(p,q)$ containing α by L_α. Then,

(a) $\alpha \in AM(p,q)$ implies $L_\alpha = \{\beta \in AE(p,q) : \beta \in AM(p,q) \text{ and } \operatorname{ran} \beta = \operatorname{ran} \alpha\}$;

(b) $\alpha \notin AM(p,q)$ implies $L_\alpha = \{\beta \in AE(p,q) : \operatorname{ran} \beta = \operatorname{ran} \alpha \text{ and } n(\beta) = n(\alpha)\}$.

Proof. Let $\beta \in AE(p,q)$ be such that $\alpha \mathcal{L} \beta$. Then, there exist $\lambda, \mu \in AE(p,q)$ such that $\alpha = \lambda\beta$ and $\beta = \mu\alpha$, since $\mathrm{id}_V \in AE(p,q)$, and so $\operatorname{ran} \alpha = \operatorname{ran} \beta$. If $\alpha \in AM(p,q)$, then $\beta \in AM(p,q)$ (by Lemma 4.2.7). If not, then $\beta \notin AM(p,q)$ (again, by Lemma 4.2.7) and so $n(\alpha) \geq q$ and $n(\beta) \geq q$. From Lemma 4.2.6, we know that $n(\beta) \leq d(\lambda) + n(\alpha)$ and, similarly, $n(\alpha) \leq d(\mu) + n(\beta)$. Since $d(\lambda) < q \leq n(\alpha)$ and $d(\mu) < q \leq n(\beta)$, it follows that $d(\lambda) + n(\alpha) = n(\alpha)$ and $d(\mu) + n(\beta) = n(\beta)$. Hence, $n(\beta) = n(\alpha)$.

Conversely, suppose $\beta \in AM(p,q) \cap AE(p,q)$ and $\operatorname{ran}\beta = \operatorname{ran}\alpha$. Since $AM(p,q) \cap AE(p,q)$ is a regular subsemigroup of $AM(p,q)$ (by Theorem 4.1.4), Hall's Theorem ([11] Proposition II.4.5) implies that the $\mathcal{L}$ relation on $AM(p,q) \cap AE(p,q)$ is the restriction of the $\mathcal{L}$ relation on $AM(p,q)$ to $AM(p,q) \cap AE(p,q)$. In other words, since $\alpha, \beta \in AM(p,q) \cap AE(p,q)$ and $\operatorname{ran}\alpha = \operatorname{ran}\beta$, we deduce from Lemma 4.2.1 that $\alpha \mathcal{L} \beta$ in $AM(p,q) \cap AE(p,q)$ and hence $\alpha \mathcal{L} \beta$ in $AE(p,q)$. That is, $\beta \in L_\alpha$ as required.

Finally, suppose $\alpha \notin AM(p,q)$, $\beta \in AE(p,q)$, $\operatorname{ran}\beta = \operatorname{ran}\alpha$ and $n(\beta) = n(\alpha)$. Let $\operatorname{ran}\alpha = \langle e_i \rangle$, and choose $a_i, b_i \in V$ such that $a_i\alpha = e_i$ and $b_i\beta = e_i$ for each i. Clearly, $\{a_i\}$ is linearly independent. Moreover, if $\ker\alpha = \langle a_k \rangle$ then $V = \langle a_i \rangle \oplus \langle a_k \rangle$: if $u \in V$ then $u\alpha = \sum x_i e_i = (\sum x_i a_i)\alpha$ for some scalars x_i, so $u - \sum x_i a_i \in \ker\alpha$; and clearly $\{a_i\} \dot\cup \{a_k\}$ is linearly independent. Similarly, $V = \langle b_i \rangle \oplus \langle b_k \rangle$ where $\ker\beta = \langle b_k \rangle$ and $|K| = n(\beta) = n(\alpha)$. Now write

$$\{a_k\} = \{u_k\} \dot\cup \{u_r\}, \quad \{b_k\} = \{v_k\} \dot\cup \{v_r\},$$

where $|R| < q$ (this is possible since $\alpha \notin AM(p,q)$ implies $|K| \geq \aleph_0$), and define $\lambda, \mu \in T(V)$ by

$$\lambda = \begin{pmatrix} a_i & u_k & u_r \\ b_i & v_k & 0 \end{pmatrix}, \quad \mu = \begin{pmatrix} b_i & v_k & v_r \\ a_i & u_k & 0 \end{pmatrix}.$$

Then $d(\lambda) = |R| < q$, so $\lambda \in AE(p,q)$, and likewise $\mu \in AE(p,q)$. Moreover, $\alpha = \lambda\beta$ and $\beta = \mu\alpha$, so $\alpha \mathcal{L} \beta$ in $AE(p,q)$ as required. □

Sullivan (personal communication) described the $\mathcal{D}$ and $\mathcal{J}$ relations on $AM(p,q)$. The characterisation of its ideals follows from his results.

Theorem 4.2.9. If $\alpha, \beta \in AM(p,q)$ then $\alpha \mathcal{D} \beta$ in $AM(p,q)$ if and only if one of the following occurs.

(a) $\alpha, \beta \in AE(p,q)$,

(b) $\alpha, \beta \notin AE(p,q)$ and $d(\alpha) = d(\beta)$.

Proof. Suppose $\alpha \mathcal{L} \gamma \mathcal{R} \beta$ in $AM(p,q)$. If $\beta \in AE(p,q)$ then $\gamma \in AE(p,q)$ (by Lemma 4.2.4): that is, $d(\gamma) < q$ and, since $\operatorname{ran}\alpha = \operatorname{ran}\gamma$, this implies $d(\alpha) < q$

and hence $\alpha \in AE(p,q)$. Conversely, suppose $\alpha, \beta \in AM(p,q) \cap AE(p,q)$ and, in the usual way, write

$$\alpha = \begin{pmatrix} e_j & e_i \\ 0 & a_i \end{pmatrix}, \quad \beta = \begin{pmatrix} f_k & f_i \\ 0 & b_i \end{pmatrix}$$

(this is possible since $\alpha, \beta \in AM(p,q)$ implies $r(\alpha) = r(\beta) = p$). Now define $\gamma \in T(V)$ by

$$\gamma = \begin{pmatrix} f_k & f_i \\ 0 & a_i \end{pmatrix}.$$

Then $n(\gamma) = n(\beta) < q$ and $d(\gamma) = d(\alpha) < q$, so $\gamma \in AM(p,q) \cap AE(p,q)$. In fact, $\operatorname{ran} \gamma = \operatorname{ran} \alpha$ and $\ker \gamma = \ker \beta$, so $\alpha \mathcal{L} \gamma$ and $\gamma \mathcal{R} \beta$, hence $\alpha \mathcal{D} \beta$ in $AM(p,q)$.

Now suppose $\alpha \mathcal{L} \gamma \mathcal{R} \beta$ in $AM(p,q)$ and $\beta \notin AE(p,q)$. Then $\operatorname{ran} \alpha = \operatorname{ran} \gamma$ and so, by Lemma 4.2.5(b), $d(\alpha) = d(\gamma) = d(\beta) \geq q$, hence $\alpha \notin AE(p,q)$. Conversely, suppose $\alpha, \beta \notin AE(p,q)$ and $d(\alpha) = d(\beta)$ and, in the usual way, write

$$\alpha = \begin{pmatrix} e_j & e_i \\ 0 & a_i \end{pmatrix}, \quad \beta = \begin{pmatrix} f_k & f_i \\ 0 & b_i \end{pmatrix}$$

(as before, this is possible since $\alpha, \beta \in AM(p,q)$). Define γ in $T(V)$ by

$$\gamma = \begin{pmatrix} f_k & f_i \\ 0 & a_i \end{pmatrix}$$

Then $\gamma \in AM(p,q)$ (as before) and $\operatorname{ran} \gamma = \operatorname{ran} \alpha$, so $\alpha \mathcal{L} \gamma$. Also, $\ker \gamma = \ker \beta$ and $d(\gamma) = d(\alpha) = d(\beta)$, so $\gamma \mathcal{R} \beta$. In other words, we have shown that $\alpha \mathcal{D} \beta$ in $AM(p,q)$. □

Corollary 4.2.10. $\mathcal{D} = \mathcal{J}$ on $AM(p,q)$.

Proof. We know $\mathcal{D} \subseteq \mathcal{J}$. Therefore, since $\mathcal{D}$ is universal on $AM(p,q) \cap AE(p,q)$ by Theorem 4.2.9(a), $\mathcal{J}$ is also. Now suppose $\alpha = \lambda\beta.\mu$ and $\beta = \lambda'\alpha\mu'$ for some $\lambda, \mu, \lambda', \mu' \in AM(p,q)$. By Lemma 4.2.3, we have

$$d(\beta) \leq d(\lambda\beta) \leq n(\mu) + \dim(\operatorname{ran} \mu / \operatorname{ran} \alpha) \leq n(\mu) + d(\alpha).$$

Hence, if $\beta \notin AE(p,q)$ then $q \leq d(\beta) \leq n(\mu) + d(\alpha)$, and $n(\mu) < q$, so $d(\alpha) \geq q$ and thus $\alpha \notin AE(p,q)$. Likewise, using $\beta = \lambda'\alpha\mu'$, we find that $\alpha \notin AE(p,q)$ implies $\beta \notin AE(p,q)$. That is, if $\alpha \mathcal{J} \beta$ in $AM(p,q)$ then either $\alpha, \beta \in AM(p,q) \cap AE(p,q)$ or $\alpha, \beta \notin AE(p,q)$. In the latter case, we have $d(\beta) \leq n(\mu) + d(\alpha) = d(\alpha)$ since

$n(\mu) < q \leq d(\alpha)$. Similarly, $\beta = \lambda'\alpha\mu'$ implies $d(\alpha) \leq d(\beta)$, and equality follows. In other words, $\alpha \mathcal{J} \beta$ implies $\alpha \mathcal{D} \beta$. □

Theorem 4.2.11. The proper ideals of $AM(p,q)$ are precisely the sets $AM(p,q) \cap OE(p,\xi)$ where $q \leq \xi \leq p$. In fact, each of these sets is a principal ideal of $AM(p,q)$ generated by an element with defect ξ.

Proof. Let ξ be a cardinal such that $q \leq \xi \leq p$. By Lemma 4.2.3, given $\alpha \in AM(p,q) \cap OE(p,\xi)$ and $\lambda, \mu \in AM(p,q)$, we have

$$\xi \leq d(\alpha) \leq d(\lambda\alpha) \leq n(\mu) + \dim(\text{ran } \mu / \text{ran } (\lambda\alpha\mu)) \leq n(\mu) + d(\lambda\alpha\mu).$$

Since $n(\mu) < q$ and $\xi \geq q$, we see that $d(\lambda\alpha\mu) \geq \xi$. Therefore, $\lambda\alpha\mu \in AM(p,q) \cap OE(p,\xi)$ and so $AM(p,q) \cap OE(p,\xi)$ is an ideal of $AM(p,q)$ (note that λ and μ can equal $\text{id}_V \in AM(p,q)$).

Conversely, let I be an ideal of $AM(p,q)$. If there exists $\alpha \in I \cap AE(p,q)$ then $\alpha \in AM(p,q) \cap AE(p,q)$ and, since $\text{id}_V \in AM(p,q) \cap AE(p,q)$, Theorem 4.2.9(a) implies $\text{id}_V \ \mathcal{D} \ \alpha$. Consequently, by Corollary 4.2.10, we have $\text{id}_V \in J(\alpha)$, the principal ideal of $AM(p,q)$ generated by α, so $\text{id}_V \in I$ and hence $I = AM(p,q)$. Now suppose $I \cap AE(p,q) = \emptyset$ and choose $\gamma \in I$ with minimal defect ξ. Note that $d(\beta) \geq d(\gamma) = \xi$ for every $\beta \in I$ and, clearly, $q \leq \xi \leq p$. Hence,

$$AM(p,q).\gamma.AM(p,q) \subseteq I \subseteq AM(p,q) \cap OE(p,\xi).$$

Given $\alpha \in AM(p,q) \cap OE(p,\xi)$, we have $d(\alpha) \geq \xi = d(\gamma)$. In the usual way, write

$$\alpha = \begin{pmatrix} e_j & e_i \\ 0 & a_i \end{pmatrix}, \quad \gamma = \begin{pmatrix} f_k & f_i \\ 0 & b_i \end{pmatrix}$$

(note that this is possible since $\alpha, \gamma \in AM(p,q)$ implies $r(\alpha) = r(\gamma) = p$). Since $\{b_i\}$ is a basis for ran γ, it can be expanded to a basis for V, say $\{b_i\} \dot\cup \{b_\ell\}$, with $|L| = d(\gamma) = \xi$. Similarly, $\{a_i\}$ is a basis for ran α and it can be expanded to a basis $\{a_i\} \dot\cup \{a_r\} \dot\cup \{a_\ell\}$ for V, where $|R| + |L| = d(\alpha)$ (note that $d(\alpha) \geq d(\gamma) = |L|$). Now define $\lambda, \mu \in T(V)$ by

$$\lambda = \begin{pmatrix} e_j & e_i \\ 0 & f_i \end{pmatrix}, \quad \mu = \begin{pmatrix} b_i & b_\ell \\ a_i & a_\ell \end{pmatrix}.$$

Clearly, $n(\lambda) = n(\alpha) < q$ and $n(\mu) = 0$, hence $\lambda, \mu \in AM(p,q)$. Also $\alpha = \lambda\gamma\mu$ and, since I is an ideal, $\gamma \in I$ implies $\alpha \in I$. So, $I = AM(p,q) \cap OE(p,\xi)$ and, in effect, we have shown that I is a principal ideal generated by an element with defect ξ. $\square$

Clearly, the proper ideals of $AM(p,q)$ form a chain under $\subseteq$, with the smallest being $AM(p,q) \cap OE(p,p)$ and the largest being $AM(p,q) \cap OE(p,q)$.

Now we proceed to characterise the $\mathcal{D}$ and $\mathcal{J}$ relations on $AE(p,q)$ and, using this, we describe the ideal structure of $AE(p,q)$.

Theorem 4.2.12. If $\alpha, \beta \in AE(p,q)$ then $\alpha \ \mathcal{D} \ \beta$ in $AE(p,q)$ if and only if one of the following occurs.

(a) $\alpha, \beta \in AM(p,q)$,

(b) $\alpha, \beta \notin AM(p,q)$ and $n(\alpha) = n(\beta)$.

Proof. Suppose $\alpha \ \mathcal{L} \ \gamma \ \mathcal{R} \ \beta$ in $AE(p,q)$. If $\alpha \in AM(p,q)$ then $\gamma \in AM(p,q)$ (by Lemma 4.2.7) and hence $n(\gamma) < q$. Since $\ker \gamma = \ker \beta$, we have $n(\beta) = n(\gamma) < q$ and so $\beta \in AM(p,q)$. Conversely, if $\alpha, \beta \in AM(p,q) \cap AE(p,q)$ then the same argument as that used in the proof of Theorem 4.2.9(a) shows that $\alpha \ \mathcal{D} \ \beta$ in $AE(p,q)$.

Now assume $\alpha \ \mathcal{L} \ \gamma \ \mathcal{R} \ \beta$ in $AE(p,q)$ and $\alpha \notin AM(p,q)$. Then, $\ker \beta = \ker \gamma$ and so, by Lemma 4.2.8(b), $n(\beta) = n(\gamma) = n(\alpha) \geq q$, hence $\beta \notin AM(p,q)$. Conversely, suppose $\alpha, \beta \notin AM(p,q)$ and $n(\alpha) = n(\beta)$ and, in the usual way, write

$$\alpha = \begin{pmatrix} e_j & e_i \\ 0 & a_i \end{pmatrix}, \quad \beta = \begin{pmatrix} f_j & f_i \\ 0 & b_i \end{pmatrix}$$

(note that this is possible since $d(\alpha) < q$ and $d(\beta) < q$ imply $r(\alpha) = r(\beta) = p$). Now define $\gamma \in T(V)$ by

$$\gamma = \begin{pmatrix} f_j & f_i \\ 0 & a_i \end{pmatrix}.$$

Then, $d(\gamma) = d(\alpha) < q$, so $\gamma \in AE(p,q)$. In fact, $\ker \gamma = \ker \beta$ and so $\gamma \ \mathcal{R} \ \beta$. Also, $\operatorname{ran} \gamma = \operatorname{ran} \alpha$ and $n(\gamma) = n(\beta) = n(\alpha)$, hence $\alpha \ \mathcal{L} \ \gamma$. In other words, we have shown that $\alpha \ \mathcal{D} \ \beta$ in $AE(p,q)$. $\square$

Corollary 4.2.13. $\mathcal{D} = \mathcal{J}$ on $AE(p,q)$.

Proof. Since $\mathcal{D} \subseteq \mathcal{J}$ and $\mathcal{D}$ is universal on $AM(p,q) \cap AE(p,q)$, so is $\mathcal{J}$. Now suppose $\alpha = \lambda.\beta\mu$ and $\beta = \lambda'\alpha\mu'$ for some $\lambda, \mu, \lambda', \mu' \in AE(p,q)$. By Lemma 4.2.6, it follows that

$$n(\beta) \leq n(\beta\mu) \leq d(\lambda) + \dim(\ker \alpha / \ker \lambda) \leq d(\lambda) + n(\alpha).$$

Therefore, if $\beta \notin AM(p,q)$ then $q \leq n(\beta) \leq d(\lambda) + n(\alpha)$, and $d(\lambda) < q$, so $n(\alpha) \geq q$. Hence $\alpha \notin AM(p,q)$. Likewise, using $\beta = \lambda'\alpha\mu'$, we conclude that $\alpha \notin AM(p,q)$ implies $\beta \notin AM(p,q)$. Thus, if $\alpha \mathcal{J} \beta$ in $AE(p,q)$, then $\alpha \in AM(p,q)$ if and only if $\beta \in AM(p,q)$. Moreover, if $\alpha, \beta \notin AM(p,q)$ then $n(\beta) \leq d(\lambda) + n(\alpha) = n(\alpha)$ and $n(\alpha) \leq d(\lambda') + n(\beta) = n(\beta)$, hence $n(\alpha) = n(\beta)$ and so $\alpha \mathcal{D} \beta$. In other words, $\mathcal{J} \subseteq \mathcal{D}$ on $AE(p,q)$. □

Theorem 4.2.14. The proper ideals of $AE(p,q)$ are precisely the sets $OM(p,\xi) \cap AE(p,q)$ where $q \leq \xi \leq p$. In fact, each of these sets is a principal ideal of $AE(p,q)$ generated by an element with nullity ξ.

Proof. Let ξ be an infinite cardinal such that $q \leq \xi \leq p$, and suppose $\alpha \in OM(p,\xi) \cap AE(p,q)$ and $\lambda, \mu \in AE(p,q)$. By Lemma 4.2.6, we have

$$\xi \leq n(\alpha) \leq n(\alpha\mu) \leq d(\lambda) + \dim(\ker(\lambda\alpha\mu)/\ker \lambda) \leq d(\lambda) + n(\lambda\alpha\mu).$$

Since $\lambda \in AE(p,q)$, we know $d(\lambda) < q$, and $q \leq \xi$ by supposition, hence $n(\lambda\alpha\mu) \geq \xi$ and so $\lambda\alpha\mu \in OM(p,\xi) \cap AE(p,q)$. Therefore, $OM(p,\xi) \cap AE(p,q)$ is an ideal of $AE(p,q)$, since $\mathrm{id}_V \in AE(p,q)$.

Conversely, let I be an ideal of $AE(p,q)$. If there exists $\alpha \in I \cap AM(p,q)$ then $\alpha \in AE(p,q) \cap AM(p,q)$ and, since $\mathrm{id}_V \in AE(p,q) \cap AM(p,q)$, Theorem 4.2.12(a) implies $\mathrm{id}_V \ \mathcal{D}\ \alpha$. Consequently, by Corollary 4.2.13, we have $\mathrm{id}_V \in J(\alpha)$, the principal ideal of $AE(p,q)$ generated by α, so $\mathrm{id}_V \in I$ and hence $I = AE(p,q)$. Finally, suppose $I \cap AM(p,q) = \emptyset$ and choose $\varepsilon \in I$ with minimal nullity ξ. Then, $q \leq \xi \leq p$ and $n(\beta) \geq n(\varepsilon) = \xi$ for every $\beta \in I$. Therefore,

$$AE(p,q).\varepsilon.AE(p,q) \subseteq I \subseteq OM(p,\xi) \cap AE(p,q).$$

If $\alpha \in OM(p,\xi) \cap AE(p,q)$ then $n(\alpha) \geq \xi = n(\varepsilon)$. Now let $\{f_k\}$ be a basis for $\ker \varepsilon$, with $|K| = \xi$, and expand it to a basis for V, say $\{f_k\} \dot{\cup} \{f_i\}$. For every i, write

$f_i\varepsilon = b_i$. Clearly, $\{b_i\}$ is a basis for ran ε, and $\varepsilon \in AE(p,q)$ implies $|I| = r(\varepsilon) = p$. Likewise, let $\{e_j\} \dot\cup \{e_k\}$ be a basis for $\ker \alpha$, with $|J| + |K| = n(\alpha) \geq n(\varepsilon) = |K|$, and expand it to a basis $\{e_j\} \dot\cup \{e_k\} \dot\cup \{e_r\}$ for V. For each r, write $e_r\alpha = a_r$. Since $\alpha \in AE(p,q)$ and $\{a_r\}$ is a basis for ran α, we know $r(\alpha) = p$, hence we can write $\{e_i\}$ and $\{a_i\}$ instead of $\{e_r\}$ and $\{a_r\}$, respectively. Expand $\{b_i\}$ to a basis for V, say $\{b_i\} \dot\cup \{b_\ell\}$, and define $\lambda, \mu \in T(V)$ by

$$\lambda = \begin{pmatrix} e_j & e_k & e_i \\ 0 & f_k & f_i \end{pmatrix}, \quad \mu = \begin{pmatrix} b_i & b_\ell \\ a_i & 0 \end{pmatrix}.$$

Clearly, $d(\lambda) = 0$ and $d(\mu) = d(\alpha) < q$, hence $\lambda, \mu \in AE(p,q)$. Also, $\alpha = \lambda\varepsilon\mu$ and so $\alpha \in I$, since I is an ideal of $AE(p,q)$ and $\varepsilon \in I$. Therefore, $I = OM(p,\xi) \cap AE(p,q)$ and, in effect, we have shown that I is a principal ideal generated by an element with nullity ξ. □

It is now easy to see that the proper ideals of $AE(p,q)$ form a chain under $\subseteq$, with the smallest being $OM(p,p) \cap AE(p,q)$ and the largest being $OM(p,q) \cap AE(p,q)$.

4.3. Green's relations and ideals in $OM(p,q)$ and $OE(p,q)$

In this section, we consider the semigroups $OM(p,q)$ and $OE(p,q)$, as defined in $(S3)$–$(S4)$, and we begin with analogues of Lemmas 4.2.1 and 4.2.2, respectively: the proofs of those Lemmas hold for the present situation, so we omit them.

Lemma 4.3.1. Let $\alpha, \beta \in OM(p,q)$. Then $\alpha \mathcal{L} \beta$ if and only if ran α = ran β.

Lemma 4.3.2. Let $\alpha, \beta \in OE(p,q)$. Then $\alpha \mathcal{R} \beta$ if and only if $\ker \alpha = \ker \beta$.

Next, we characterise the $\mathcal{R}$ relation on $OM(p,q)$. Note that the $\mathcal{L}$ relations on $AM(p,q)$ and on $OM(p,q)$ have identical characterisations (compare Lemmas 4.2.1 and 4.3.1), but the same does not happen for the $\mathcal{R}$ relations on these two semigroups.

Lemma 4.3.3. Let $\alpha \in OM(p,q)$ and denote the $\mathcal{R}$–class of $OM(p,q)$ containing α by R_α. Then,

(a) $\alpha \notin OE(p,q)$ implies $R_\alpha = \{\alpha\}$;

(b) $\alpha \in OE(p,q)$ implies $R_\alpha = \{\beta \in OM(p,q) : \beta \in OE(p,q) \text{ and } \ker \beta = \ker \alpha\}$.

Proof. Suppose $\alpha \notin OE(p,q)$ and $\alpha \mathcal{R} \beta$ in $OM(p,q)$ for some $\beta \neq \alpha$. Then, as before, this implies $\ker \alpha = \ker \beta$ and $\beta = \alpha\mu$ for some $\mu \in OM(p,q)$. That is, $d(\alpha) < q$ and $n(\mu) \geq q$, and by Lemma 4.2.3 we have a contradiction: namely, $d(\alpha) = n(\mu) + \dim(\operatorname{ran} \mu / \operatorname{ran} \beta) \geq q$.

To see that (b) holds, suppose $\alpha \in OM(p,q) \cap OE(p,q)$ and $\alpha \mathcal{R} \beta$ in $OM(p,q)$. Then, as usual, this implies $\ker \alpha = \ker \beta$. Moreover, if $\beta \notin OE(p,q)$ then (a) implies $\alpha \in R_\alpha = R_\beta = \{\beta\}$, hence $\alpha = \beta$, contradicting our supposition.

Conversely, suppose $\beta \in OM(p,q) \cap OE(p,q)$ and $\ker \beta = \ker \alpha$. Since $OM(p,q) \cap OE(p,q)$ is a regular subsemigroup of $OE(p,q)$ (by Theorem 4.1.10), Hall's Theorem implies that the $\mathcal{R}$ relation on $OM(p,q) \cap OE(p,q)$ is the restriction of the $\mathcal{R}$ relation on $OE(p,q)$ to $OM(p,q) \cap OE(p,q)$. In other words, since $\alpha, \beta \in OM(p,q) \cap OE(p,q)$ and $\ker \alpha = \ker \beta$, we deduce from Lemma 4.3.2 that $\alpha \mathcal{R} \beta$ in $OM(p,q) \cap OE(p,q)$ and hence $\alpha \mathcal{R} \beta$ in $OM(p,q)$. That is, $\beta \in R_\alpha$ as required. □

Analogously, it is easy to see the similarity between the characterisations of the $\mathcal{R}$ relations on $AE(p,q)$ and $OE(p,q)$, but there is a substantial difference between the characterisations of the $\mathcal{L}$ relations on these semigroups.

Lemma 4.3.4. Let $\alpha \in OE(p,q)$ and denote the $\mathcal{L}$–class of $OE(p,q)$ containing α by L_α. Then,

(a) $\alpha \notin OM(p,q)$ implies $L_\alpha = \{\alpha\}$;

(b) $\alpha \in OM(p,q)$ implies $L_\alpha = \{\beta \in OE(p,q) : \beta \in OM(p,q) \text{ and } \operatorname{ran} \beta = \operatorname{ran} \alpha\}$.

Proof. First suppose $\alpha \notin OM(p,q)$. If $\beta \in OE(p,q)$ is such that $\alpha \mathcal{L} \beta$ and $\beta \neq \alpha$, then there exist $\lambda, \mu \in OE(p,q)$ such that $\alpha = \lambda\beta$ and $\beta = \mu\alpha$, and so $\operatorname{ran} \alpha = \operatorname{ran} \beta$. By Lemma 4.2.6, we have $q > n(\alpha) = d(\mu) + \dim(\ker \beta / \ker \mu) \geq q$, a contradiction. Thus, (a) holds.

To see that (b) holds, suppose $\alpha \in OM(p,q) \cap OE(p,q)$ and $\alpha \mathcal{L} \beta$ in $OE(p,q)$. Then, as usual, this implies $\operatorname{ran} \alpha = \operatorname{ran} \beta$. Moreover, if $\beta \notin OM(p,q)$ then (a) implies $\alpha \in L_\alpha = L_\beta = \{\beta\}$, hence $\alpha = \beta$, contradicting our supposition.

Now suppose $\beta \in OM(p,q) \cap OE(p,q)$ and $\operatorname{ran} \beta = \operatorname{ran} \alpha$. Since $OM(p,q) \cap OE(p,q)$ is a regular subsemigroup of $OM(p,q)$, Hall's Theorem implies that the $\mathcal{L}$ relation on

$OM(p,q) \cap OE(p,q)$ is the restriction of the $\mathcal{L}$ relation on $OM(p,q)$ to $OM(p,q) \cap OE(p,q)$. In other words, since $\alpha, \beta \in OM(p,q) \cap OE(p,q)$ and $\operatorname{ran} \alpha = \operatorname{ran} \beta$, we deduce from Lemma 4.3.1 that $\alpha \ \mathcal{L} \ \beta$ in $OM(p,q) \cap OE(p,q)$ and hence $\alpha \ \mathcal{L} \ \beta$ in $OE(p,q)$. That is, $\beta \in L_\alpha$ as required. □

We proceed to describe the $\mathcal{D}$ and $\mathcal{J}$ relations on $OM(p,q)$, and (as for the 'almost' case) the characterisation of its ideals follows from this.

Theorem 4.3.5. If $\alpha, \beta \in OM(p,q)$ then $\alpha \ \mathcal{D} \ \beta$ in $OM(p,q)$ if and only if one of the following occurs.

(a) $\alpha, \beta \in OE(p,q)$ and $r(\alpha) = r(\beta)$,

(b) $\alpha, \beta \notin OE(p,q)$ and $\operatorname{ran} \alpha = \operatorname{ran} \beta$.

Proof. Suppose $\alpha \ \mathcal{L} \ \gamma \ \mathcal{R} \ \beta$ in $OM(p,q)$. By Lemma 4.3.3(b), if $\beta \in OE(p,q)$ then $\gamma \in OE(p,q)$ and $\ker \beta = \ker \gamma$. Suppose $\{e_j\}$ is a basis for $\ker \beta = \ker \gamma$ and expand it to a basis $\{e_j\} \dot{\cup} \{e_i\}$ for V. Then $\{e_i\beta\}$ and $\{e_i\gamma\}$ are bases for $\operatorname{ran} \beta$ and $\operatorname{ran} \gamma$, respectively, and hence $r(\beta) = r(\gamma)$. By Lemma 4.3.1, $\operatorname{ran} \alpha = \operatorname{ran} \gamma$, so $r(\alpha) = r(\beta)$; also $d(\alpha) = d(\gamma) \geq q$, so $\alpha \in OE(p,q)$. Conversely, suppose $\alpha, \beta \in OM(p,q) \cap OE(p,q)$ and $r(\alpha) = r(\beta)$. Let $\{e_k\}$ and $\{f_j\}$ be bases for $\ker \alpha$ and $\ker \beta$, respectively, with $|K| = n(\alpha) \geq q$ and $|J| = n(\beta) \geq q$. Expand these sets to two bases for V, say $\{e_k\} \dot{\cup} \{e_i\}$ and $\{f_j\} \dot{\cup} \{f_\ell\}$, respectively. Then, $\{e_i\alpha\}$ is a basis for $\operatorname{ran} \alpha$ and $\{f_\ell\beta\}$ is a basis for $\operatorname{ran} \beta$. Since $r(\alpha) = r(\beta)$, we have $|I| = |L|$, so we can write $\{f_i\}$ and $\{f_i\beta\}$ instead of $\{f_\ell\}$ and $\{f_\ell\beta\}$, respectively. Now define $\lambda \in T(V)$ by

$$\lambda = \begin{pmatrix} f_j & f_i \\ 0 & e_i\alpha \end{pmatrix}.$$

Since $n(\lambda) = n(\beta) \geq q$ and $d(\lambda) = d(\alpha) \geq q$, we have $\lambda \in OM(p,q) \cap OE(p,q)$. In fact, $\operatorname{ran} \alpha = \operatorname{ran} \lambda$ and $\ker \lambda = \ker \beta$, hence $\alpha \ \mathcal{L} \ \lambda \ \mathcal{R} \ \beta$ by Lemmas 4.3.1 and 4.3.3(b). In other words, we have shown that $\alpha \ \mathcal{D} \ \beta$ in $OM(p,q)$.

Now suppose $\alpha \ \mathcal{L} \ \gamma \ \mathcal{R} \ \beta$ in $OM(p,q)$ and $\beta \notin OE(p,q)$. Then, $\beta = \gamma$ by Lemma 4.3.3(a), and so $\alpha \ \mathcal{L} \ \beta$. Hence, $\operatorname{ran} \alpha = \operatorname{ran} \beta$ and $\alpha \notin OE(p,q)$. Conversely, if $\alpha, \beta \notin OE(p,q)$ and $\operatorname{ran} \alpha = \operatorname{ran} \beta$ then $\alpha \ \mathcal{L} \ \beta$ (by Lemma 4.3.1), and the result follows. □

Theorem 4.3.6. If $\alpha, \beta \in OM(p,q)$ then $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in OM(p,q)^1$ if and only if $r(\alpha) \leq r(\beta)$. Consequently, $\alpha \; \mathcal{J} \; \beta$ in $OM(p,q)$ if and only if $r(\alpha) = r(\beta)$.

Proof. Let $\alpha, \beta \in OM(p,q)$ and suppose $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in OM(p,q)^1$. Then, $r(\alpha) \leq r(\beta\mu) \leq r(\beta)$. Conversely, suppose $r(\alpha) \leq r(\beta)$ and let $\{e_j\}$ and $\{f_k\}$ be bases for $\ker \alpha$ and $\ker \beta$, respectively, with $|J| = n(\alpha) \geq q$ and $|K| = n(\beta) \geq q$. Expand $\{e_j\}$ to a basis $\{e_j\} \dot{\cup} \{e_i\}$ for V and write $e_i\alpha = a_i$ for each i. Clearly, $\{a_i\}$ is a basis for $\operatorname{ran} \alpha$. Similarly, expand $\{f_k\}$ to a basis for V, say $\{f_k\} \dot{\cup} \{f_\ell\}$, and write $\{f_\ell\} = \{g_i\} \dot{\cup} \{g_m\}$ (note that $\{f_\ell\beta\}$ is a basis for $\operatorname{ran} \beta$ and $r(\beta) \geq r(\alpha) = |I|$). Now write $g_i\beta = b_i$ and $g_m\beta = b_m$ for each i and each m. Since $\{b_i\} \dot{\cup} \{b_m\}$ is a basis for $\operatorname{ran} \beta$, it can be extended to a basis $\{b_i\} \dot{\cup} \{b_m\} \dot{\cup} \{b_r\}$ for V, where $|R| = d(\beta)$.

If $|M| \geq q$ or $|R| \geq q$, define $\lambda, \mu \in T(V)$ by

$$\lambda = \begin{pmatrix} e_j & e_i \\ 0 & g_i \end{pmatrix}, \quad \mu = \begin{pmatrix} b_i & b_m & b_r \\ a_i & 0 & 0 \end{pmatrix}.$$

Then $n(\lambda) = n(\alpha) \geq q$ and $n(\mu) = |M| + |R| \geq q$, so $\lambda, \mu \in OM(p,q)$. Also, $\alpha = \lambda\beta\mu$. On the other hand, suppose that $|M| < q$ and $|R| < q$. Then, $r(\beta) + d(\beta) = p$ implies $|I| = p$. Since p is infinite, we can write $\{g_i\} = \{u_i\} \dot{\cup} \{v_i\}$. For every i, write $u_i\beta = c_i$ and $v_i\beta = d_i$, and define λ, μ in $T(V)$ by

$$\lambda = \begin{pmatrix} e_j & e_i \\ 0 & u_i \end{pmatrix}, \quad \mu = \begin{pmatrix} c_i & d_i & b_m & b_r \\ a_i & 0 & 0 & 0 \end{pmatrix}.$$

Clearly, $n(\lambda) = n(\alpha) \geq q$ and $n(\mu) = \dim\langle d_i, b_m, b_r \rangle = p \geq q$. Hence, $\lambda, \mu \in OM(p,q)$. Also, $\alpha = \lambda\beta\mu$. □

As remarked in chapter 1, section 1.3, it is well known that the ideals of $T(V)$ are precisely the sets

$$T_\xi = \{\alpha \in T(V) : r(\alpha) < \xi\}$$

where $1 \leq \xi \leq p'$ and p' denotes the successor of p. By Lemma 4.1.1, each $OM(p,\xi)$, with $\aleph_0 \leq \xi \leq p$, is a right ideal of $T(V)$ and each $OE(p,\zeta)$, with $\aleph_0 \leq \zeta \leq p$, is a left ideal of $T(V)$. Hence, $OE(p,\zeta).OM(p,\xi)$ is an ideal of $T(V)$ for all cardinals ζ and ξ such that $\aleph_0 \leq \zeta, \xi \leq p$. Next we show that, in fact, $OE(p,\zeta).OM(p,\xi) = T(V)$. To do this, let $\alpha \in T(V)$ and write, in the usual way,

$$\alpha = \begin{pmatrix} e_j & e_i \\ 0 & a_i \end{pmatrix}.$$

If $|J| \geq \zeta$ and $|J| \geq \xi$ then define $\beta \in T(V)$ by

$$\beta = \begin{pmatrix} e_j & e_i \\ 0 & e_i \end{pmatrix}.$$

Clearly, $\beta \in OE(p,\zeta)$ and $\alpha = \beta\alpha$. Since $\alpha \in OM(p,\xi)$ in this case, it follows that $\alpha \in OE(p,\zeta).OM(p,\xi)$. Now if $|J| < \zeta \leq p$ or $|J| < \xi \leq p$, then $|I| = p$, hence we can write $\{e_i\} = \{f_i\} \dot{\cup} \{g_i\}$. Define $\delta, \gamma \in T(V)$ by

$$\delta = \begin{pmatrix} e_j & e_i \\ 0 & g_i \end{pmatrix}, \quad \gamma = \begin{pmatrix} e_j & f_i & g_i \\ 0 & 0 & a_i \end{pmatrix}.$$

Then, $d(\delta) = \dim\langle e_j, f_i\rangle = p \geq \zeta$ and $n(\gamma) = \dim\langle e_j, f_i\rangle = p \geq \xi$. Since $\alpha = \delta\gamma$, we have $\alpha \in OE(p,\zeta).OM(p,\xi)$. Therefore, $OE(p,\zeta).OM(p,\xi) = T(V)$.

If $\xi \leq p$ then $T_\xi \subseteq OM(p,q)$ (since $r(\alpha) < p$ and $p = n(\alpha)+r(\alpha)$ imply $n(\alpha) = p \geq q$). Hence, each T_ξ, with $1 \leq \xi \leq p$, is an ideal of $OM(p,q)$, and clearly it is a proper subset of $OM(p,q)$. The next result shows that these are exactly the proper ideals of $OM(p,q)$.

Theorem 4.3.7. The proper ideals of $OM(p,q)$ are precisely the sets T_ξ where $1 \leq \xi \leq p$. Moreover, the set T_ξ is a principal ideal of $OM(p,q)$ if and only if ξ is a successor cardinal.

Proof. By the remark above, each T_ξ, with $1 \leq \xi \leq p$, is a proper ideal of $OM(p,q)$. Conversely, let I be any proper ideal of $OM(p,q)$ and let ξ be the least cardinal greater than $r(\beta)$ for every $\beta \in I$ (possible since the cardinals are well-ordered). Then, $1 \leq \xi \leq p'$ and $I \subseteq OM(p,q) \cap T_\xi$. Given $\alpha \in OM(p,q) \cap T_\xi$, we know $n(\alpha) \geq q$ and $r(\alpha) < \xi$. Thus, there exists $\beta \in I$ such that $r(\alpha) \leq r(\beta)$: otherwise, $r(\beta) < r(\alpha) < \xi$ for every $\beta \in I$, and this contradicts our choice of ξ. Therefore, by Theorem 4.3.6, $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in OM(p,q)^1$ and so $\alpha \in I$. Hence $I = OM(p,q) \cap T_\xi$, and this equals T_ξ precisely when $\xi \neq p'$ (since $T_\xi = T(V)$ when $\xi = p'$).

Next we determine all principal ideals of $OM(p,q)$. To do this, let ξ be a successor cardinal, say $\xi = \eta'$, and choose $\alpha \in T_\xi$ with $r(\alpha) = \eta$. Then $r(\beta) \leq \eta$ for every $\beta \in T_\xi$ (otherwise, $r(\beta) > \eta$ implies $r(\beta) \geq \eta' = \xi$, a contradiction). Therefore, by Theorem 4.3.6, $\beta \in J(\alpha)$, the principal ideal of $OM(p,q)$ generated by α. Hence, $T_\xi \subseteq J(\alpha)$

and clearly the reverse inclusion also holds. Thus, T_ξ is principal. Conversely, suppose $T_\xi = J(\alpha)$ for some $\alpha \in OM(p,q)$. Let $r(\alpha) = \eta$ and suppose $\eta < \chi < \xi \leq p$ for some cardinal χ. Clearly, $\chi = r(\beta)$ for some $\beta \in OM(p,q)$ and, by Theorem 4.3.6, $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in OM(p,q)^1$. Therefore, $J(\alpha) \subsetneq J(\beta) \subseteq T_\xi$, contradicting our supposition. In other words, ξ is the least cardinal greater than η, so $\xi = \eta'$. □

Similarly, we can determine the ideals of $OE(p,q)$. To do so, we first describe the $\mathcal{D}$ and $\mathcal{J}$ relations on this semigroup.

Theorem 4.3.8. If $\alpha, \beta \in OE(p,q)$ then $\alpha \mathcal{D} \beta$ in $OE(p,q)$ if and only if one of the following occurs.

(a) $\alpha, \beta \in OM(p,q)$ and $r(\alpha) = r(\beta)$,

(b) $\alpha, \beta \notin OM(p,q)$ and $\ker \alpha = \ker \beta$.

Proof. Suppose $\alpha \mathcal{L} \gamma \mathcal{R} \beta$ in $OE(p,q)$. By Lemma 4.3.4(b), if $\alpha \in OM(p,q)$ then $\gamma \in OM(p,q)$ and $\operatorname{ran} \alpha = \operatorname{ran} \gamma$. By Lemma 4.3.2, $\ker \beta = \ker \gamma$, so $r(\beta) = r(\gamma)$ and $n(\beta) = n(\gamma) \geq q$. Therefore, $\beta \in OM(p,q)$ and $r(\alpha) = r(\beta)$. Conversely, if $\alpha, \beta \in OM(p,q) \cap OE(p,q)$ and $r(\alpha) = r(\beta)$, then the same argument as that used in the proof of Theorem 4.3.5(a) shows that $\alpha \mathcal{D} \beta$ in $OE(p,q)$.

Now suppose $\alpha \mathcal{L} \gamma \mathcal{R} \beta$ in $OE(p,q)$ and $\alpha \notin OM(p,q)$. Then, $\alpha = \gamma$ by Lemma 4.3.4(a), and so $\alpha \mathcal{R} \beta$. By Lemma 4.3.2, $\ker \alpha = \ker \beta$, hence $\beta \notin OM(p,q)$. Conversely, if $\alpha, \beta \notin OM(p,q)$ and $\ker \alpha = \ker \beta$ then $\alpha \mathcal{R} \beta$ (by Lemma 4.3.2), and the result follows. □

Theorem 4.3.9. If $\alpha, \beta \in OE(p,q)$ then $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in OE(p,q)^1$ if and only if $r(\alpha) \leq r(\beta)$. Consequently, $\alpha \mathcal{J} \beta$ in $OE(p,q)$ if and only if $r(\alpha) = r(\beta)$.

Proof. Suppose $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in OE(p,q)^1$. Then, as before, $r(\alpha) \leq r(\beta)$. Conversely, assume $r(\alpha) \leq r(\beta)$ and write, in the usual way,

$$\alpha = \begin{pmatrix} e_j & e_i \\ 0 & a_i \end{pmatrix}, \quad \beta = \begin{pmatrix} f_k & g_i & g_m \\ 0 & b_i & b_m \end{pmatrix}$$

(note that this is possible since $r(\beta) \geq r(\alpha) = |I|$). Clearly, $\{a_i\}$ and $\{b_i\} \dot\cup \{b_m\}$ are bases for $\operatorname{ran} \alpha$ and $\operatorname{ran} \beta$, respectively. Hence, they can be expanded to bases

for V, say $\{a_i\}\dot{\cup}\{a_s\}$ and $\{b_i\}\dot{\cup}\{b_m\}\dot{\cup}\{b_r\}$, respectively, where $|S| = d(\alpha) \geq q$ and $|R| = d(\beta) \geq q$. If $|K| \geq q$ or $|M| \geq q$, then define $\lambda, \mu \in T(V)$ by

$$\lambda = \begin{pmatrix} e_j & e_i \\ 0 & g_i \end{pmatrix}, \quad \mu = \begin{pmatrix} b_i & b_m & b_r \\ a_i & 0 & 0 \end{pmatrix}.$$

Since $d(\lambda) = \dim\langle f_k, g_m\rangle \geq q$ and $d(\mu) = d(\alpha) \geq q$, it follows that $\lambda, \mu \in OE(p,q)$. Also, $\alpha = \lambda\beta\mu$. Now suppose $|K| < q$ and $|M| < q$. Then, $n(\beta) + r(\beta) = p$ implies $|I| = p$. Therefore, we can write $\{g_i\} = \{u_i\}\dot{\cup}\{v_i\}$ (because p is infinite). Let $u_i\beta = c_i$ and $v_i\beta = d_i$ for each i and define $\lambda, \mu \in T(V)$ by

$$\lambda = \begin{pmatrix} e_j & e_i \\ 0 & u_i \end{pmatrix}, \quad \mu = \begin{pmatrix} c_i & d_i & b_m & b_r \\ a_i & 0 & 0 & 0 \end{pmatrix}.$$

Clearly, $d(\lambda) = \dim\langle f_k, v_i, g_m\rangle = p \geq q$ and $d(\mu) = d(\alpha) \geq q$. Hence, $\lambda, \mu \in OE(p,q)$. Also, $\alpha = \lambda\beta\mu$. □

The following result determines the proper ideals of $OE(p,q)$: they are exactly the proper ideals of $T(V)$.

Theorem 4.3.10. The proper ideals of $OE(p,q)$ are precisely the sets

$$T_\xi = \{\alpha \in T(V) : r(\alpha) < \xi\},$$

where $1 \leq \xi \leq p$. Moreover, the set T_ξ is a principal ideal of $OE(p,q)$ if and only if ξ is a successor cardinal.

Proof. If $1 \leq \xi \leq p$ then $T_\xi \subseteq OE(p,q)$ (since $r(\alpha) < \xi \leq p$ and $p = r(\alpha) + d(\alpha)$ imply $d(\alpha) = p \geq q$). Since each T_ξ, with $1 \leq \xi \leq p$, is an ideal of $T(V)$ (see a remark before Theorem 4.3.7) it is an ideal of $OE(p,q)$. For the converse, let I be an ideal of $OE(p,q)$ and let ξ be the least cardinal greater than $r(\beta)$ for every $\beta \in I$ (this is possible since the cardinals are well-ordered). Then, $1 \leq \xi \leq p'$ and $I \subseteq OE(p,q) \cap T_\xi$. Let $\alpha \in OE(p,q) \cap T_\xi$. Then, $d(\alpha) \geq q$ and $r(\alpha) < \xi$, hence (as before) there exists $\beta \in I$ such that $r(\alpha) \leq r(\beta)$. Therefore, by Theorem 4.3.9, $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in OE(p,q)^1$ and so $\alpha \in I$. Hence, $I = OE(p,q) \cap T_\xi$ and this equals T_ξ precisely when $\xi \neq p'$ (since $T_\xi = T(V)$ when $\xi = p'$). Using an argument similar to that in the proof of Theorem 4.3.7, we conclude that T_ξ is principal if and

only if ξ is a successor cardinal; and in this case, $T_\xi = J(\alpha)$ for some α such that $r(\alpha)' = \xi$. $\square$

We end this section by illustrating the ideal structure of the four semigroups we have considered. In the following diagram, we write $AM = AM(p,q)$, $AE = AE(p,q)$, $OM = OM(p,q)$ and $OE = OE(p,q)$. Clearly, the three columns below the first row are mutually disjoint.

$$
\begin{array}{cccc}
AM & OE & OM & AE \\
\searrow \quad \swarrow & \searrow \quad \swarrow & \searrow \quad \swarrow & \\
AM \cap OE(p,q) & OE(p,q) \cap OM(p,q) & OM(p,q) \cap AE & \\
\vdots & \downarrow & \vdots & \\
AM \cap OE(p,\xi) & T_p & OM(p,\xi) \cap AE & \\
\vdots & \vdots & \vdots & \\
AM \cap OE(p,p) & T_1 & OM(p,p) \cap AE &
\end{array}
$$

4.4. Group $\mathcal{H}$–classes

According to Clifford and Preston [5] Vol. 1, p. 51, in 1955 Miller and Doss studied the $\mathcal{D}$–structure of $T(X)$ for an arbitrary non-empty set X. In particular, they determined all $\mathcal{D}$–classes of $T(X)$: they are precisely the sets $D_\xi = \{\alpha \in T(X) : r(\alpha) = \xi\}$, with $1 \leq \xi \leq |X|$. Also, Miller and Doss characterised all $\mathcal{H}$–classes in each D_ξ and they showed that every group $\mathcal{H}$–class of $T(X)$ is isomorphic to a symmetric group $G(Y)$ for some $Y \subseteq X$. As observed by Clifford and Preston in [5] Vol. 1, p. 51, similar results hold for $T(V)$, where V is an arbitrary vector space over a field F (also, see [5] Vol. 1, p. 57, Exercise 6). By analogy with this, here we determine the $\mathcal{D}$–classes and the $\mathcal{H}$–classes of each of the semigroups in $(S1)$–$(S4)$. Also, we describe the group $\mathcal{H}$–classes of each of the four semigroups studied.

It is not difficult to see (by Theorem 4.2.9) that $AM(p,q) \cap AE(p,q)$ is a $\mathcal{D}$–class of $AM(p,q)$ and that there exists a bijection between the set of all cardinals $q \leq \xi \leq p$ and the set of all remaining $\mathcal{D}$–classes of $AM(p,q)$: namely, the $\mathcal{D}$–class D_ξ^{AM} corresponding to ξ consists of all $\alpha \in AM(p,q)$ with defect ξ.

By Lemmas 4.2.1 and 4.2.5(a), there exists a bijection between the set of all pairs (A,B) of subspaces of V, where $\dim A < q$ and $\operatorname{codim} B < q$, and the set of all

$\mathcal{H}$–classes of $AM(p,q)$ in $AM(p,q) \cap AE(p,q)$, defined as follows: the $\mathcal{H}$–class $H_{A,B}$ corresponding to (A,B) consists of all $\alpha \in T(V)$ such that $\ker \alpha = A$ and $\operatorname{ran} \alpha = B$. Similarly, for each cardinal ξ such that $q \leq \xi \leq p$, there exists (by Lemmas 4.2.1 and 4.2.5(b)), a bijection between the set of all pairs (C,D) of subspaces of V, with $\dim C < q$, $\dim D = p$ and $\operatorname{codim} D = \xi$, and the set of all $\mathcal{H}$–classes of $AM(p,q)$ in D_ξ^{AM}: namely, the $\mathcal{H}$–class $H_{C,D}$ corresponding to (C,D) consists of all $\alpha \in T(V)$ such that $\ker \alpha = C$ and $\operatorname{ran} \alpha = D$.

Next we determine all group $\mathcal{H}$–classes of $AM(p,q)$. By Green's Theorem ([5] Vol. 1, Theorem 2.16) if S is a semigroup and H is an $\mathcal{H}$–class of S, then H is a subgroup of S if and only if it contains an idempotent. By [37] Lemma 1, $\lambda = \lambda^2 \in T(V)$ implies $d(\lambda) = n(\lambda)$, hence the idempotents of $AM(p,q)$ belong to $AM(p,q) \cap AE(p,q)$. Thus, the group $\mathcal{H}$–classes of $AM(p,q)$ are in $AM(p,q) \cap AE(p,q)$. Let (A,B) be a pair of subspaces of V where $\dim A < q$ and $\operatorname{codim} B < q$ and let $H_{A,B}$ be the $\mathcal{H}$–class consisting of all $\alpha \in T(V)$ such that $\ker \alpha = A$ and $\operatorname{ran} \alpha = B$. By [5] Vol. 1, p. 57, Exercise 6, $H_{A,B}$ contains an idempotent ε if and only if $V = A \oplus B$; moreover,

$$\varepsilon = \begin{pmatrix} a_j & b_i \\ 0 & b_i \end{pmatrix},$$

where $\{a_j\}$ and $\{b_i\}$ are bases for A and B, respectively. Also, in this case, $H_{A,B}$ is isomorphic to the full linear group $G(B)$ on the vector space B.

Similar results hold for $AE(p,q)$, as we proceed to show. From Theorem 4.2.12, it follows that $AM(p,q) \cap AE(p,q)$ is a $\mathcal{D}$–class of $AE(p,q)$ and there exists a bijection between the set of all cardinals $q \leq \xi \leq p$ and the set of all remaining $\mathcal{D}$–classes of $AE(p,q)$: namely, the $\mathcal{D}$–class D_ξ^{AE} corresponding to ξ consists of all $\alpha \in AE(p,q)$ with nullity ξ.

We have just described the group $\mathcal{H}$–classes of $AM(p,q)$. Likewise, it is not difficult to show (by Lemmas 4.2.2 and 4.2.8(b)) that for each cardinal ξ such that $q \leq \xi \leq p$, there exists a bijection between the set of all pairs (E,F) of subspaces of V, with $\dim E = \xi$ and $\operatorname{codim} F < q$, and the set of all $\mathcal{H}$–classes of $AE(p,q)$ in D_ξ^{AE}, defined as follows: the $\mathcal{H}$–class $H_{E,F}$ corresponding to (E,F) consists of all $\alpha \in T(V)$ such that $\ker \alpha = E$ and $\operatorname{ran} \alpha = F$. Clearly, the $\mathcal{H}$–classes of $AE(p,q)$ in $AM(p,q) \cap AE(p,q)$ and the $\mathcal{H}$–classes of $AM(p,q)$ in $AM(p,q) \cap AE(p,q)$

coincide. In particular, the group $\mathcal{H}$–classes of $AE(p,q)$ are precisely the $\mathcal{H}$–classes $H_{A,B}$ corresponding to pairs (A,B) of subspaces of V such that $\dim A < q$ and $V = A \oplus B$; and we showed that, in this case, $H_{A,B}$ is isomorphic to $G(B)$.

Next, we study the $\mathcal{D}$–classes and the $\mathcal{H}$–classes of the semigroups in $(S3)$–$(S4)$. By Theorem 4.3.5(a), the $\mathcal{D}$–classes of $OM(p,q)$ contained in $OE(p,q)$ are exactly the sets

$$D_k = \{\alpha \in T(V) : n(\alpha) \geq q,\ d(\alpha) \geq q,\ r(\alpha) = k\},$$

with $0 \leq k \leq p$; and by Lemmas 4.3.1 and 4.3.3(b), there exists a bijection between the set of all $\mathcal{H}$–classes of $OM(p,q)$ in each D_k and the set of all pairs (A,B) of subspaces of V where $\dim A \geq q$, $\dim B = k$ and $\operatorname{codim} B \geq q$: namely, the $\mathcal{H}$–class $H^{OM}_{A,B}$ corresponding to the pair (A,B) consists of all $\alpha \in T(V)$ with $\ker \alpha = A$ and $\operatorname{ran} \alpha = B$. As before, the $\mathcal{H}$–class $H^{OM}_{A,B}$ is a subgroup of $OM(p,q)$ if and only if $V = A \oplus B$ and, in this case, it is isomorphic to $G(B)$. By Theorem 4.3.5(b), there exists a bijection between the set of all remaining $\mathcal{D}$–classes of $OM(p,q)$ and the set of all subspaces A of V with $\operatorname{codim} A < q$, defined as follows: the $\mathcal{D}$–class corresponding to A consists of all $\alpha \in T(V)$ such that $n(\alpha) \geq q$ and $\operatorname{ran} \alpha = A$. Note that, in this case, α is not an idempotent, since $n(\alpha) \neq d(\alpha)$ (see a remark before Theorem 4.1.4). By Lemma 4.3.3(b), the $\mathcal{H}$–classes of $OM(p,q)$ contained in such $\mathcal{D}$–classes are singletons, and none of them is a group $\mathcal{H}$–class of $OM(p,q)$ since it contains no idempotents.

As for the 'opposite one-to-one' case, by Theorem 4.3.8(a), the $\mathcal{D}$–classes of $OE(p,q)$ contained in $OM(p,q)$ are precisely the sets D_k, with $0 \leq k \leq p$; and the $\mathcal{H}$–classes of $OE(p,q)$ and of $OM(p,q)$ in each D_k coincide. In particular, each group $\mathcal{H}$–class of $OE(p,q)$ corresponds to a pair (A,B) of subspaces of V such that $\dim A \geq q$, $\dim B = k$ and $V = A \oplus B$: in this case, the $\mathcal{H}$–class is isomorphic to $G(B)$. By Theorem 4.3.8(b), there exists a bijection between the set of all remaining $\mathcal{D}$–classes of $OE(p,q)$ and the set of all subspaces A of V with $\dim A < q$: namely, the $\mathcal{D}$–class corresponding to A consists of all $\alpha \in T(V)$ such that $d(\alpha) \geq q$ and $\ker \alpha = A$ (note that, in this case, no α is an idempotent since $n(\alpha) \neq d(\alpha)$). By Lemma 4.3.4(a), the $\mathcal{H}$–classes of $OE(p,q)$ contained in such $\mathcal{D}$–classes are singletons and, as before, none of them is a group $\mathcal{H}$–class of $OE(p,q)$.

Example 4.4.1. Suppose $p = q = \aleph_0$ and consider the $\mathcal{D}$–class $AM(p,q) \cap AE(p,q)$ of $AM(p,q)$. We note that, by Theorem 4.1.4, this is a regular $\mathcal{D}$–class of $AM(p,q)$. Let U be a subspace of V with codimension 1 and suppose $V = U \oplus \langle a \rangle = U \oplus \langle b \rangle$, where a, b are linearly independent. Let $\{e_i\}$ be a basis for U, with $|I| = \aleph_0$, and choose $c \in \{e_i\}$. Now put $\{e_i\} \setminus \{c\} = \{f_i\}$ and define $\alpha, \beta \in T(V)$ by

$$\alpha = \begin{pmatrix} a & e_i \\ 0 & e_{i\pi} \end{pmatrix}, \quad \beta = \begin{pmatrix} b & e_i \\ 0 & f_i \end{pmatrix},$$

where π is a permutation of the index set I. Clearly, $\ker \alpha = \langle a \rangle$ and $\operatorname{ran} \alpha = \langle e_i \rangle$, hence $V = \ker \alpha \oplus \operatorname{ran} \alpha$. Also, $\ker \beta = \langle b \rangle$ and $\operatorname{ran} \beta = \langle f_i \rangle$, so $V = \ker \beta \oplus \operatorname{ran} \beta \oplus \langle c \rangle$. By Lemmas 4.2.1 and 4.2.2, we know that α and β are neither $\mathcal{L}$–related nor $\mathcal{R}$–related in $AM(p,q) \cap AE(p,q)$ (since $\operatorname{ran} \beta \subsetneq \operatorname{ran} \alpha$ and $\ker \alpha \neq \ker \beta$). In effect, from Lemmas 4.2.1 and 4.2.5, the $\mathcal{H}$–class H_α of $AM(p,q)$ which contains α consists of all $\lambda \in T(V)$ such that $\ker \lambda = \langle a \rangle$ and $\operatorname{ran} \lambda = \langle e_i \rangle$. In view of the above, since $V = \langle a \rangle \oplus \langle e_i \rangle$, it follows that H_α is a group $\mathcal{H}$–class and its idempotent is the transformation ϵ_1^* defined in $T(V)$ by

$$\epsilon_1^* = \begin{pmatrix} a & e_i \\ 0 & e_i \end{pmatrix}.$$

On the other hand, the $\mathcal{H}$–class H_β which contains β is a non-group $\mathcal{H}$–class (since $\ker \beta \oplus \operatorname{ran} \beta \subsetneq V$). In fact, we may visualise the $\mathcal{D}$–class $AM(p,q) \cap AE(p,q)$ as an infinite 'eggbox', in which each row represents an $\mathcal{R}$–class, each column an $\mathcal{L}$–class, and each cell an $\mathcal{H}$–class (compare [5] Vol. 1, p. 56). Here, we display two group and two non-group $\mathcal{H}$–classes, although there are infinitely many $\mathcal{H}$-classes of each kind. Also, we identify one element of each of the four $\mathcal{H}$–classes represented in the picture: starred elements are idempotents and these show which of the four $\mathcal{H}$–classes considered are groups.

	L_α		L_β	
R_α	ϵ_1^*		γ	
R_β	ϵ_2^*		β	

It is easy to see that, by (4.1.2), the idempotent in the group $\mathcal{H}$–class $R_\beta \cap L_\alpha$ is the transformation ϵ_2^* given by

$$\epsilon_2^* = \begin{pmatrix} b & e_i \\ 0 & e_i \end{pmatrix}.$$

Also, the transformation γ defined in $T(V)$ by

$$\gamma = \begin{pmatrix} a & e_i \\ 0 & f_i \end{pmatrix}$$

is contained in $R_\alpha \cap L_\beta$, since $\ker \gamma = \ker \alpha$ and $\operatorname{ran} \gamma = \operatorname{ran} \beta$. But $\ker \gamma \oplus \operatorname{ran} \gamma \subsetneq V$, hence the $\mathcal{H}$–class $H_\gamma = R_\alpha \cap L_\beta$ is a non-group $\mathcal{H}$–class.

4.5. Maximal right cancellative (or right simple) subsemigroups

In chapter 3, we studied basic properties of the semigroup $KN(p,q)$ consisting of all injective $\alpha \in T(V)$ for which $d(\alpha) \geq q$; and we showed that $KN(p,q)$ is a right cancellative semigroup without idempotents. Clearly, $KN(p,q) \subseteq AM(p,q) \cap OE(p,q)$ and, in fact, $KN(p,q)$ is a maximal right cancellative subsemigroup of $OE(p,q)$, as we proceed to show.

Lemma 4.5.1. If S is a subsemigroup of $T(V)$ which contains $KN(p,q)$ and is not contained in $M(V)$, then there exists some $\alpha \in S$ such that $n(\alpha) \geq 2$.

Proof. Suppose $n(\alpha) < 2$ for every $\alpha \in S$. Since $S \not\subseteq M(V)$, there exists $\gamma \in S$ such that $n(\gamma) = 1$. Let $a \in \ker \gamma$ be non-zero and suppose $\{a\} \dot\cup \{a_i\}$ is a basis for V, with $|I| = p$. Write $\{a_i\} = \{b\} \dot\cup \{c_i\}$ (note that this is possible since p is infinite). Clearly, $\{b\gamma\} \dot\cup \{c_i\gamma\}$ is a basis for $\operatorname{ran} \gamma$, hence it can be extended to a basis for V, say $\{b\gamma\} \dot\cup \{c_i\gamma\} \dot\cup \{d_\ell\}$, where $|L| = d(\gamma) \leq p$. Now write $\{c_i\} = \{e_i\} \dot\cup \{f_i\} \dot\cup \{g_\ell\}$ and define $\lambda, \mu \in T(V)$ by

$$\lambda = \begin{pmatrix} a & b & c_i \\ b & a & e_i \end{pmatrix}, \quad \mu = \begin{pmatrix} b\gamma & c_i\gamma & d_\ell \\ a & e_i & g_\ell \end{pmatrix}.$$

Then $n(\lambda) = 0 = n(\mu)$ and $d(\lambda) = p = d(\mu)$, so $\lambda, \mu \in KN(p,q) \subseteq S$. Therefore, $\lambda\gamma\mu\gamma \in S$ and we have $a\lambda\gamma\mu\gamma = 0 = b\lambda\gamma\mu\gamma$. Since $\{a, b\}$ is linearly independent, it follows that $n(\lambda\gamma\mu\gamma) \geq 2$, a contradiction. Hence, there exists some $\alpha \in S$ such that $n(\alpha) \geq 2$. □

Theorem 4.5.2. The semigroup $KN(p,q)$ is a maximal right cancellative subsemigroup of $OE(p,q)$.

Proof. Suppose $KN(p,q) \subseteq M \subseteq OE(p,q)$, where M is a right cancellative subsemigroup of $OE(p,q)$. If M is not contained in $M(V)$, then there exists some $\beta \in M$ such that $n(\beta) \geq 2$ (by Lemma 4.5.1). Suppose $\{e_j\}$ is a basis for $\ker \beta$ and let $a, b \in \{e_j\}$, $a \neq b$. Now expand $\{a,b\}$ to a basis $\{a,b\} \dot{\cup} \{e_i\}$ for V, with $|I| = p$, and write $\{e_i\} = \{f_i\} \dot{\cup} \{g_i\}$ (possible since p is infinite). Define $\lambda, \mu \in T(V)$ by

$$\lambda = \begin{pmatrix} a & b & e_i \\ a & b & f_i \end{pmatrix}, \quad \mu = \begin{pmatrix} a & b & e_i \\ b & a & f_i \end{pmatrix}.$$

Then $n(\lambda) = 0 = n(\mu)$ and $d(\lambda) = p = d(\mu)$, hence $\lambda, \mu \in KN(p,q) \subseteq M$. Clearly $\lambda\beta = \mu\beta$ and, since M is right cancellative, it follows that $\lambda = \mu$, a contradiction. Therefore, $M \subseteq M(V) \cap OE(p,q)$, and hence $M \subseteq KN(p,q)$, so the result follows.□

Example 4.5.3. Here we see that $OE(p,q)$ has maximal right cancellative subsemigroups which do not equal $KN(p,q)$. This was observed by Sullivan (personal communication). Let $V = U \oplus \langle x_i \rangle$ where $\dim U = q$ and $|I| = p$, and let H denote the set of all $\alpha \in T(V)$ with the form:

$$\alpha = \begin{pmatrix} U & x_i \\ 0 & x_{i\pi} \end{pmatrix}$$

for some $\pi \in G(I)$, the symmetric group on the set I. It is easy to see that H is a subgroup of $OE(p,q)$, and clearly $H \cap M(V) = \emptyset$. Moreover, if $\mathcal{F}$ denotes the family of all right cancellative subsemigroups of $OE(p,q)$ that contain H, then $\mathcal{F}$ is non-empty. Thus, we can use Zorn's Lemma to show that $\mathcal{F}$ contains a maximal element, H' say. Then H' is a maximal right cancellative subsemigroup of $OE(p,q)$ which does not equal $KN(p,q)$.

By chapter 3, Theorem 3.6.1, if $q < p$ then the maximal right simple subsemigroups of $KN(p,q)$ are precisely the linear Baer-Levi semigroups $GS(p,\xi)$ where $q < \xi < p$ or ξ equals p or q, but not both. It is not difficult to show that each $GS(p,\xi)$ is a maximal right simple subsemigroup of $AM(p,q)$ (even if $p = q$). In fact, we will determine all maximal right simple subsemigroups of $AM(p,q)$. To do this, we need two preliminary results.

Lemma 4.5.4. For each infinite cardinal ξ such that $\xi \leq p$, and for each subspace A of V with $\dim A < q$, the set

$$M(A,\xi) = \{\alpha \in T(V) : \ker \alpha = A,\ \text{ran}\ \alpha \cap A = \{0\},\ \dim V/(\text{ran}\ \alpha \oplus A) = \xi\}$$

is a maximal right simple subsemigroup of $AM(p,q)$.

Proof. Clearly, $M(A,\xi) \subseteq AM(p,q)$ and it is non-empty. For example, if $V = \langle a_j\rangle \oplus \langle a_i\rangle$ where $A = \langle a_j\rangle$ and $|I| = p$ (possible since $\dim A < q \leq p$), we can write $\{a_i\} = \{b_i\} \dot\cup \{b_k\}$ where $|K| = \xi$ and define $\pi \in M(A,\xi)$ by

$$\pi = \begin{pmatrix} a_j & a_i \\ 0 & b_i \end{pmatrix}.$$

Let $\alpha, \beta \in M(A,\xi)$. Then, by (4.1.1), $(\ker(\alpha\beta))\alpha = \text{ran}\ \alpha \cap \ker \beta = \{0\}$ and so $\ker(\alpha\beta) \subseteq \ker \alpha$. Since $\ker \alpha \subseteq \ker(\alpha\beta)$ always, it follows that $\ker(\alpha\beta) = A$. Also $\text{ran}\,(\alpha\beta) \subseteq \text{ran}\ \beta$ implies $\text{ran}\,(\alpha\beta) \cap A \subseteq \{0\}$, and equality follows. Now suppose $\{a_j\}$ is a basis for A and expand it to a basis $\{a_j\} \dot\cup \{a_i\}$ for V, with $|I| = \text{codim}\ A = p$. For each i, write $a_i\alpha = e_i$. Then $\{e_i\}$ is a basis for $\text{ran}\ \alpha$, and $\text{ran}\ \alpha \cap A = \{0\}$ implies $V = \langle a_j\rangle \oplus \langle e_i\rangle \oplus \langle e_k\rangle$ for some linearly independent $\{e_k\} \subseteq V$, where $|K| = \dim V/(\text{ran}\ \alpha \oplus A) = \xi$. Now write $e_i\beta = f_i$ and $e_k\beta = f_k$ for every i and every k, respectively. Since $\ker \beta = A$, we know that $\{f_i\} \dot\cup \{f_k\}$ is a basis for $\text{ran}\ \beta$, hence it can be expanded to a basis for V, say $\{f_i\} \dot\cup \{f_k\} \dot\cup \{c_k\} \dot\cup \{a_j\}$ (recall that $\text{ran}\ \beta \cap A = \{0\}$ and $\dim V/(\text{ran}\ \beta \oplus A) = \xi = |K|$). Clearly, we have

$$\alpha\beta = \begin{pmatrix} a_j & a_i \\ 0 & f_i \end{pmatrix},$$

hence $\dim V/(\text{ran}\,(\alpha\beta) \oplus A) = \dim\langle f_k, c_k\rangle = \xi + \xi = \xi$ (since ξ is infinite). Therefore, $\alpha\beta \in M(A,\xi)$ and so $M(A,\xi)$ is a subsemigroup of $AM(p,q)$.

Next we show that $M(A,\xi)$ is right simple. To do this, write $a_i\beta = c_i$ for every i. Since $\ker \beta = A$, we know $\{c_i\}$ is a basis for $\text{ran}\ \beta$, hence it can be expanded to a basis for V, say $\{c_i\} \dot\cup \{g_k\} \dot\cup \{a_j\}$ (note that $\text{ran}\ \beta \cap A = \{0\}$ and $\dim V/(\text{ran}\ \beta \oplus A) = \xi = |K|$). Now write $\{g_k\} = \{u_k\} \dot\cup \{v_k\}$ (possible since $|K| = \xi \geq \aleph_0$) and define λ in $T(V)$ by

$$\lambda = \begin{pmatrix} a_j & e_i & e_k \\ 0 & c_i & u_k \end{pmatrix}.$$

Then, $\ker \lambda = A$, $\operatorname{ran} \lambda \cap A = \{0\}$ and $\dim V/(\operatorname{ran} \lambda \oplus A) = \xi$, so $\lambda \in M(A, \xi)$. Also $\beta = \alpha\lambda$, and we have shown that $M(A, \xi)$ is right simple.

Next suppose $M(A, \xi) \subseteq M \subseteq AM(p, q)$ where M is a right simple subsemigroup of $AM(p, q)$. Since $AM(p, q)$ is not right simple (see a remark before Theorem 4.1.4), it follows that $M \neq AM(p, q)$. Let $\alpha \in M$ and $\gamma \in M(A, \xi)$. If $\alpha = \gamma$ then $\alpha \in M(A, \xi)$. If not, then both α and γ are elements of M and, since this semigroup is right simple, there exist $\lambda, \mu \in M$ such that $\alpha = \gamma\lambda$ and $\gamma = \alpha\mu$: that is, $\alpha \mathcal{R} \gamma$ in M, and hence in $AM(p, q)$ also. By Lemma 4.2.5 we have $\ker \alpha = \ker \gamma = A$. Now suppose there exists a non-zero $v = u\alpha \in \operatorname{ran} \alpha \cap A$. Then $u \notin A = \ker \alpha$ and so $\ker \gamma \subsetneq A \oplus \langle u \rangle \subseteq \ker(\alpha\gamma)$. From Lemma 4.2.5, we deduce that γ and $\alpha\gamma$ are not $\mathcal{R}$–related in $AM(p, q)$, and hence $\alpha\gamma \notin M$ since M is right simple. But this contradicts the fact that M is closed, so $\operatorname{ran} \alpha \cap A = \{0\}$. Next, we claim that $\dim V/(\operatorname{ran} \alpha \oplus A) = \dim V/(\operatorname{ran} \gamma \oplus A)$.

First, since $\lambda, \mu \in M$, an argument similar to the one above shows that $\ker \lambda = A = \ker \mu$ and $\operatorname{ran} \lambda \cap A = \{0\} = \operatorname{ran} \mu \cap A$. Next, we adopt the same notation as in the second paragraph of this proof, although for a different α. Now write $a_i\gamma = g_i$ for each i. Then $\{g_i\}$ is a basis for $\operatorname{ran} \gamma$ (since $\ker \gamma = A = \langle a_j \rangle$) and it can be expanded to a basis for V, say $\{g_i\} \dot\cup \{a_j\} \dot\cup \{g_\ell\}$, where $|L| = \dim V/(\operatorname{ran} \gamma \oplus A) = \xi$ since $\gamma \in M(A, \xi)$. Clearly, $e_i = a_i\alpha = g_i\lambda$ for each i and, since $\ker \lambda = A$, we deduce that $\operatorname{ran} \lambda = \langle e_i \rangle \oplus \langle g_\ell\lambda \rangle$. Consequently, since $\operatorname{ran} \alpha = \langle e_i \rangle$ and $\operatorname{ran} \lambda \cap A = \{0\}$, we obtain

$$\dim V/(\operatorname{ran} \alpha \oplus A) = \operatorname{codim} \langle e_i, a_j \rangle = |K| \geq |L|.$$

Likewise, $\gamma = \alpha\mu$ implies $|K| \leq |L|$. Thus, our claim is valid, hence α belongs to $M(A, \xi)$, and so $M(A, \xi) = M$. Therefore, $M(A, \xi)$ is a maximal right simple subsemigroup of $AM(p, q)$. □

Note that for each cardinal ξ such that $q \leq \xi \leq p$, we have $GS(p, \xi) = M(\{0\}, \xi)$, hence each $GS(p, \xi)$ is a maximal right simple subsemigroup of $AM(p, q)$, as observed before.

Clearly, the group $G(V)$, consisting of all invertible linear transformations of V, is a right simple subsemigroup of $AM(p, q)$. In fact, it is maximal under these conditions.

For, suppose $G(V) \subseteq M \subseteq AM(p,q)$ for some right simple subsemigroup M of $AM(p,q)$. Then, given $\alpha \in M$ and $\gamma \in G(V)$, we have $\alpha \mathcal{R} \gamma$ in M and hence also in $AM(p,q)$, so $\ker \alpha = \ker \gamma = \{0\}$ by Lemma 4.2.5. If $\alpha = \gamma$ then $\alpha \in G(V)$. If not then, in fact, $\alpha = \gamma\lambda$ and $\gamma = \alpha\mu$ for some $\lambda, \mu \in M$ imply λ and μ are $\mathcal{R}$–related to $\gamma \in M$ (since M is right simple), and so $\ker \lambda = \{0\} = \ker \mu$ as before. Therefore, by Lemma 4.2.3,

$$d(\gamma) \leq n(\lambda) + d(\alpha) = d(\alpha) \leq n(\mu) + d(\gamma) = d(\gamma).$$

Hence, $d(\alpha) = 0 = n(\alpha)$ and $\alpha \in G(V)$. In fact, the next result gives a class of maximal right simple subsemigroups of $AM(p,q)$ which contains $G(V)$ (with a slight abuse of terminology, we observe that $G(V) = N(B,\zeta)$ precisely when $\zeta = 0$ and $B = \{0\}$).

Lemma 4.5.5. For every cardinal $\zeta < q$ and every subspace B of V with dimension ζ, the set

$$N(B,\zeta) = \{\alpha \in T(V) : \ker \alpha = B,\ V = \operatorname{ran} \alpha \oplus B\}$$

is a maximal right simple subsemigroup of $AM(p,q)$.

Proof. Clearly, $N(B,\zeta) \subseteq AM(p,q)$ and it is non-empty. For example, if $V = \langle b_j \rangle \oplus \langle b_i \rangle$ where $B = \langle b_j \rangle$, $|J| = \zeta$ and $|I| = \operatorname{codim} B$, we can define $\alpha \in N(B,\zeta)$ by

$$\alpha = \begin{pmatrix} b_j & b_i \\ 0 & b_i \end{pmatrix}.$$

Let $\alpha, \beta \in N(B,\zeta)$. Then, $\operatorname{ran} \alpha \cap B = \{0\}$ and (4.1.1) imply $(\ker(\alpha\beta))\alpha = \operatorname{ran} \alpha \cap \ker \beta = \{0\}$, hence $\ker(\alpha\beta) \subseteq B$. Since $B = \ker \alpha \subseteq \ker(\alpha\beta)$, we have $\ker(\alpha\beta) = B$. Clearly $\operatorname{ran}(\alpha\beta) \subseteq \operatorname{ran} \beta$. Now, if $v \in V$, then $v = a + b$ for some $a \in \ker \alpha = \ker \beta$ and $b \in \operatorname{ran} \alpha$. Therefore, there exists $u \in V$ such that $b = u\alpha$ and $v\beta = a\beta + b\beta = u(\alpha\beta)$. Hence, $\operatorname{ran}(\alpha\beta) = \operatorname{ran} \beta$ and so $\alpha\beta \in N(B,\zeta)$. That is, $N(B,\zeta)$ is a semigroup.

Now suppose $\{b_j\}$ is a basis for B and expand it to a basis $\{b_j\} \dot{\cup} \{b_i\}$ for V. For each i, write $b_i\alpha = e_i$ and $b_i\beta = f_i$. Since $\{e_i\}$ and $\{f_i\}$ are bases for $\operatorname{ran} \alpha$ and $\operatorname{ran} \beta$, respectively, we have $V = \langle e_i \rangle \oplus \langle b_j \rangle = \langle f_i \rangle \oplus \langle b_j \rangle$. Define $\lambda \in T(V)$ by

$$\lambda = \begin{pmatrix} e_i & b_j \\ f_i & 0 \end{pmatrix}.$$

Clearly, $\lambda \in N(B,\zeta)$ and $\beta = \alpha\lambda$. In other words, $N(B,\zeta)$ is right simple.

We have just proved that $N(B,\zeta)$ is a right simple subsemigroup of $AM(p,q)$: next we show it is maximal under these conditions. To do this, suppose $N(B,\zeta) \subseteq M \subseteq AM(p,q)$, where M is a right simple subsemigroup of $AM(p,q)$. As before, $M \neq AM(p,q)$ since the latter is not right simple. Now let $\alpha \in M$ and $\gamma \in N(B,\zeta)$. If $\alpha = \gamma$ then $\alpha \in N(B,\zeta)$. Now suppose $\alpha \neq \gamma$. Clearly, $\alpha, \gamma \in M$ and so $\alpha = \gamma\lambda$ and $\gamma = \alpha\mu$ for some $\lambda, \mu \in M$. Since $d(\gamma) = \zeta < q$, we have $\gamma \in AE(p,q)$, hence Lemma 4.2.5(a) implies $d(\alpha) < q$ and $\ker \alpha = \ker \gamma = B$. As in the proof of Lemma 4.5.4, if $\operatorname{ran} \alpha \cap B \neq \{0\}$ then γ and $\alpha\gamma$ are not $\mathcal{R}$–related in $AM(p,q)$, which implies $\alpha\gamma \notin M$, a contradiction. Therefore $\operatorname{ran} \alpha \cap B = \{0\}$. Likewise, by considering $\lambda, \gamma \in M$ and $\mu, \gamma \in M$, we deduce that $\ker \lambda = B = \ker \mu$ and $\operatorname{ran} \lambda \cap B = \{0\} = \operatorname{ran} \mu \cap B$. Suppose $\operatorname{ran} \alpha \oplus B \subsetneq V$ and write $V = \langle e_i \rangle \oplus \langle b_j \rangle \oplus \langle v_s \rangle$, where $\{b_j\}$ is a basis for B, $\{b_j\} \dot\cup \{b_i\}$ is a basis for V and $e_i = b_i\alpha$ for each i. Since $b_i\gamma = (b_i\alpha)\mu = e_i\mu$ and $V = \langle b_i\gamma \rangle \oplus \langle b_j \rangle$, we have $V = \langle e_i\mu \rangle \oplus \langle b_j \rangle \subsetneq \langle e_i\mu \rangle \oplus \langle v_s\mu \rangle \oplus \langle b_j \rangle \subseteq V$, a contradiction since $S \neq \emptyset$ and $\langle v_s \rangle \cap \ker \mu = \{0\}$. Hence $\operatorname{ran} \alpha \oplus B = V$. Thus, $\alpha \in N(B,\zeta)$ and $M = N(B,\zeta)$. Therefore, $N(B,\zeta)$ is a maximal right simple subsemigroup of $AM(p,q)$. □

Theorem 4.5.6. The maximal right simple subsemigroups of $AM(p,q)$ are exactly the sets $M(A,\xi)$, where A is a subspace of V with $\dim A < q$ and ξ is an infinite cardinal such that $\xi \leq p$, and the sets $N(B,\zeta)$, where ζ is a cardinal such that $\zeta < q$ and B is a subspace of V with $\dim B = \zeta$.

Proof. By Lemma 4.5.4, each $M(A,\xi)$ is a maximal right simple subsemigroup of $AM(p,q)$; and by Lemma 4.5.5, so is each $N(B,\zeta)$. Now suppose M is a maximal right simple subsemigroup of $AM(p,q)$ and let $\alpha \in M$. For every $\beta \in M$, α and β are $\mathcal{R}$–related in $AM(p,q)$, hence $\ker \alpha = \ker \beta$. Let $A = \ker \alpha$. As in the proof of Lemma 4.5.4, if $\operatorname{ran} \beta \cap A \neq \{0\}$ for some $\beta \in M$, then $A \subsetneq \ker(\beta\alpha)$ and so $\beta\alpha \notin M$, a contradiction. Therefore, $\operatorname{ran} \beta \cap A = \{0\}$ for every $\beta \in M$: in particular, we have $d(\beta) \geq \dim A$. Since M is right simple, if $\beta \neq \alpha$ then there exist $\lambda, \mu \in M$ such that $\alpha = \beta\lambda$ and $\beta = \alpha\mu$. Since $\lambda, \mu \in M$, we have $\ker \lambda = A = \ker \mu$ and $\operatorname{ran} \lambda \cap A = \{0\} = \operatorname{ran} \mu \cap A$. In fact, using an argument similar to that in the proof of Lemma 4.5.4, we can show that $\dim V/(\operatorname{ran} \alpha \oplus A) = \dim V/(\operatorname{ran} \beta \oplus A)$.

Let $\xi = \dim V/(\operatorname{ran}\alpha \oplus A)$ and suppose $\xi \geq \aleph_0$. Then, $M \subseteq M(A,\xi)$ and, by the maximality of M, it follows that $M = M(A,\xi)$.

On the other hand, if ξ is finite then it must be 0: that is, we claim that in this case $V = \operatorname{ran}\beta \oplus A$ for every $\beta \in M$. For, suppose $\operatorname{ran}\alpha \oplus A \subsetneq V$ and write, in the usual way,

$$\alpha = \begin{pmatrix} a_j & a_i \\ 0 & e_i \end{pmatrix}.$$

Now expand $\{e_i\}$ to a basis $\{e_i\}\dot\cup\{a_j\}\dot\cup\{e_k\}$ for V, with $|K| = \xi < \aleph_0$. Write $e_i\alpha = v_i$ and $e_k\alpha = v_k$ for every i and every k. Since $\{v_i\}\dot\cup\{v_k\}$ is a basis for $\operatorname{ran}\alpha$, it can be expanded to a basis for V, say $\{v_i\}\dot\cup\{v_k\}\dot\cup\{a_j\}\dot\cup\{f_k\}$ (this is possible since $\operatorname{ran}\alpha \cap A = \{0\}$ and $\dim V/(\operatorname{ran}\alpha \oplus A) = |K|$). Clearly, $\dim V/(\operatorname{ran}\alpha^2 \oplus A) = \dim\langle v_k, f_k\rangle = 2\xi \neq \xi$, a contradiction. Therefore, $V = \operatorname{ran}\alpha \oplus A$ and $V/(\operatorname{ran}\alpha \oplus A) = \{0\}$. Hence, $V/(\operatorname{ran}\beta \oplus A) = \{0\}$ for every $\beta \in M$, and this implies $V = \operatorname{ran}\beta \oplus A$. Thus, $M \subseteq N(A, \dim A)$ and by the maximality of M, we have $M = N(A, \dim A)$, and the result follows. □

4.6. Products of idempotents and products of nilpotents

In this section, we determine the subsemigroup of $AM(p,q) \cap AE(p,q)$ generated by its *proper* idempotents (that is, idempotents different from the identity), as well as the subsemigroup of $OM(p,q) \cap OE(p,q)$ generated by its idempotents. For this, we use work by Reynolds and Sullivan [37]. Also, we describe the subsemigroup of $OM(p,q) \cap OE(p,q)$ generated by all nilpotent linear transformations, using work by Sullivan [43] (clearly, $0 \notin AM(p,q) \cap AE(p,q)$ for every cardinal q, hence this semigroup has no nilpotents).

In [37], section 2, Reynolds and Sullivan showed that $\mathcal{E}(V)$, the subsemigroup of $T(V)$ generated by the proper idempotents of $T(V)$, is a disjoint union of two subsemigroups, F and I, defined by

$$F = \{\alpha \in T(V) : 0 < n(\alpha) = d(\alpha) \leq s(\alpha) < \aleph_0\},$$

$$I = \{\alpha \in T(V) : d(\alpha) = n(\alpha) = s(\alpha) \geq \aleph_0\}.$$

To do so, for each infinite cardinal $\xi \leq p = \dim V$, they considered the set

$$I(\xi) = \{\alpha \in T(V) : d(\alpha) = n(\alpha) = s(\alpha) = \xi\}.$$

From [37] Lemma 1, if $\lambda = \lambda^2 \in T(V)$, then $d(\lambda) = n(\lambda) = s(\lambda)$. Therefore, the idempotents of $T(V)$ belong either to $AM(p,q) \cap AE(p,q)$ or $OM(p,q) \cap OE(p,q)$. Thus, we consider the subsemigroup $\mathcal{EA}(V)$ of $AM(p,q) \cap AE(p,q)$ generated by the proper idempotents of $AM(p,q) \cap AE(p,q)$ and the subsemigroup $\mathcal{EO}(V)$ of $OM(p,q) \cap OE(p,q)$ generated by the idempotents of $OM(p,q) \cap OE(p,q)$.

Suppose $q > \aleph_0$ and write I as a disjoint union of two sets: namely,

$$I_{<q} = \{\alpha \in T(V) : \aleph_0 \leq d(\alpha) = n(\alpha) = s(\alpha) < q\},$$
$$I_{\geq q} = \{\alpha \in T(V) : d(\alpha) = n(\alpha) = s(\alpha) \geq q\}.$$

From Lemma 4.1.1 and [37] Lemma 5, $I_{<q}$ and $I_{\geq q}$ are subsemigroups of $T(V)$. From [37] Theorems 1 and 2, F and $I_{<q}$ are regular subsemigroups of $AM(p,q) \cap AE(p,q)$ and $I_{\geq q}$ is a regular subsemigroup of $OM(p,q) \cap OE(p,q)$. Clearly,

$$\mathcal{EA}(V) \subseteq F \,\dot\cup\, I_{<q}, \quad \mathcal{EO}(V) \subseteq I_{\geq q}.$$

In [37] Theorem 3, the authors proved that F is generated by the proper idempotents of $T(V)$ with finite shift, and this implies those idempotents belong to $\mathcal{EA}(V)$. In [37] Theorem 4, they showed each $\alpha \in I(\xi)$ can be written as a product of three idempotents in $I(\xi)$, with $\aleph_0 \leq \xi \leq \dim V$, and the number '3' is best possible. Therefore, every element in $I_{<q}$ can be written as a product of three idempotents in $\mathcal{EA}(V)$, and so $\mathcal{EA}(V)$ equals $F \,\dot\cup\, I_{<q}$. Also it follows from [37] Theorem 4 that each $\alpha \in I_{\geq q}$ can be expressed as a product of three idempotents in $\mathcal{EO}(V)$. Thus, $\mathcal{EO}(V) = I_{\geq q}$.

If $q = \aleph_0$ then $I = \{\alpha \in T(V) : d(\alpha) = n(\alpha) = s(\alpha) \geq q\}$, and so

$$\mathcal{EA}(V) = F, \qquad \mathcal{EO}(V) = I.$$

Example 4.6.1. Here we illustrate part of the above discussion. Suppose $q > \aleph_0$ and let $\{e_i\} \,\dot\cup\, \{a_j\} \,\dot\cup\, \{b_j\} \,\dot\cup\, \{c_j\}$ be a basis for V, with $p = |I| \geq |J| \geq q$. Now define $\alpha \in T(V)$ by

$$\alpha = \begin{pmatrix} e_i & a_j & b_j & c_j \\ e_i & b_j & c_j & 0 \end{pmatrix}.$$

It is easy to see that $d(\alpha) = n(\alpha) = s(\alpha) = |J|$, hence $\alpha \in \mathcal{EO}(V)$. Next, we define $\mu_1, \mu_2, \mu_3 \in T(V)$ by

$$\mu_1 = \begin{pmatrix} e_i & a_j & b_j & c_j \\ e_i & a_j & b_j & 0 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} e_i & a_j & b_j & c_j \\ e_i & a_j & c_j & c_j \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} e_i & a_j & b_j & c_j \\ e_i & b_j & b_j & c_j \end{pmatrix}.$$

Clearly, μ_1, μ_2 and μ_3 are idempotents whose kernels are $\langle c_j \rangle$, $\langle b_j - c_j \rangle$ and $\langle a_j - b_j \rangle$, respectively. It follows that, for each $n \in \{1, 2, 3\}$, $d(\mu_n) = n(\mu_n) = s(\mu_n) = |J|$, and hence $\mu_1, \mu_2, \mu_3 \in \mathcal{EO}(V)$. Also we have $\alpha = \mu_1\mu_2\mu_3$: that is, α is a product of three idempotents in $\mathcal{EO}(V)$. In fact, as in [37] p. 129,

$$\ker \alpha \subseteq \text{ran } \alpha^2 \subsetneq \text{ran } \alpha,$$

so α cannot be written as a product of two idempotents in $I(\xi)$ where $\xi = |J|$.

In [43] Corollary 3.4, Sullivan showed that $N(V)$, the subsemigroup of $T(V)$ generated by all nilpotent linear transformations of V, is a regular semigroup and

$$N(V) = \{\alpha \in T(V) : n(\alpha) = d(\alpha) = p\}.$$

Note that $N(V) = I(p)$, since $d(\alpha) = p$ and $d(\alpha) \leq s(\alpha)$ imply $s(\alpha) = p$. Clearly $N(V)$ is a subsemigroup of $OM(p, q) \cap OE(p, q)$. Therefore, the subsemigroup of $OM(p, q) \cap OE(p, q)$ generated by all of its nilpotent linear transformations equals $I(p)$. Also, from [43] Theorems 3.3 and 3.8, each element of this semigroup is a product of three nilpotents, each with index 2, and the number '3' is best possible. On the other hand, by [43] p. 1140, any $\alpha \in T(V)$ with rank less than p is a product of two nilpotents with index 2.

Example 4.6.2. To emphasise one aspect of the above discussion, we show how any nilpotent with index 2 can be written as a product of three nilpotents with index 2. In fact, if $\alpha^2 = 0$, then ran $\alpha \subseteq \ker \alpha$. Moreover, if we write α in the usual way as

$$\alpha = \begin{pmatrix} e_j & e_i \\ 0 & a_i \end{pmatrix},$$

then either $|I| = p$ or $|J| = p$ (since $\dim V = p$) and therefore, in each case, we have $n(\alpha) = p$. Consequently, $d(\alpha) = n(\alpha) = p$ by [43] Lemma 3.2. Since $\{a_i\}$ is a basis for ran α, we can expand it to a basis $\{a_i\} \dot\cup \{a_j\}$ for V, where $|J| = p$. Now write $\{a_j\} = \{b_j\} \dot\cup \{b_i\}$ (this is possible since p is infinite) and define λ_1, λ_2 in $T(V)$ by

$$\lambda_1 = \begin{pmatrix} a_i & b_j & b_i \\ b_i & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} a_i & b_j & b_i \\ 0 & 0 & a_i \end{pmatrix}.$$

Clearly, λ_1 and λ_2 are nilpotents with index 2, and $\alpha\lambda_1\lambda_2 = \alpha$.

Chapter 5

Injective partial linear transformation semigroups

Suppose V is a vector space over a field F with dimension m and let $P(V)$ denote the semigroup (under composition) of all *partial linear transformations* of V: that is, all linear transformations α whose domain and range are subspaces of V. There is a natural subsemigroup of $P(V)$ – namely, $I(V)$ – which consists of all one-to-one partial linear transformations of V; and it is easy to see that $I(V)$ is an inverse semigroup. In 1994, Lima studied some properties of $I(V)$ in the context of 'strong independence algebras' ('sets' and 'vector spaces' are prime examples of a strong independence algebra). In fact, $I(V)$ can be regarded as a linear version of the well-known symmetric inverse semigroup $I(X)$ defined on a set X, and we adopt this approach in this chapter.

According to Sullivan [41] p. 82, in 1953 Liber described all ideals of $I(X)$. Likewise, in section 5.1 we determine all ideals of $I(V)$: they are precisely the sets $I_r(V)$ consisting of all $\alpha \in I(V)$ with rank less than r, where $1 \leq r \leq m'$ and m' denotes the successor of m. We also describe all $\mathcal{D}$–classes of $I(V)$ and we show that the semigroups $I(X)$ and $I(V)$ are almost never isomorphic, for arbitrary X and V (this was not considered by Lima). Finally, in the same section, we give a linear version of the Vagner-Preston Theorem: that is, we prove that any inverse semigroup can be embedded in some $I(V)$.

In 1999, Yang described all maximal inverse subsemigroups of each ideal of $I(X)$, for an arbitrary finite set X, in terms of the maximal subgroups of $G(X)$, the symmetric group on X, and all of these are known. In sections 5.2 and 5.3, we do the same for $I(V)$ where V is finite-dimensional, although not all maximal subgroups of G_m, the general linear group of degree m over F, are known at present. To do so, we first determine all maximal inverse subsemigroups of the Brandt semigroup $B(G, J)$, where G is an arbitrary group and J is an arbitrary index set, and we establish a Rees matrix representation for each principal factor $I_r(V)/I_{r-1}(V)$ of $I(V)$, with $2 \leq r \leq m + 1$.

Given an infinite set X with cardinal p and a cardinal q such that $\aleph_0 \leq q \leq p$, let B denote the Baer-Levi semigroup of type (p, q) on the set X, as discussed in chapter 2, section 2.1. In 1984, Howie and Marques-Smith showed that, if $p = q$, then B^{-1} consists of all onto $\alpha \in I(X)$ such that the complement of its domain in X has cardinal p, and they proved that $BB^{-1} = I(X)$. They also described the subsemigroup of $I(X)$ generated by $B^{-1}B$ and they showed that this is, in fact, the inverse subsemigroup of $I(X)$ generated by the nilpotent elements of index 2. A decade later, Lima extended these results to strong independence algebras, and thus also to vector spaces. In section 5.4, we consider the case when $p > q$ and we show that the analogues BB^{-1} for sets and GG^{-1} for vector spaces (where G denotes the linear Baer-Levi semigroup of type (m, n) on V, with $\aleph_0 \leq n \leq m$) are never isomorphic, despite their apparent similarities. All work in sections 5.1–5.3 has appeared in [32] and the work in section 5.4 has appeared in [31]

5.1. Basic properties

Let V be a vector space over a field F with dimension m and let X be an arbitrary non-empty set with cardinal p. For simplicity, if $a, b \in X$, we write a_b for the map in $I(X)$ with domain $\{a\}$ and range $\{b\}$.

Following [5] Vol. 1, p. 57, Exercise 6 (compare [1] Corollary 4 and [26] section 3, p. 427, for endomorphisms of an independence algebra), it is easy to see that, for every $\alpha, \beta \in I(V)$, there exist $\lambda, \mu \in I(V)$ such that $\alpha = \lambda\beta\mu$ if and only if $r(\alpha) \leq r(\beta)$. Also, $\alpha \mathcal{D} \beta$ in $I(V)$ if and only if $r(\alpha) = r(\beta)$ (that is, $\mathcal{D} = \mathcal{J}$). Consequently, we denote each $\mathcal{D}$–class of $I(V)$ by

$$D_r = \{\alpha \in I(V) : r(\alpha) = r\}$$

where $0 \leq r \leq m$ (when necessary, we use $D_r(V)$ to emphasise our use of linear maps). It is also easy to see that the idempotents of $I(V)$ are precisely the identity maps id_A whose domains A are subspaces of V. Similar facts and notation are well-known for $I(X)$: compare [28] pp. 309-310. In particular, we use $D_r(X)$ (or simply D_r within context) to denote the $\mathcal{D}$–class of $I(X)$ consisting of all elements of $I(X)$ with rank r where $0 \leq r \leq p$.

It is well-known that the ideals of $I(X)$ are the sets

$$I_r(X) = \{\alpha \in I(X) : r(\alpha) < r\}$$

where $1 \leq r \leq p'$. Likewise, following [5] Vol. 1, p. 57, Exercise 6, we determine all ideals of $I(V)$ (in a certain way, this extends a remark in [26] p. 427).

Lemma 5.1.1. Let V be a vector space over a field F with dimension m. Then, the ideals of $I(V)$ are precisely the sets

$$I_r(V) = \{\alpha \in I(V) : r(\alpha) < r\}$$

where $1 \leq r \leq m'$. Moreover, the set $I_r(V)$ is a principal ideal of $I(V)$ if and only if r is a successor cardinal.

Proof. Suppose r is a cardinal such that $1 \leq r \leq m'$. Let $\alpha \in I_r(V)$ and $\lambda, \mu \in I(V)$. Clearly, $\lambda\alpha\mu \in I(V)$ and $r(\lambda\alpha\mu) \leq r(\alpha\mu) \leq r(\alpha) < r$, hence $\lambda\alpha\mu \in I_r(V)$. In other words, we have shown that each $I_r(V)$ is an ideal of $I(V)$.

Conversely, let I be any ideal of $I(V)$ and let r be the least cardinal greater than $r(\beta)$ for every $\beta \in I$ (this is possible since cardinals are well-ordered). Then, $1 \leq r \leq m'$ and $I \subseteq I_r(V)$. Now, given $\alpha \in I_r(V)$, we know $r(\alpha) < r$, hence there exists some $\beta \in I$ such that $r(\alpha) \leq r(\beta)$: otherwise, $r(\beta) < r(\alpha) < r$ for every $\beta \in I$, and this contradicts our choice of r. Therefore, by a remark above, there exist $\lambda, \mu \in I(V)$ such that $\alpha = \lambda\beta\mu$, so $\alpha \in I$. Hence, $I = I_r(V)$ as required.

Next we determine all principal ideals of $I(V)$. To do this, suppose r is a successor cardinal, say $r = \xi'$, and choose $\alpha \in I_r(V)$ with $r(\alpha) = \xi$. If $r(\beta) > \xi$ for some $\beta \in I_r(V)$, then $r(\beta) \geq \xi' = r$, which contradicts our choice of r. Therefore, $r(\beta) \leq \xi = r(\alpha)$, for every $\beta \in I_r(V)$, hence $\beta = \lambda\alpha\mu$ with $\lambda, \mu \in I(V)$: that is, $\beta \in J(\alpha)$, the principal ideal of $I(V)$ generated by α. Thus, $I_r(V) \subseteq J(\alpha)$ and clearly the reverse inclusion also holds. Therefore, $I_r(V)$ is a principal ideal of $I(V)$. Conversely, suppose $I_r(V) = J(\alpha)$ for some α in $I(V)$ and let $r(\alpha) = \xi$. If $\xi < \chi < r$ for some cardinal χ, then there exists $\beta \in I_r(V)$ such that $r(\beta) = \chi$. As before, $r(\alpha) \leq r(\beta)$ implies $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in I(V)$ and, since $r(\alpha) < r(\beta)$, we have $J(\alpha) \subsetneq J(\beta) \subseteq I_r(V)$, contradicting our assumption. Hence, r is the least cardinal greater than ξ, and so $r = \xi'$. □

Note that $I_1(V) = \{0\}$ where 0 denotes the linear map with domain $\{0\}$ in V. As observed in chapter 1, section 1.3, this map acts as a zero for the semigroup $I(V)$.

We proceed by showing that $I(X)$ and $I(V)$ are almost never isomorphic (this question was not considered in [26]). Sullivan (personal communication) proved that $I(X)$ and $I(V)$ are not isomorphic if '$p \geq 1$, $m \geq 1$ and $|F| \geq 3$' or '$p \geq 3$, $m \geq 3$ and $|F| = 2$'. First suppose $p \geq 1$, $m \geq 1$ and $|F| \geq 3$. If such an isomorphism exists, then $I_2(X)$ is isomorphic to $I_2(V)$ since these are the smallest non-zero ideals of $I(X)$ and $I(V)$, respectively. But this is impossible since every element of $I_2(X)$ is either an idempotent or a nilpotent of index 2, whereas if $k \neq 0, 1$ in F and $u \neq 0$ in V then $I_2(V)$ contains an element α with domain $\langle u \rangle$ such that $\alpha : u \mapsto ku$, and this is neither idempotent nor nilpotent.

To handle the case when $|F| = 2$, we suppose $p \geq 3$ and assert that $I(X)$ satisfies:

$$\begin{gathered} \text{if } \alpha \in D_2(X) \text{ where } \alpha^2 \text{ is not idempotent and } \alpha\gamma, \alpha^2\gamma \text{ are non-zero} \\ \text{for some idempotent } \gamma \in D_1(X), \text{ then } \gamma\alpha = \emptyset. \end{gathered} \tag{5.1.1}$$

To see this, suppose $\alpha \in D_2(X)$ satisfies the initial condition and write

$$\alpha = \begin{pmatrix} a & b \\ x & y \end{pmatrix}.$$

Without loss of generality, let $\gamma = x_x$. If $x = a$ then $\alpha^2 = a_a$ (if $b \neq y$) or $\alpha^2 = \mathrm{id}_{\{a,b\}}$ (if $b = y$), contradicting our supposition that α^2 is not idempotent. On the other hand, if $x = b$ then $y \neq b$ (since α is injective) and $y \neq a$ (since α^2 is not idempotent). Hence $\alpha^2 = a_y$ and so $\alpha^2\gamma = \emptyset$, another contradiction. Therefore, $x \neq a, b$ and so $\gamma\alpha = \emptyset$. Finally, note that (5.1.1) is not vacuous in $I(X)$ if $p \geq 3$, since the map $b \to a \to x$, where $x \neq a, b$, satisfies (5.1.1).

Clearly, property (5.1.1) will be preserved under an isomorphism φ from $I(X)$ onto $I(V)$. In fact, φ will map the $\mathcal{D}$–classes of $I(X)$ onto the $\mathcal{D}$–classes of $I(V)$ in an obvious way. However, $I(V)$ does not satisfy (5.1.1) if $m \geq 3$ and $|F| = 2$. For, in this case, if a, b are two non-zero vectors in V then they are linearly independent, and the same is true for $\{b, a + b\}$. Let

$$\alpha = \begin{pmatrix} a & b \\ b & a+b \end{pmatrix} \quad \text{and} \quad \alpha^2 = \begin{pmatrix} a & b \\ a+b & a \end{pmatrix}$$

be linear maps defined and obtained as shown, and observe that α^2 is not idempotent. Moreover, if $c = a+b$ and $\gamma \in D_1(V)$ is the idempotent with domain $\langle c \rangle$, then $\alpha\gamma$ and $\alpha^2\gamma$ are non-zero. However, $\gamma\alpha$ is also non-zero, so property (5.1.1) fails to hold in $I(V)$. Therefore, $I(X)$ and $I(V)$ are not isomorphic when $p \geq 3$, $m \geq 3$ and $|F| = 2$. Thus we have proved most of the following result.

Theorem 5.1.2. The semigroups $I(X)$ and $I(V)$ are isomorphic only when $|X| = 1 = \dim V$ and $|F| = 2$.

Proof. For the remaining cases, we suppose $1 < |X| = p < \aleph_0$, $1 < \dim V = m < \aleph_0$ and $|F| = 2$. Clearly, the above discussion covers almost all of these situations but the argument we use here gives a more direct proof in the finite case. In fact, since V has finite dimension m over the field F, we know $V \cong F^m$ (as vector spaces) and so $|V| = 2^m$. Also, the ideals of $I(X)$ form a chain with length $p+1$; and we have a similar result for $I(V)$: its ideals form a chain with length $m+1$. Hence, if $I(X)$ and $I(V)$ are isomorphic then $p = m$ and, as mentioned above, $I_2(X)$ is isomorphic to $I_2(V)$. But the non-zero elements of $I_2(X)$ are the maps a_b where $a, b \in X$, hence $|I_2(X)| = p^2 + 1$. And similarly, since $|F| = 2$, the non-zero elements of $I_2(V)$ are the maps $u \mapsto v$ where u, v are non-zero elements in V. Hence, if $m = p$ then $|I_2(V)| = (2^p - 1)^2 + 1 = 2^{p+1}(2^{p-1} - 1) + 2$ and, since $2^{p+1} > p^2 + 1$ for every $p \geq 2$, it follows that $|I_2(V)| > |I_2(X)|$. Finally, observe that if $|X| = 1 = \dim V$ and $|F| = 2$, then $I(X)$ and $I(V)$ are simply the group of order 1 with a zero adjoined.□

According to Clifford and Preston [5] Vol. 1, p. 30, Vagner (in 1952) and Preston (in 1954) proved that every inverse semigroup can be embedded in some symmetric inverse semigroup. In view of the above, it is interesting to give a linear version of the Vagner-Preston Theorem: our proof closely follows that of [5] Vol. 1, Theorem 1.20.

Theorem 5.1.3. Any inverse semigroup can be embedded in $I(V)$ for some vector space V.

Proof. Let S be an inverse semigroup with $|S| = k$ (finite or infinite) and write $S = \{a_i\}$ with $|I| = k$. Let F be any field and let F_i be a copy of F for each $i \in I$. As in chapter 2, Theorem 2.3.10, we let V be the vector space $\sum F_i$ over F whose basis

can be identified in a natural way with $\{a_i\}$: that is, $\sum F_i$ is the set of all $(r_i)_{i\in I}$ where $r_i \in F_i$ and at most finitely many r_i are non-zero.

Let $x \in S$ and suppose ρ_x is the partial map with domain $Sx^{-1} = Sxx^{-1}$ such that $a\rho_x = ax$ ($a \in Sx^{-1}$). The range of ρ_x is $Sx^{-1}x = Sx$ and, in fact, ρ_x is one-to-one: if $a, b \in Sx^{-1}$ and $a\rho_x = b\rho_x$, then there exist $j, k \in I$ such that $a = a_j x^{-1}$ and $b = a_k x^{-1}$, so $a = a_j x^{-1} = a_j x^{-1}xx^{-1} = a_k x^{-1}xx^{-1} = a_k x^{-1} = b$. Since Sxx^{-1} is a subset of S, it is linearly independent. Hence ρ_x can be extended by linearity to an injective linear map $\rho_x : \langle a_i xx^{-1}\rangle \to V$ with range $\langle a_i x\rangle$. Since $\{a_i x^{-1}\}$ and $\{a_i xx^{-1}\}$ are bases for the domain of ρ_x, we can display ρ_x as an element of $I(V)$ in two different ways,

$$\rho_x = \begin{pmatrix} a_i x^{-1} \\ a_i x^{-1} x \end{pmatrix} \quad \text{and} \quad \rho_x = \begin{pmatrix} a_i xx^{-1} \\ a_i x \end{pmatrix}.$$

We assert that $\phi : S \to I(V)$, $x \mapsto \rho_x$, is an injective homomorphism.

Let $x, y \in S$ and suppose $x\phi = y\phi$, that is, $\rho_x = \rho_y$. Then, $\langle a_i xx^{-1}\rangle = \text{dom}\ \rho_x = \text{dom}\ \rho_y = \langle a_i yy^{-1}\rangle$ and $a\rho_x = a\rho_y$ for all $a \in \text{dom}\ \rho_x$. Since $x^{-1} \in Sxx^{-1} = \{a_i xx^{-1}\}$, it follows that $x^{-1}x = x^{-1}\rho_x = x^{-1}\rho_y = x^{-1}y$. On the other hand, $xx^{-1} \in \{a_i x^{-1}\} \subseteq \langle a_i x^{-1}\rangle = \langle a_i y^{-1}\rangle$ and so $xx^{-1} = \sum r_i(a_i y^{-1})$ for some scalars $r_i \in F$. Since $\{xx^{-1}\} \cup \{a_i y^{-1}\} \subseteq S$ and S is linearly independent, this implies that there exists some $i_0 \in I$ such that $r_{i_0} = 1$ and $xx^{-1} = a_{i_0} y^{-1}$. Similarly we find that $yy^{-1} = a_{i_1} x^{-1}$ for some $i_1 \in I$. Therefore, since idempotents commute in inverse semigroups, we have:

$$\begin{aligned} xx^{-1} = a_{i_0} y^{-1} &= a_{i_0} y^{-1} yy^{-1} \\ &= xx^{-1} yy^{-1} = yy^{-1} xx^{-1} = a_{i_1} x^{-1} xx^{-1} = a_{i_1} x^{-1} = yy^{-1}. \end{aligned}$$

Hence, $x = xx^{-1}x = xx^{-1}y = yy^{-1}y = y$.

To see that ϕ is a homomorphism, we let $x, y \in S$ and recall that ρ_x, ρ_y are defined in $I(V)$ by

$$\rho_x = \begin{pmatrix} a_i x^{-1} \\ a_i x^{-1} x \end{pmatrix}, \quad \rho_y = \begin{pmatrix} a_i yy^{-1} \\ a_i y \end{pmatrix}.$$

It is easy to see that $\rho_x^{-1} = \rho_{x^{-1}}$. Since $\{a_i x^{-1}x\} \cap \{a_i yy^{-1}\} = Sx^{-1}x \cap Syy^{-1} = Sx^{-1}xyy^{-1} = Sxyy^{-1} = \{a_i xyy^{-1}\}$, it follows that

$$\rho_x\rho_y = \begin{pmatrix} a_i xyy^{-1}x^{-1} \\ a_i xyy^{-1} y \end{pmatrix} = \begin{pmatrix} a_i (xy)(xy)^{-1} \\ a_i (xy) \end{pmatrix} = \rho_{xy}.$$

Hence $(xy)\phi = x\phi y\phi$. □

5.2. Maximal inverse subsemigroups of $I(V)$

In [48] Theorem 2.3, Yang determined the maximal inverse subsemigroups of $I(X)$ for an arbitrary finite set X, in terms of the maximal subgroups of $G(X)$. In this section we do the same for $I(V)$, where V is finite-dimensional: that is, we assume $m = \dim V < \aleph_0$. As in [48], we begin with two Lemmas: the second of these is comparable with [9] Theorem 3.1, and both Lemmas are due to Sullivan (personal communication).

In the introduction to this chapter, we denoted the group of all $m \times m$ invertible matrices over F by G_m. It is well-known that G_m is isomorphic to the group of all invertible linear transformations of V, hence we regard these two groups as interchangeable: that is, throughout this chapter G_m denotes either the general linear group of degree m over F or the group of all invertible linear transformations of V.

Lemma 5.2.1. For $1 \leq r \leq m-2$, $D_r \subseteq D_{r+1}D_{r+1}$.

Proof. Let $\alpha \in D_r$ and choose bases $\{a_1, \ldots, a_r\}$ and $\{b_1, \ldots, b_r\}$ for dom α and ran α respectively, where $a_i\alpha = b_i$ for each i. Since $r \leq m-2$, there exist $u_0, u_1 \in V$ such that $\{a_1, \ldots, a_r, u_0, u_1\}$ is linearly independent; and likewise $\{b_1, \ldots, b_r, v_1\}$ is linearly independent for some $v_1 \in V$. Then we have

$$\alpha = \begin{pmatrix} a_1 & \ldots & a_r \\ b_1 & \ldots & b_r \end{pmatrix} = \begin{pmatrix} a_1 & \ldots & a_r & u_0 \\ a_1 & \ldots & a_r & u_0 \end{pmatrix} \circ \begin{pmatrix} a_1 & \ldots & a_r & u_1 \\ b_1 & \ldots & b_r & v_1 \end{pmatrix}$$

where the two mappings on the right can be suitably extended by linearity to become elements of D_{r+1}. □

Lemma 5.2.2. If $\alpha \in D_{m-1}$ then $I(V) = \langle \alpha, G_m \rangle$.

Proof. Suppose $a_1, \ldots, a_{m-1}$ and $b_1, \ldots, b_{m-1}$ are bases for dom α and ran α respectively, and assume that

$$\alpha = \begin{pmatrix} a_1 & \ldots & a_{m-1} \\ b_1 & \ldots & b_{m-1} \end{pmatrix}.$$

To show that $\langle \alpha, G_m \rangle$ contains a mapping with rank $m-2$, we consider two cases.

Case 1. Suppose dom α = ran $\alpha = A$ say. Then there exists $u \notin A$ and $g \in G_m$ defined by

$$g = \begin{pmatrix} b_1 & \ldots & b_{m-2} & b_{m-1} & u \\ b_1 & \ldots & b_{m-2} & u & b_{m-1} \end{pmatrix}.$$

Since $\langle b_1, \ldots, b_{m-2} \rangle \subseteq \text{dom } \alpha$ and α is injective, we obtain

$$\alpha g.\alpha = \begin{pmatrix} a_1 & \ldots & a_{m-2} & a_{m-1} \\ b_1 & \ldots & b_{m-2} & u \end{pmatrix} \circ \begin{pmatrix} a_1 & \ldots & a_{m-2} & a_{m-1} \\ b_1 & \ldots & b_{m-2} & b_{m-1} \end{pmatrix} = \begin{pmatrix} a_1 & \ldots & a_{m-2} \\ c_1 & \ldots & c_{m-2} \end{pmatrix}$$

for some linearly independent $c_1, \ldots, c_{m-2} \in V$.

Case 2. Suppose $A = \text{dom } \alpha \neq \text{ran } \alpha = B$. Since A and B have the same finite dimension, this implies $A \not\subseteq B$. Choose $u \in A \setminus B$, let $\{c_1, \ldots, c_{m-2}, u\}$ be a basis for A and write

$$\alpha = \begin{pmatrix} c_1 & \ldots & c_{m-2} & u \\ d_1 & \ldots & d_{m-2} & v \end{pmatrix}$$

where $\{d_1, \ldots, d_{m-2}, v\}$ is a basis for B (hence $\{d_1, \ldots, d_{m-2}, v, u\}$ is a basis for V). Choose $w \notin A$ and define $g \in G_m$ by

$$g = \begin{pmatrix} d_1 & \ldots & d_{m-2} & v & u \\ c_1 & \ldots & c_{m-2} & u & w \end{pmatrix}.$$

Then

$$\alpha g = \begin{pmatrix} c_1 & \ldots & c_{m-2} & u \\ c_1 & \ldots & c_{m-2} & u \end{pmatrix}$$

and, as in Case 1, we can use this to obtain a mapping in $\langle \alpha, G_m \rangle$ with rank $m-2$.

Clearly we can repeat the above process to obtain an element of $\langle \alpha, G_m \rangle$ with rank r where $0 \leq r \leq m-1$. Then if

$$\beta = \begin{pmatrix} e_1 & \ldots & e_r \\ f_1 & \ldots & f_r \end{pmatrix}$$

is any element of $\langle \alpha, G_m \rangle$ with rank r, we can expand dom β and ran β to bases

$$\{e_1, \ldots, e_r, e_{r+1}, \ldots, e_m\} \quad \text{and} \quad \{f_1, \ldots, f_r, f_{r+1}, \ldots, f_m\}$$

for V. Also, if $\{u_1, \ldots, u_r\}$ and $\{v_1, \ldots, v_r\}$ are any linearly independent subsets of V, we can expand them to bases $\{u_1, \ldots, u_r, u_{r+1}, \ldots, u_m\}$ and $\{v_1, \ldots, v_r, v_{r+1}, \ldots, v_m\}$ for V and then define $g, h \in G_m$ by $u_i g = e_i$ and $f_i h = v_i$ for each i. Then

$$g\beta h = \begin{pmatrix} u_1 & \ldots & u_r \\ v_1 & \ldots & v_r \end{pmatrix} \in \langle \alpha, G_m \rangle.$$

Since this is an arbitrary element of $I(V)$ with rank r, we conclude that $I(V) = \langle \alpha, G_m \rangle$. □

The next result is also due to Sullivan (personal communication). Its proof follows that of [48] Theorem 2.3. For convenience, here we write $I_r(V)$ more simply as I_r.

Theorem 5.2.3. If $\dim V = m < \aleph_0$, the maximal inverse subsemigroups of $I(V)$ are precisely the following sets.

(a) $I_{m-1} \cup G_m$,

(b) $I_m \cup H$, where H is a maximal subgroup of G_m.

Proof. Clearly, $I_m \cup H$ is a maximal inverse subsemigroup of $I(V)$ in (b). Also, $I_{m-1} \cup G_m$ is an inverse semigroup. In fact, if $I_{m-1} \cup G_m \subsetneq T \subseteq I(V)$ for some inverse semigroup T, then $T \cap D_{m-1} \neq \emptyset$ and so Lemma 5.2.2 implies $T = I(V)$.

Conversely, suppose S is a maximal inverse subsemigroup of $I(V)$. Then $S \cap G_m \neq \emptyset$: otherwise, $S \subseteq I_m \subsetneq I_m \cup H$ for any maximal subgroup H of G_m, contradicting the maximality of S.

Case 1. $S \cap G_m = G_m$. This means $G_m \subseteq S$ and hence $S \cap D_{m-1} = \emptyset$: otherwise, $\langle \alpha, G_m \rangle \subseteq S$ for some $\alpha \in D_{m-1}$ and then Lemma 5.2.2 implies $S = I(V)$, a contradiction as before. Consequently, $S \subseteq I_{m-1} \cup G_m$ which is an inverse subsemigroup of $I(V)$, hence $S = I_{m-1} \cup G_m$ by the maximality of S.

Case 2. $S \cap G_m = H \neq G_m$. Note that H is a group since S is inverse (that is, $\alpha \in H$ implies $\alpha^{-1} \in H$ and clearly H is a semigroup). Now, $S \subseteq I_m \cup H$ which is a proper inverse subsemigroup of $I(V)$, hence $S = I_m \cup H$. Moreover, if $H \subsetneq H' \subsetneq G_m$ for some subgroup H' of G_m, we have

$$S = I_m \cup H \subsetneq I_m \cup H' \subsetneq I(V),$$

which contradicts the maximality of S. Hence, H is a maximal subgroup of G_m. □

In passing, we observe that all maximal subgroups of the general linear group G_m are not known at present. As stated in [49] p. 162, "Currently, there is a fairly rich collection of examples of maximal subgroups of the general linear group. However, it is unclear... how to find a description of all of them."

For example, in [4] Theorem 7, Borevich states that, if $m \geq 2$ and F is any field with $|F| \geq 7$, then all maximal subgroups of G_m that contain the group of diagonal

matrices over F are known. On the other hand, we recall that $Z(G_m)$, the centre of G_m, is the set of (non-zero) scalar matrices in G_m and that S_m is the *special linear group* over F: that is, the group of all matrices in G_m with determinant 1. Also, when $F = GF(q)$ is finite, $PG_m = G_m/Z(G_m)$ is the *projective general linear group* over F. It can be shown that, for $3 \leq m \leq 12$, PG_m satisfies the criteria of [20] Theorem 1.2.2, and so all of its maximal subgroups are known. From this, Sullivan (personal communication) deduced that, if $3 \leq m \leq 12$ and $F = GF(q)$ is finite, then all subgroups of G_m which are maximal and contain all the scalar matrices are known.

However, as observed by Sullivan (personal communication), a maximal subgroup of G_m need not contain all the scalar matrices; and moreover, if one contains all the scalar matrices, then it need not contain all the diagonal matrices. For example, suppose $m = 3$ and $F = \mathbb{Z}_5$, so $F^* = \langle a \rangle$ is a cyclic group of order 4. Let H be the group generated by S_3 and all diagonal matrices in G_3 whose determinant equals a^2. Note that $aI \notin H$ since $\det(aI) = a^3$ whereas the determinant of every matrix in H equals 1 or a^2 (since $(a^2)^k$ equals 1 or a^2 for each $k \geq 1$). Write $E(x) = \mathrm{diag}(1, 1, x)$ for each $x \in F^*$ and recall that each $C \in G_3$ equals $D.E(x)$ for some $D \in S_3$ and some $x \in F^*$ (see [2] Theorem 4.1 for a proof, in general, for any m and $q = |F|$). In particular, if $B \in G_3$ and $\det B = a^2$ then $B = D.E(x)$ and it follows that $x = a^2$. Thus, H contains all elements of G_3 with determinant a^2. To see that H is maximal in G_3, choose $B' \notin H$ and let K be the group generated by H and B'. Then $B' = D'.E(y)$ for some $D' \in S_3$ and some $y \in F^*$, and we deduce that $E(y) \in K$. Now $y = \det B' \neq 1$ and $y \neq a^2$ (since $B' \notin H$). Also, if $y = a^3$ then $y^3 = a$ and K contains $\mathrm{diag}(1, 1, a) = [\mathrm{diag}(1, 1, a^3)]^3$. That is, we may conclude that K contains $E(x)$ for each $x \in F^*$ and thus it equals G_3.

Finally, let L be the group generated by S_4 and all diagonal matrices in G_4 whose determinant equals a^2. Then L contains all the scalar matrices (since $x^4 = 1$ for all $x \in F^*$) but not all the diagonal matrices of G_4. Moreover, as before, L is maximal in G_4: that is, L is a group determined by Sullivan's result, but not by Borevich's result (both cited above).

5.3. Maximal inverse subsemigroups of I_r

Given a group G, two arbitrary sets J and Λ, and an arbitrary but fixed $\Lambda \times J$ matrix $P = (p_{\lambda j})$ over G^0, the group with 0 adjoined, then the *Rees* $J \times \Lambda$ *matrix semigroup over* G^0 *with sandwich matrix* P is the semigroup $\mathcal{M}^0(G; J, \Lambda; P)$ of all triples (j, a, λ) with $a \in G^0$, $j \in J$ and $\lambda \in \Lambda$ and a product defined as follows: given $(j, a, \lambda), (i, b, \mu) \in \mathcal{M}^0(G; J, \Lambda; P)$,

$$(j, a, \lambda)(i, b, \mu) = (j, ap_{\lambda i}b, \mu)$$

(compare [5] Vol. 1, p. 88: we use J instead of the more usual I to avoid any confusion with earlier notation; and we use (j, a, λ) rather than (a, j, λ) to maintain some uniformity with the notation in [48]). If $J = \Lambda$ and P is the identity $J \times J$ matrix over G^0, then we say the semigroup $\mathcal{M}^0(G; J, \Lambda; P)$ is a *Brandt semigroup* and we denote it more simply by $B(G, J)$. In this section, we describe the maximal inverse subsemigroups of $B(G, J)$ for arbitrary G and J; and, using this, we determine all maximal inverse subsemigroups of each proper ideal of $I(V)$ where V is finite-dimensional.

From [5] Vol. 1, Theorem 3.9, the semigroup $B(G, J)$ is a completely 0-simple inverse semigroup. Also, by [5] Vol. 1, Lemma 3.2, Green's $\mathcal{L}$ and $\mathcal{R}$ relations on $B(G, J)$ are characterised as follows: given (j, a, λ), (i, b, μ) in $B(G, J)$, we have $(j, a, \lambda)\ \mathcal{L}\ (i, b, \mu)$ if and only if $\lambda = \mu$, and $(j, a, \lambda)\ \mathcal{R}\ (i, b, \mu)$ if and only if $j = i$.

As in [48], if S is an inverse semigroup and A is a non-empty subset of S, then we write $\langle\langle A \rangle\rangle$ for the inverse subsemigroup of S generated by A: that is, the set consisting of all finite products of elements of A and inverses of elements of A. (compare [5] Vol. 1, p. 31)

In [48] section 3, Yang first determines all maximal inverse subsemigroups of the Brandt semigroup $B(G, J)$ where G is a finite group and $J = \{1, \ldots, n\}$. He requires two preliminary results ([48] Lemmas 3.2 and 3.3) to do this, and both of these depend on the following result. However, Yang's version of this result covers only the finite case. Since we need to know all maximal inverse subsemigroups of $B(G, J)$ where G is an arbitrary group with identity e and J is an arbitrary index set, we include a proof of the following result for clarity and completeness (for the reason

why we need such generality, see the discussion after Lemma 5.3.5 below). Here, we write B^A for the set of all maps $g : A \to B$, and represent g by $[g_a]$ (that is, A acts as an index set for the images of g in B, hence the g_a's are not necessarily distinct).

Lemma 5.3.1. For each $i \in J$, let H_{ii} denote the $\mathcal{H}$–class of $B(G, J)$ containing the element (i, e, i), put $K_i = J \setminus \{i\}$ and let $[g_j] \in G^{K_i}$. Then,

$$B(G, J) = \langle\langle H_{ii} \cup \{(i, g_j, j) : j \in K_i\}\rangle\rangle.$$

Proof. Let $(j, g, k) \in B(G, J)$. If $j = k = i$, then $(j, g, k) \in H_{ii}$. If $j = i$ and $k \neq i$, then $(j, g, k) = (i, gg_k^{-1}, i)(i, g_k, k)$ where $(i, gg_k^{-1}, i) \in H_{ii}$; and similarly, if $j \neq i$ and $k = i$, then we can write (j, g, k) as $(i, g_j, j)^{-1}(i, g_j g, i)$. If $j \neq i$ and $k \neq i$, then $(j, g, k) = (i, g_j, j)^{-1}(i, g_j g g_k^{-1}, i)(i, g_k, k)$. Therefore, each (j, g, k) can be written as a finite product of elements of $H_{ii} \cup \{(i, g_j, j) : j \in K_i\}$ and their inverses, and the result follows. □

The proofs of [48] Lemmas 3.2 and 3.3 are stated for finite groups and finite index sets and, in some places, the notation and argument is not entirely clear (for example, in [48] Lemma 3.2, $I = \{1, \ldots, m\}$ and $g_1, g_2, \ldots, g_m \in G$ where $g_1 = e$, the identity of the group G, all of which appears to require G to have order at least m, but this is not intended). For this reason, and since we must deal with arbitrary groups and arbitrary index sets, we prove the following Lemmas in detail, but use the basic idea of Yang's proofs to do so.

The first Lemma determines some maximal inverse subsemigroups of $B(G, J)$.

Lemma 5.3.2. Let H be a maximal subgroup of G and let $1 \in J$. Put $H_{11}^* = \{(1, h, 1) \in B(G, J) : h \in H\}$. Let $[g_j] \in G^J$ with $g_1 = e$ and let

$$B_H = \langle\langle H_{11}^* \cup \{(1, g_j, j) : j \in J\}\rangle\rangle.$$

Then B_H is isomorphic to the Brandt semigroup $B(H, J)$ and it is a maximal inverse subsemigroup of $B(G, J)$.

Proof. Let H_{ij} denote the $\mathcal{H}$–class of $B(G, J)$ containing the element (i, e, j), that is, $H_{ij} = \{(i, g, j) : g \in G\}$, and write $\{(i, g_i^{-1} h g_j, j) : h \in H\}$ as $(i, g_i^{-1} H g_j, j)$.

Consider the set $H_{ij} \cap B_H$ and let $(i,g,j) \in B_H$. Then, (i,g,j) can be written as a finite product, say $y_1 \cdots y_r$, of elements of $H_{11}^* \cup \{(1,g_j,j) : j \in J\}$ and their inverses. Clearly, $y_1 = (i,g_i^{-1},1)$ and $y_r = (1,g_j,j)$ and so $y_2 \cdots y_{r-1} = (1,g_i g g_j^{-1},1)$ (for example, $(i,g,j) = y_1 y_2 \cdots y_r$ and $y_1 = (i,g_i^{-1},1)$ imply $(1,g_i g,j) = (1,g_i,i)(i,g,j) = (1,e,1)\cdot y_2 \cdots y_r = y_2 \cdots y_r$). Now, if $y_2 = (1,g_k,k)$ for some $k \in J \setminus \{1\}$, then y_3 must equal $(k,g_k^{-1},1)$, so $y_2 y_3 = (1,e,1)$ and we then consider $y_4 \cdots y_{r-1}$; alternatively, if $y_2 = (1,h_2,1)$ for some $h_2 \in H$, then $y_3 \cdots y_{r-1}$ equals $(1,g',1)$ for some $g' \in G$ and we repeat this argument by starting with y_3. It follows that $y_2 \cdots y_{r-1} \in H_{11}^*$ and there exists some h in H such that $y_2 \cdots y_{r-1} = (1,h,1)$. Thus $(i,g,j) = (i,g_i^{-1},1)(1,h,1)(1,g_j,j) = (i,g_i^{-1}hg_j,j) \in (i,g_i^{-1}Hg_j,j)$. Conversely, given $h \in H$, we have $(i,g_i^{-1}hg_j,j) = (i,g_i^{-1},1)(1,h,1)(1,g_j,j) \in B_H$ and so $(i,g_i^{-1}hg_j,j) \in H_{ij} \cap B_H$. Hence, $H_{ij} \cap B_H = (i,g_i^{-1}Hg_j,j)$.

Now define a mapping $\theta : B(H,J) \to B_H$ as follows: $0\theta = 0$ and $(i,h,j)\theta = (i,g_i^{-1}hg_j,j)$. We assert that θ is an isomorphism from $B(H,J)$ onto B_H.

To see that θ is a morphism, let $(i,h_1,j),(r,h_2,s) \in B(H,J)$. If $j \neq r$, then

$$(i,h_1,j)\theta(r,h_2,s)\theta = (i,g_i^{-1}h_1g_j,j)(r,g_r^{-1}h_2g_s,s) = 0 = 0\theta = [(i,h_1,j)(r,h_2,s)]\theta.$$

On the other hand, if $j = r$, then

$$\begin{aligned}(i,h_1,j)\theta(r,h_2,s)\theta &= (i,g_i^{-1}h_1g_j,j)(j,g_j^{-1}h_2g_s,s) = (i,g_i^{-1}h_1g_jg_j^{-1}h_2g_s,s)\\ &= (i,g_i^{-1}h_1h_2g_s,s) = (i,h_1h_2,s)\theta\\ &= [(i,h_1,j)(j,h_2,s)]\theta = [(i,h_1,j)(r,h_2,s)]\theta.\end{aligned}$$

Clearly, given $x \in B(H,J)$, $x\theta = 0$ if and only if $x = 0$. If (i,h_1,j) and (r,h_2,s) are such that $(i,h_1,j)\theta = (r,h_2,s)\theta$, then $i = r$, $j = s$ and $g_i^{-1}h_1g_j = g_i^{-1}h_2g_j$, and so $h_1 = h_2$. Hence, $(i,h_1,j) = (r,h_2,s)$ and θ is one-to-one.

Next, let $x \in B_H \setminus \{0\}$. Since $x \in B(G,J)$, there exist $i,j \in J$ and $g \in G$ such that $x = (i,g,j)$. Thus, $x \in H_{ij} \cap B_H$ and so there is some h in H such that $x = (i,g_i^{-1}hg_j,j) = (i,h,j)\theta$. Hence, θ is an isomorphism from $B(H,J)$ onto B_H.

To show that B_H is maximal, it suffices to show that $B(H,J)$ is maximal. Let S be an inverse subsemigroup of $B(G,J)$ properly containing $B(H,J)$. Then, there exists

some $(i,a,j) \in S \setminus B(H,J)$. It follows that $a \in G \setminus H$ and, since H is a maximal subgroup of G, we have $G = \langle\langle H \cup \{a\}\rangle\rangle$. Also,

$$(1,a,1) = (1,e,i)(i,a,j)(j,e,1) \in B(H,J).S.B(H,J) \subseteq S$$

and so $(1,a^{-1},1) = (1,a,1)^{-1} \in S$. Since H is a group, every $g \in G$ can be written as a finite product $x_1 \cdots x_\ell$ of elements of $H \cup \{a, a^{-1}\}$. Since $B(H,J) \subseteq S$, it follows that $(1,x_t,1) \in S$ for every $t \in \{1,\ldots,\ell\}$, and so $(1,g,1) = (1,x_1,1)\cdots(1,x_\ell,1) \in S$. Hence, $H_{11} \subseteq S$. From Lemma 5.3.1, $B(G,J) = \langle\langle H_{11} \cup \{(1,g_j,j) : j \in K_1\}\rangle\rangle$ for some $[g_j] \in G^{K_1}$ with $K_1 = J \setminus \{1\}$. Let $j \in J$. Then,

$$(1,g_j,j) = (1,g_j,1)(1,e,j) \in S.B(H,J) \subseteq S.$$

Therefore, $B(G,J) \subseteq S$ and so $B(H,J)$ is maximal. □

As in [48], a pair (J_1,J_2) is said to be a *two-partition of* J if $J = J_1 \cup J_2$ and $J_1 \cap J_2 = \emptyset$. The next Lemma determines another family of maximal inverse subsemigroups of $B(G,J)$.

Lemma 5.3.3. Let (J_1,J_2) be a two-partition of J and suppose $B_{J_1J_2} = B(G,J_1) \cup B(G,J_2)$. Then, $B_{J_1J_2}$ is a maximal inverse subsemigroup of $B(G,J)$.

Proof. Clearly, $B(G,J_1) \cap B(G,J_2) = \{0\}$ and $B(G,J_1), B(G,J_2)$ are inverse subsemigroups of $B(G,J)$. Also, $(i_1,g,j_1)(i_2,g',j_2) = 0$ for all $(i_1,g,j_1) \in B(G,J_1)$ and $(i_2,g',j_2) \in B(G,J_2)$. Thus, $B_{J_1J_2}$ is an inverse subsemigroup of $B(G,J)$.

To see that $B_{J_1J_2}$ is maximal, let S be an inverse subsemigroup of $B(G,J)$ properly containing $B_{J_1J_2}$. Then, there exists $(i,b,j) \in S \setminus B_{J_1J_2}$. Since S is an inverse semigroup, we have $(j,b^{-1},i) = (i,b,j)^{-1} \in S$. Also, since $(i,b,j) \notin B_{J_1J_2}$, it follows that either ($i \in J_1$ and $j \in J_2$) or ($i \in J_2$ and $j \in J_1$). Hence, there exist $j_1 \in J_1$, $a \in G$ and $j_2 \in J_2$ such that $(j_1,a,j_2) \in S$. Let R_{j_1} denote the $\mathcal{R}$–class of $B(G,J)$ containing (j_1,a,j_2), that is, $R_{j_1} = \{(j_1,g,j) : g \in G,\ j \in J\}$. Given $(j_1,g,j) \in R_{j_1}$, if $j \in J_1$, then $(j_1,g,j) \in B(G,J_1) \subseteq S$; and if $j \in J_2$, then $(j_1,g,j) = (j_1,a,j_2)(j_2,a^{-1}g,j) \in S.B(G,J_2) \subseteq S$. Therefore, $R_{j_1} \subseteq S$. From Lemma 5.3.1, if $K_{j_1} = J \setminus \{j_1\}$, then there exists $[g_j] \in G^{K_{j_1}}$ such that $B(G,J) =$

$\langle\langle H_{j_1 j_1} \cup \{(j_1, g_j, j) : j \in K_{j_1}\}\rangle\rangle$. Since $H_{j_1 j_1} \cup \{(j_1, g_j, j) : j \in K_{j_1}\} \subseteq R_{j_1} \subseteq S$ and S is an inverse semigroup, it follows that $B(G, J) \subseteq S$, as required. □

In essence, the next result is [48] Proposition 3.4, but in a more general setting, although we restrict J to being an index set with at least two elements. Once again, we follow most of Yang's argument closely.

Theorem 5.3.4. Let S be a subsemigroup of the Brandt semigroup $B(G, J)$, where G is an arbitrary group with identity e and J is an index set with at least two elements. Then, S is a maximal inverse subsemigroup of $B(G, J)$ if and only if S is one of the following:

(a) $S = B(G, J_1) \cup B(G, J_2)$, where (J_1, J_2) is a two-partition of J,

(b) $S = \langle\langle H_{11}^* \cup \{(1, g_j, j) : j \in K_1\}\rangle\rangle$, where $1 \in J$, $K_1 = J \setminus \{1\}$, $H_{11}^* = \{(1, h, 1) \in B(G, J) : h \in H\}$, H is a maximal subgroup of G and $[g_j] \in G^{K_1}$.

Proof. From Lemmas 5.3.2 and 5.3.3, if S is of type (a) or (b) then it is a maximal inverse subsemigroup of $B(G, J)$. Conversely, let S be a maximal inverse subsemigroup of $B(G, J)$. We assert that $S \cap H_{jj} \neq \emptyset$ for each $j \in J$. For, suppose $S \cap H_{ii} = \emptyset$ for some $i \in J$. Then, $S \cap L_i = \emptyset$ where L_i is the $\mathcal{L}$–class of $B(G, J)$ which contains the element (i, e, i). For, if $S \cap L_i \neq \emptyset$, then there exist some $j \in J$ and $a \in G$ such that $(j, a, i) \in S$, and so $(i, e, i) = (i, a^{-1}, j)(j, a, i) = (j, a, i)^{-1}(j, a, i) \in S \cap H_{ii}$, a contradiction. Similarly, $S \cap R_i = \emptyset$ where R_i denotes the $\mathcal{R}$–class of $B(G, J)$ containing (i, e, i). Hence,

$$S \subseteq B(G, J \setminus \{i\}) \subsetneq B(G, J \setminus \{i\}) \cup H_{ii} \subsetneq B(G, J).$$

Clearly, $B(G, J \setminus \{i\}) \cup H_{ii}$ is an inverse subsemigroup of $B(G, J)$, which contradicts our assumption on S. Note that, in the above argument, we use the fact that $|J| \geq 2$.

We now consider two cases.

Case 1. Suppose $S \cap H_{11} = H_{11}$. Let $J_1 = \{j \in J : S \cap H_{1j} \neq \emptyset\}$ and $J_2 = J \setminus J_1$, and note that $J_1 \neq \emptyset$ by supposition. If $S \cap H_{1j} \neq \emptyset$, then there exists some $a \in G$ such that $(1, a, j) \in S$, and so we have $(1, g, j) = (1, ga^{-1}, 1)(1, a, j) \in H_{11}.S \subseteq S$, for every $(1, g, j) \in H_{1j}$. Hence, $S \cap H_{1j} = H_{1j}$ for each $j \in J_1$. Thus, given

$(i,g,j) \in B(G,J_1)$, we have $S \cap H_{1i} = H_{1i}$ and $S \cap H_{1j} = H_{1j}$, and so $(i,g,j) = (1,g^{-1},i)^{-1}(1,e,j) \in H_{1i}^{-1}.H_{1j} \subseteq S$. Therefore, $B(G,J_1) \subseteq S$.

Suppose there exist $j_1 \in J_1$, $a \in G$ and $j_2 \in J_2$ such that $(j_1,a,j_2) \in S$. Then, $(1,a,j_2) = (1,e,j_1)(j_1,a,j_2) \in H_{1j_1}.S \subseteq S$, since $j_1 \in J_1$ implies $S \cap H_{1j_1} \neq \emptyset$ and so $H_{1j_1} \subseteq S$ by a remark in the last paragraph. Hence $(1,a,j_2) \in S \cap H_{1j_2}$, so this set is non-empty and therefore $j_2 \in J_1$, a contradiction. Thus, since S is inverse, given $j_1 \in J_1$, $a \in G$ and $j_2 \in J_2$, both $(j_1,a,j_2) \notin S$ and $(j_2,a,j_1) \notin S$. We then have $S \subseteq B(G,J_1) \cup B(G,J_2) \subsetneq B(G,J)$. By maximality of S, it follows that $S = B(G,J_1) \cup B(G,J_2)$.

Case 2. Suppose $S \cap H_{11} \neq H_{11}$. Let $H_{11}^* = S \cap H_{11}$ and $H = \{h \in G : (1,h,1) \in H_{11}^*\}$. We assert that H is a proper subgroup of G. Since $S \cap H_{11} \neq \emptyset$, there exists $a \in G$ such that $(1,a,1) \in S \cap H_{11} = H_{11}^*$. Then, $(1,e,1) = (1,a,1)(1,a,1)^{-1} \in H_{11}^*$ and so $e \in H$. Now suppose $h_1, h_2 \in H$. Then, $(1,h_1,1),(1,h_2,1) \in H_{11}^*$. Since H_{11}^* is an inverse subsemigroup of $B(G,J)$, it follows that $(1,h_1,1)(1,h_2,1)^{-1} = (1,h_1h_2^{-1},1) \in H_{11}^*$ and so $h_1h_2^{-1} \in H$. Since $S \cap H_{11} \neq H_{11}$, there exists some $g \in G$ such that $(1,g,1) \notin S \cap H_{11} = H_{11}^*$, and so $g \notin H$ and $H \subsetneq G$.

Still in Case 2, let $J^* = \{j \in J : S \cap H_{1j} \neq \emptyset\}$ and choose $(1,g_j,j) \in S \cap H_{1j}$ for each $j \in J^*$. Clearly, $\langle\langle H_{11}^* \cup \{(1,g_j,j) : j \in J^*\}\rangle\rangle \subseteq S$. We assert that $J = J^*$. For, if $J \setminus J^* \neq \emptyset$, then $S \cap H_{1i} = \emptyset$ for each $i \in J \setminus J^*$. Now suppose $(i,a,j) \in S$ with $i \notin J^*$ and $j \in J^*$. Then, $(1,g_ja^{-1},i) = (1,g_j,j)(j,a^{-1},i) = (1,g_j,j)(i,a,j)^{-1} \in S$ (since S is inverse) which contradicts $S \cap H_{1i} = \emptyset$. In other words, if $(i,a,j) \in S$ then either ($i \in J^*$ and $j \in J^*$) or ($i \notin J^*$ and $j \notin J^*$). Therefore, since $1 \in J^*$ (by a remark at the start of this proof) and $H_{11} \setminus S$ is non-empty (by assumption in this case), it follows that

$$S \subsetneq B(G,J^*) \cup B(G,J \setminus J^*) \subsetneq B(G,J). \tag{5.3.1}$$

But $B(G,J^*) \cup B(G,J \setminus J^*)$ is an inverse subsemigroup of $B(G,J)$, so (5.3.1) contradicts the maximality of S. Therefore, $J = J^*$.

Next we see that $S = \langle\langle H_{11}^* \cup \{(1,g_j,j) : j \in J\}\rangle\rangle$. For convenience, we let $g_1 = e$ and for every i and j in J, we denote the subset $\{(i,g_i^{-1}hg_j,j) : h \in H\}$ of $B(G,J)$

by $(i, g_i^{-1}Hg_j, j)$. We assert that $S \cap H_{ij} = (i, g_i^{-1}Hg_j, j)$. For, if $h \in H$, we have

$$(i, g_i^{-1}hg_j, j) = (1, g_i, i)^{-1}(1, h, 1)(1, g_j, j) \in S.H_{11}^*.(S \cap H_{1j}) \subseteq S$$

and so $(i, g_i^{-1}hg_j, j) \in S \cap H_{ij}$. Conversely, let $(i, a, j) \in S \cap H_{ij}$. Then,

$$(1, g_i a g_j^{-1}, 1) = (1, g_i, i)(i, a, j)(1, g_j, j)^{-1} \in S \cap H_{11} = H_{11}^*$$

and so $g_i a g_j^{-1} \in H$. Thus, $(i, a, j) = (i, g_i^{-1}hg_j, j)$ where $h = g_i a g_j^{-1} \in H$, and so $S \cap H_{ij} = (i, g_i^{-1}Hg_j, j)$. Now, if $(i, a, j) \in S$, then $(i, a, j) = (i, g_i^{-1}hg_j, j) = (1, g_i, i)^{-1}(1, h, 1)(1, g_j, j)$ for some $h \in H$, and so (i, a, j) can be written as a finite product of elements of $H_{11}^* \cup \{(1, g_j, j) : j \in J\}$ and inverses of such elements. Therefore, $S \subseteq \langle\langle H_{11}^* \cup \{(1, g_j, j) : j \in J\}\rangle\rangle$, and equality follows from the containment observed earlier.

It remains to show that H is a maximal subgroup of G. Suppose this is not true. Then, there exists a proper subgroup K of G which properly contains H. Let $K_{11}^* = \{(1, k, 1) : k \in K\}$. Clearly, $H_{11}^* \subsetneq K_{11}^*$ and so S is a proper subsemigroup of the inverse semigroup $\langle\langle K_{11}^* \cup \{(1, g_j, j) : j \in J\}\rangle\rangle$, a contradiction. Therefore, H is a maximal subgroup of G, and the result follows. □

Now we turn our attention to the inverse semigroup $I_r = \{\alpha \in I(V) : r(\alpha) < r\}$ for $2 \leq r \leq m + 1$ where $m = \dim V < \aleph_0$. From Lemma 5.2.1, the $\mathcal{D}$–class D_{r-1} generates I_r since, if $s \in \{1, \ldots, r-1\}$, then $D_s \subseteq D_{r-1}^k$ where $k = 2^{r-1-s}$. Since $D_{r-1} = I_r \setminus I_{r-1}$ is a $\mathcal{D}$–class of $I(V)$, the principal factor I_r/I_{r-1} is 0-simple. Moreover, I_r/I_{r-1} contains non-zero idempotents: namely, the identity maps on subspaces of V with dimension $r-1$. In fact, if η, ε are two such idempotents and $\eta\varepsilon = \varepsilon\eta = \varepsilon$, then $\operatorname{ran}\varepsilon = \operatorname{ran}(\varepsilon\eta) \subseteq \operatorname{ran}\eta$. Since $r(\varepsilon) = r - 1 = r(\eta)$ and $r-1$ is finite, it follows that $\operatorname{ran}\varepsilon = \operatorname{ran}\eta$. Therefore, $\varepsilon = \mathrm{id}_{\operatorname{ran}\varepsilon} = \mathrm{id}_{\operatorname{ran}\eta} = \eta$, and this proves that η is a primitive idempotent of I_r/I_{r-1}. Hence, I_r/I_{r-1} is completely 0-simple.

In [5] Vol. 1, pp. 95-96, Clifford and Preston give a Rees matrix representation for the completely 0-simple semigroup T_k/T_{k-1}, where T_k is the set of all total transformations α on a finite set X with cardinal n such that $r(\alpha) \leq k$ and $2 \leq k \leq n$.

And, in effect, [10] Theorem 4.12 provides the Rees matrix representation for the corresponding principal factor in $T(V)$, the semigroup of all total linear transformations of a finite-dimensional vector space V. However, that result is expressed in terms of 'strong independence algebras', an unnecessary complication in the present situation. Therefore, we adopt a simpler approach and proceed to establish a Rees matrix representation for the principal factor I_r/I_{r-1} of $I(V)$.

Green's $\mathcal{L}$ and $\mathcal{R}$ relations on $I(X)$ are well-known for any set X: namely, $\alpha \mathcal{L} \beta$ if and only if $\operatorname{ran} \alpha = \operatorname{ran} \beta$; and $\alpha \mathcal{R} \beta$ if and only if $\operatorname{dom} \alpha = \operatorname{dom} \beta$ (see chapter 1, section 1.2). As observed in chapter 1, section 1.3, it is easy to show that similar statements hold for $I(V)$ and any vector space V (compare [1] Corollary 4 and [26] p. 427, for independence algebras). Thus, we have the following result for I_r/I_{r-1}.

Lemma 5.3.5. If $2 \leq r \leq m+1$ and $\alpha, \beta \in I_r/I_{r-1}$ are non-zero, then

(a) $\alpha \mathcal{L} \beta$ if and only if $\operatorname{ran} \alpha = \operatorname{ran} \beta$,

(b) $\alpha \mathcal{R} \beta$ if and only if $\operatorname{dom} \alpha = \operatorname{dom} \beta$.

Let r_V denote the family of all subspaces of V with dimension $r-1$, and note that r_V is infinite if the field F is infinite, hence our need to consider *arbitrary* Brandt semigroups. For every A in r_V, fix a basis $\{v_1^A, \ldots, v_{r-1}^A\}$ of A. From Lemma 5.3.5, I_r/I_{r-1} has $|r_V|$ non-zero $\mathcal{R}$–classes corresponding to the possible domains of dimension $r-1$ and $|r_V|$ non-zero $\mathcal{L}$–classes corresponding to the possible ranges of dimension $r-1$. Let $A, B \in r_V$ and put $H_{A,B} = R_A \cap L_B$. From Lemma 5.3.5, $\alpha \in H_{A,B}$ if and only if $\operatorname{dom} \alpha = A$ and $\operatorname{ran} \alpha = B$, that is, if and only if

$$\alpha = \begin{pmatrix} v_1^A & \ldots & v_{r-1}^A \\ \sum_{j=1}^{r-1} x_{1j} v_j^B & \ldots & \sum_{j=1}^{r-1} x_{r-1\,j} v_j^B \end{pmatrix}$$

for some invertible $(r-1) \times (r-1)$ matrix $[x_{ij}]$ over F. In this case, we write $\alpha = (A, [x_{ij}], B)$.

Now let $\beta \in D_{r-1}$. Since $\operatorname{ran} \alpha \cap \operatorname{dom} \beta$ is a subspace of the $(r-1)$-dimensional vector spaces $\operatorname{ran} \alpha$ and $\operatorname{dom} \beta$ (where $r-1$ is finite and β is one-to-one), we have $r(\alpha\beta) = r-1$ if and only if $\dim(\operatorname{ran} \alpha \cap \operatorname{dom} \beta) = r-1$, and this is true if and only if $B = \operatorname{ran} \alpha = \operatorname{ran} \alpha \cap \operatorname{dom} \beta = \operatorname{dom} \beta$.

Suppose $r(\alpha\beta) = r-1$. Then $B = \text{dom } \beta$ and for some subspace D of V and some invertible $(r-1)\times(r-1)$ matrix $[y_{ij}]$ over F,

$$\beta = \begin{pmatrix} v_1^B & \cdots & v_{r-1}^B \\ \sum_{j=1}^{r-1} y_{1j} v_j^D & \cdots & \sum_{j=1}^{r-1} y_{r-1j} v_j^D \end{pmatrix} = (B, [y_{ij}], D).$$

Also, in this case, since $\text{ran}\,(\alpha\beta) \subseteq \text{ran } \beta$ and $r-1$ is finite, we have $\text{ran}\,(\alpha\beta) = \text{ran } \beta = D$. Analogously, we conclude that $\text{dom}\,(\alpha\beta) = \text{dom } \alpha = A$. If $i \in \{1, \ldots, r-1\}$, then

$$v_i^A(\alpha\beta) = \Big(\sum_{j=1}^{r-1} x_{ij} v_j^B\Big)\beta = \sum_{j=1}^{r-1} x_{ij} \Big(\sum_{k=1}^{r-1} y_{jk} v_k^D\Big) = \sum_{k=1}^{r-1} \Big(\sum_{j=1}^{r-1} x_{ij} y_{jk}\Big) v_k^D.$$

Therefore, we have $\alpha\beta = (A, [z_{ij}], D)$ where $z_{ij} = \sum_{k=1}^{r-1} x_{ik} y_{kj}$ and so

$$\alpha\beta = (A, [x_{ij}][y_{ij}], D) = (A, [x_{ij}], B) \circ (\text{dom } \beta, [y_{ij}], D).$$

If $r(\alpha\beta) < r-1$, then $B \neq \text{dom } \beta$. Also, $\alpha\beta \notin I_r \setminus I_{r-1}$, and hence $\alpha\beta = 0$ in I_r/I_{r-1}.

Let $G = G_{r-1}$, the group of all invertible $(r-1)\times(r-1)$ matrices over F, and let $J = r_V$. Define the mapping $\theta : I_r/I_{r-1} \to B(G, J)$ by $0\theta = 0$ and $\alpha\theta = (A, [x_{ij}], B)$ where $v_i^A\alpha = \sum_{j=1}^{r-1} x_{ij} v_j^B$, for every $i \in \{1, \ldots, r-1\}$. We have shown that θ is a morphism from I_r/I_{r-1} into $B(G, J)$. Clearly, θ is one-to-one: given $\alpha, \beta \in I_r \setminus I_{r-1}$ such that $(\text{dom } \alpha, [x_{ij}], \text{ran } \alpha) = \alpha\theta = \beta\theta = (\text{dom } \beta, [y_{ij}], \text{ran } \beta)$, we have

$$v_i^{\text{dom } \alpha}\alpha = \sum_{j=1}^{r-1} x_{ij} v_j^{\text{ran } \alpha} = \sum_{j=1}^{r-1} y_{ij} v_j^{\text{ran } \beta} = v_i^{\text{dom } \beta}\beta.$$

Since $\text{dom } \alpha = \text{dom } \beta$, it follows that $\alpha = \beta$. To see that θ is onto, let $[x_{ij}]$ be an invertible $(r-1)\times(r-1)$ matrix over F and $A, B \in r_V$. Now define $\alpha \in P(V)$ by

$$\alpha = \begin{pmatrix} v_1^A & \cdots & v_{r-1}^A \\ \sum_{j=1}^{r-1} x_{1j} v_j^B & \cdots & \sum_{j=1}^{r-1} x_{r-1j} v_j^B \end{pmatrix}.$$

Then, $\text{dom } \alpha = \langle v_1^A, \ldots, v_{r-1}^A \rangle = A$ and, since $[x_{ij}]$ is invertible, α is one-to-one and $\text{ran } \alpha = \langle v_1^B, \ldots, v_{r-1}^B \rangle = B$. Therefore, $\alpha \in I_r \setminus I_{r-1}$ and $\alpha\theta = (A, [x_{ij}], B)$. Hence θ is an isomorphism from I_r/I_{r-1} onto $B(G, J)$.

Finally, we turn to the maximal inverse subsemigroups S of I_r where $2 \leq r \leq m$ (the case $r = m+1$ is handled by Theorem 5.2.3). If $I_{r-1} \setminus S \neq \emptyset$ then $S \subsetneq S \cup I_{r-1}$, which

is an inverse subsemigroup of I_r. By the maximality of S, this implies $S \cup I_{r-1} = I_r$. Hence $D_{r-1} \subseteq S$ and, by Lemma 5.2.1, this implies $I_r \subseteq S$ (see the remark after the proof of Theorem 5.3.4 above). Since this contradicts our supposition, we deduce that $I_{r-1} \subseteq S$ and thus $S = I_{r-1} \cup (S \cap D_{r-1})$. Clearly, $(S \cap D_{r-1}) \cup \{0\}$ is an inverse subsemigroup of I_r/I_{r-1}, and we assert it is a maximal one. For, if T is an inverse subsemigroup of I_r/I_{r-1} which properly contains it, then $I_{r-1} \cup (T \setminus \{0\})$ is an inverse subsemigroup of I_r (under composition) which properly contains S, a contradiction.

If $\dim V = m$ and $r = m+1$ then I_r/I_{r-1} is simply the group G_m of units in $I(V)$ with 0 adjoined. Therefore, $|J| = 1$ in the representation of G_m^0 as a Brandt semigroup $B(G, J)$, and so we cannot apply Theorem 5.3.4 in this case. However, that result describes all maximal inverse subsemigroups of the Brandt semigroup $B(G, J)$ where G is an arbitrary group and $|J| \geq 2$; hence, via the isomorphism established above, we know all maximal inverse subsemigroups of I_r/I_{r-1} where $2 \leq r \leq m$. Consequently, we have the following result.

Theorem 5.3.6. Let $2 \leq r \leq m = \dim V < \aleph_0$ and let r_V denote the family of all subspaces of V with dimension $r-1$. If S is a subsemigroup of the inverse semigroup I_r, then S is a maximal inverse subsemigroup of I_r if and only if S is one of the following:

(a) $S = I_{r-1} \cup B_{r_{V_1}} \cup B_{r_{V_2}}$, where (r_{V_1}, r_{V_2}) is a two-partition of r_V and $B_{r_{V_i}} = \{\alpha \in I_r : \text{dom } \alpha, \text{ ran } \alpha \in r_{V_i}\}$ for $i = 1, 2$,

(b) $S = I_{r-1} \cup \langle\langle H_{AA}^* \cup \{\alpha_B : B \in r'_V\}\rangle\rangle$, where $A \in r_V$, $r'_V = r_V \setminus \{A\}$, $H_{AA}^* = \{\alpha = [h_{ij}] \in I_r : \text{dom } \alpha = A = \text{ran } \alpha \text{ and } [h_{ij}] \in H\}$, H is a maximal subgroup of the group G_{r-1} of all $(r-1) \times (r-1)$ invertible matrices over F and $[\alpha_B] \in I_r^{r'_V}$ where $\text{dom } \alpha_B = A$ and $\text{ran } \alpha_B = B$ for every $B \in r'_V$.

5.4. Some inverse subsemigroups of $I(V)$ for V infinite-dimensional

Let X be an infinite set with cardinal p, and let q be a cardinal such that $\aleph_0 \leq q \leq p$. Let B denote the Baer-Levi semigroup $BL(p, q)$ of type (p, q) on the set X, as discussed in chapter 2, section 2.1. We recall that in chapter 1, section 1.2, we

defined the gap of $\alpha \in P(X)$ to be $g(\alpha) = |X \setminus \text{dom } \alpha|$. In [12] Theorems 2.3 and 2.4, Howie and Marques-Smith showed that if $p = q$, then

$$B^{-1} = \{\alpha \in I(X) : g(\alpha) = p,\ d(\alpha) = 0\},$$

$$BB^{-1} = I(X) \quad \text{and} \quad B^{-1}B = \{\alpha \in I(X) : r(\alpha) = g(\alpha) = d(\alpha) = p\}.$$

In this case (namely, $p = q$), they also showed that the subsemigroup of $I(X)$ generated by $B^{-1}B$ equals $K_p = \{\alpha \in I(X) : g(\alpha) = d(\alpha) = p\}$ ([12] Theorem 2.5). In [12] Theorem 3.2, the authors proved that K_p is, in fact, the inverse subsemigroup of $I(X)$ generated by the nilpotent elements of index 2 (that is, all $\alpha \in I(X)$ for which $\alpha \neq \emptyset$ and $\alpha^2 = \emptyset$). In [26] Propositions 4.3 and 4.5 and Theorem 5.6, Lima extended these results to strong independence algebras, and thus also to vector spaces ('sets' and 'vector spaces' are prime examples of a strong independence algebra). In this section, we answer the natural question: what happens when $p > q$? We remind the reader that what follows has appeared in [31].

Let V be a vector space over a field F and suppose $\dim V = m \geq \aleph_0$. For each cardinal n such that $\aleph_0 \leq n \leq m$, consider the linear Baer-Levi semigroup on V as defined in chapter 2, section 2.2:

$$G = GS(m, n) = \{\alpha \in T(V) : n(\alpha) = 0,\ d(\alpha) = n\}. \tag{5.4.1}$$

As shown in chapter 2, Theorem 2.2.1, this is a right simple, right cancellative subsemigroup of $T(V)$ without idempotents. Thus, it is not an inverse semigroup: in fact, $\alpha \in G$ if and only if $\alpha^{-1} \in G^{-1}$ where

$$G^{-1} = \{\beta \in I(V) : g(\beta) = n,\ d(\beta) = 0\} \tag{5.4.2}$$

(recall that in chapter 1, section 1.3, we defined the gap $g(\alpha)$ of $\alpha \in P(V)$ by $g(\alpha) = \text{codim dom } \alpha$). From the anti-isomorphism $G \to G^{-1}, \alpha \mapsto \alpha^{-1}$, we see that G^{-1} is a left simple, left cancellative subsemigroup of $I(V)$ which has no idempotents. As already remarked, Lima [26] Propositions 4.3 and 4.5, showed that if $m = n$, then

$$GG^{-1} = I(V) \quad \text{and} \quad G^{-1}G = \{\alpha \in I(V) : r(\alpha) = g(\alpha) = d(\alpha) = m\}.$$

Therefore, in this case, GG^{-1} is an inverse semigroup but $G^{-1}G$ is not even closed. We will show that if $m > n$, then both GG^{-1} and $G^{-1}G$ are inverse subsemigroups of $I(V)$.

As in section 5.1, we let 0 denote the linear map with domain $\{0\}$ in V.

Lemma 5.4.1. Suppose $\aleph_0 \leq n < m$ and let G, G^{-1} be as defined in (5.4.1) and (5.4.2). If $\alpha \in G$ and $\beta \in G^{-1}$ then $\dim(\text{ran}\,\alpha \cap \text{dom}\,\beta) = m$. In particular, $\alpha\beta \neq 0$ in $I(V)$.

Proof. Suppose $\{e_i\}$ is a basis for $\text{ran}\,\alpha \cap \text{dom}\,\beta$ and $|I| < m$. Since $n < m$, we know $\dim \text{ran}\,\alpha = m = \dim \text{dom}\,\beta$, so $\{e_i\}$ can be expanded to bases $\{e_i\} \dot\cup \{a_j\}$ for $\text{ran}\,\alpha$ and $\{e_i\} \dot\cup \{b_j\}$ for $\text{dom}\,\beta$ where $|J| = m$. Then $\{e_i\} \dot\cup \{a_j\} \dot\cup \{b_j\}$ is linearly independent: for, if $\sum x_i e_i + \sum y_j a_j + \sum z_j b_j = 0$ for some scalars x_i, y_j and z_j, then $\sum z_j b_j \in \text{ran}\,\alpha \cap \text{dom}\,\beta$, and so $\sum z_j b_j = \sum r_i e_i$ for some scalars r_i. Thus, $\sum r_i e_i - \sum z_j b_j = 0$ and so $z_j = 0$ for each j. Therefore, $\sum x_i e_i + \sum y_j a_j = 0$ and, since the set $\{e_i\} \dot\cup \{a_j\}$ is linearly independent, it follows that $x_i = 0$ for each i and $y_j = 0$ for each j. Consequently, $d(\alpha) \geq |J| = m$, a contradiction. □

Recall that $\text{dom}\,(\alpha\beta) \subseteq \text{dom}\,\alpha$ and $\text{ran}\,(\alpha\beta) \subseteq \text{ran}\,\beta$, so $g(\alpha\beta) \geq g(\alpha)$ and $d(\alpha\beta) \geq d(\beta)$. Hence the next result is a little surprising.

Theorem 5.4.2. Suppose $\aleph_0 \leq n < m$. If G and G^{-1} are as defined in (5.4.1) and (5.4.2), then

$$GG^{-1} = \{\alpha \in I(V) : g(\alpha) \leq n,\ d(\alpha) \leq n\}$$

and this is an inverse subsemigroup of $I(V)$ without nilpotents.

Proof. Let $\alpha \in G$ and $\beta \in G^{-1}$. Then, from Lemma 5.4.1, $\dim(\text{ran}\,\alpha \cap \text{dom}\,\beta) = m$. Suppose $\{e_i\}$ is a basis for $\text{ran}\,\alpha \cap \text{dom}\,\beta$ and expand it to bases $\{e_i\} \dot\cup \{a_r\}$ and $\{e_i\} \dot\cup \{b_s\}$ for $\text{ran}\,\alpha$ and $\text{dom}\,\beta$, respectively. Since α is one-to-one, there exist unique f_i and f_r such that $e_i\alpha^{-1} = f_i$ and $a_r\alpha^{-1} = f_r$. Write $e_i\beta = g_i$ for each i and $b_s\beta = g_s$ for each s. We have $V = \langle f_i, f_r\rangle = \langle g_i, g_s\rangle$ and

$$\alpha = \begin{pmatrix} f_i & f_r \\ e_i & a_r \end{pmatrix}, \quad \beta = \begin{pmatrix} e_i & b_s \\ g_i & g_s \end{pmatrix}.$$

Therefore

$$\alpha\beta = \begin{pmatrix} f_i \\ g_i \end{pmatrix}$$

and so $g(\alpha\beta) = \dim\langle f_r \rangle = |R|$ and $d(\alpha\beta) = \dim\langle g_s \rangle = |S|$. As in the proof of Lemma 5.4.1, $\{e_i\} \dot\cup \{a_r\} \dot\cup \{b_s\}$ is linearly independent, so it can be expanded to a basis for V, say $\{e_i\} \dot\cup \{a_r\} \dot\cup \{b_s\} \dot\cup \{c_\ell\}$. Thus, $|R| \leq |R| + |L| = g(\beta) = n$ and $|S| \leq |S| + |L| = d(\alpha) = n$. Hence, $\alpha\beta$ is such that $g(\alpha\beta) \leq n$ and $d(\alpha\beta) \leq n$.

Conversely, let $\alpha : A \to B$ be an injective linear map such that $g(\alpha) \leq n$ and $d(\alpha) \leq n$. Suppose $\{a_i\}$ is a basis for A and write $a_i\alpha = b_i$ for each i. Then $\{b_i\}$ is a basis for ran α. Expand $\{a_i\}$ and $\{b_i\}$ to bases $\{a_i\} \dot\cup \{a_j\}$ and $\{b_i\} \dot\cup \{b_\ell\}$ for V, respectively, with $|J| = g(\alpha) \leq n$, $|L| = d(\alpha) \leq n$. Since $|L| \leq n < m$, we may write $\{a_i\}$ as $\{u_i\} \dot\cup \{u_k\}$ with $|K| = n$ and $\{u_k\} = \{v_k\} \dot\cup \{v_\ell\}$. Now define β, γ in $P(V)$ by

$$\beta = \begin{pmatrix} a_i & a_j \\ u_i & a_j \end{pmatrix}, \quad \gamma = \begin{pmatrix} u_i & v_\ell \\ b_i & b_\ell \end{pmatrix}.$$

It is easy to see that $\beta, \gamma \in I(V)$ and $g(\beta) = 0$, $d(\beta) = \dim\langle u_k \rangle = n$, $g(\gamma) = \dim\langle v_k, a_j \rangle = n$ and $d(\gamma) = 0$. Therefore, $\beta \in G$ and $\gamma \in G^{-1}$. Since $\alpha = \beta\gamma$, it follows that $\alpha \in GG^{-1}$.

Next, we show that GG^{-1} is an inverse semigroup. To do so, let $\alpha, \beta \in GG^{-1}$. Then, $g(\alpha) \leq n$, $d(\alpha) \leq n$, $g(\beta) \leq n$ and $d(\beta) \leq n$. Since $n < m$, $\dim \operatorname{dom} \alpha = m = \dim \operatorname{dom} \beta$. Suppose $\{a_i\}$ is a basis for dom α and $\{b_i\}$ a basis for dom β. Write $a_i\alpha = u_i$ and $b_i\beta = v_i$ for each i. If ran $\alpha \cap \operatorname{dom} \beta = \{0\}$ then $\{u_i\} \dot\cup \{b_i\}$ is linearly independent, and it can be expanded to a basis $\{u_i\} \dot\cup \{b_i\} \dot\cup \{c_\ell\}$ for V. Thus, $n \geq d(\alpha) = \dim\langle b_i, c_\ell \rangle = |I| + |L| = m$, a contradiction. Therefore, ran $\alpha \cap \operatorname{dom} \beta \neq \{0\}$. Let $\{e_j\}$ be a basis for ran $\alpha \cap \operatorname{dom} \beta$ and expand it to bases $\{e_j\} \dot\cup \{u_r\}$ and $\{e_j\} \dot\cup \{b_s\}$ for ran α and dom β, respectively. Since α is one-to-one, there exist unique f_j and f_r such that $f_j\alpha = e_j$ and $f_r\alpha = u_r$. Write $e_j\beta = v_j$ and $b_s\beta = v_s$. Then, we have

$$\alpha = \begin{pmatrix} f_j & f_r \\ e_j & u_r \end{pmatrix}, \quad \beta = \begin{pmatrix} e_j & b_s \\ v_j & v_s \end{pmatrix} \quad \text{and so} \quad \alpha\beta = \begin{pmatrix} f_j \\ v_j \end{pmatrix}.$$

Since $\{e_j\} \dot\cup \{u_r\} \dot\cup \{b_s\}$ is linearly independent, it can be expanded to a basis

$$\{e_j\} \dot\cup \{u_r\} \dot\cup \{b_s\} \dot\cup \{d_t\}$$

for V. Thus, $n \geq d(\alpha) = |S| + |T|$ and $n \geq g(\beta) = |R| + |T|$, so $|S| \leq n$ and $|R| \leq n$. Hence, $g(\alpha\beta) = |R| + g(\alpha) \leq n$ and $d(\alpha\beta) = |S| + d(\beta) \leq n$. Therefore, $\alpha\beta \in GG^{-1}$ and this shows that GG^{-1} is a semigroup. Since $\operatorname{dom} \alpha^{-1} = \operatorname{ran} \alpha$ and $\operatorname{ran} \alpha^{-1} = \operatorname{dom} \alpha$, it follows that $g(\alpha^{-1}) = d(\alpha)$ and $d(\alpha^{-1}) = g(\alpha)$ and hence GG^{-1} is an inverse semigroup. Moreover, since $d(\alpha\beta) \leq n < m$ for each $\alpha, \beta \in GG^{-1}$, we deduce that $r(\alpha\beta) = m$ and thus $\alpha\beta \neq 0$: that is, GG^{-1} has no nilpotents. □

Remark 5.4.3. For the above proof, it is natural to think $GG^{-1}G \subseteq G$, but this does not hold. To see this, suppose $\{e_i\}$ is a basis for V and write $\{e_i\}$ as $\{f_i\} \dot\cup \{f_j\}$ and $\{f_j\}$ as $\{a_j\} \dot\cup \{b_j\}$ where $|J| = n$. Now define $\alpha, \beta \in I(V)$ by

$$\alpha = \begin{pmatrix} f_i & f_j \\ f_i & a_j \end{pmatrix}, \quad \beta = \begin{pmatrix} f_i & b_j \\ f_i & f_j \end{pmatrix}.$$

Clearly, $\operatorname{dom} \alpha = V = \operatorname{ran} \beta$, $d(\alpha) = \dim\langle b_j \rangle = n$ and $g(\beta) = \dim\langle a_j \rangle = n$. Thus, $\alpha \in G$ and $\beta \in G^{-1}$. Since $\operatorname{dom}(\alpha\beta\gamma) \subseteq \operatorname{dom}(\alpha\beta) = \langle f_i \rangle \neq V$ for every $\gamma \in G$, it follows that $\alpha\beta\gamma \notin G$.

Theorem 5.4.4. Suppose $\aleph_0 \leq n < m$. If G and G^{-1} are as defined in (5.4.1) and (5.4.2), then

$$G^{-1}G = \{\alpha \in I(V) : g(\alpha) = d(\alpha) = n\}$$

and this is an inverse subsemigroup of $I(V)$ without nilpotents.

Proof. Let $\alpha \in G$ and $\beta \in G^{-1}$. Then, $\operatorname{dom} \alpha = V = \operatorname{ran} \beta$ and so $\operatorname{dom}(\beta\alpha) = (\operatorname{ran} \beta \cap \operatorname{dom} \alpha)\beta^{-1} = V\beta^{-1} = \operatorname{dom} \beta$ and $\operatorname{ran}(\beta\alpha) = V\alpha = \operatorname{ran} \alpha$. Therefore, $g(\beta\alpha) = g(\beta) = n$ and $d(\beta\alpha) = d(\alpha) = n$.

Conversely, let $\alpha \in I(V)$ be such that $g(\alpha) = d(\alpha) = n$ and suppose $\{a_i\}$ is a basis for $\operatorname{dom} \alpha$. Write $a_i\alpha = b_i$ for each i and expand $\{a_i\}$ and $\{b_i\}$ to bases for V, say $\{a_i\} \dot\cup \{a_j\}$ and $\{b_i\} \dot\cup \{b_j\}$, respectively, with $|J| = n$. Since $n < m = \dim V$, it follows that $|I| = m$. Suppose $\{e_i\}$ is a basis for V and define $\beta, \gamma \in I(V)$ by

$$\beta = \begin{pmatrix} a_i \\ e_i \end{pmatrix}, \quad \gamma = \begin{pmatrix} e_i \\ b_i \end{pmatrix}.$$

Then, $g(\beta) = \dim\langle a_j \rangle = g(\alpha) = n$, $\operatorname{ran} \beta = V = \operatorname{dom} \gamma$ and $d(\gamma) = \dim\langle b_j \rangle = d(\alpha) = n$. Therefore, $\beta \in G^{-1}$ and $\gamma \in G$. Since $\alpha = \beta\gamma$, we have $\alpha \in G^{-1}G$ and the result follows.

To see that $G^{-1}G$ is a semigroup, let $\alpha, \beta \in G^{-1}G$. Then, $g(\alpha) = d(\alpha) = g(\beta) = d(\beta) = n$ and, since $n < m$, $\dim \operatorname{dom} \alpha = m = \dim \operatorname{dom} \beta$. Let $\{a_i\}$ and $\{b_i\}$ be bases for $\operatorname{dom} \alpha$ and $\operatorname{dom} \beta$, respectively, and write $a_i\alpha = e_i$ and $b_i\beta = f_i$ for each i. Suppose $\operatorname{ran} \alpha \cap \operatorname{dom} \beta = \{0\}$. Then $\{e_i\} \dot{\cup} \{b_i\}$ is linearly independent and so it can be expanded to a basis for V, say $\{e_i\} \dot{\cup} \{b_i\} \dot{\cup} \{v_j\}$. Thus, $n = d(\alpha) = \dim\langle b_i, v_j\rangle = m$, which contradicts our assumption on n and m. Therefore, $\operatorname{ran} \alpha \cap \operatorname{dom} \beta \neq \{0\}$. Let $\{c_\ell\}$ be a basis for $\operatorname{ran} \alpha \cap \operatorname{dom} \beta$ and expand it to bases $\{c_\ell\} \dot{\cup} \{e_r\}$ and $\{c_\ell\} \dot{\cup} \{b_s\}$ for $\operatorname{ran} \alpha$ and $\operatorname{dom} \beta$, respectively. Since α is injective, there exist unique u_ℓ and u_r in V such that $u_\ell\alpha = c_\ell$ and $u_r\alpha = e_r$. If we write $c_\ell\beta = f_\ell$ for each ℓ and $b_s\beta = f_s$ for each s, we then have

$$\alpha = \begin{pmatrix} u_\ell & u_r \\ c_\ell & e_r \end{pmatrix}, \quad \beta = \begin{pmatrix} c_\ell & b_s \\ f_\ell & f_s \end{pmatrix}$$

and so

$$\alpha\beta = \begin{pmatrix} u_\ell \\ f_\ell \end{pmatrix}.$$

As in the proof of Lemma 5.4.1, it follows that $\{c_\ell\} \dot{\cup} \{e_r\} \dot{\cup} \{b_s\}$ is linearly independent, so it can be expanded to a basis $\{c_\ell\} \dot{\cup} \{e_r\} \dot{\cup} \{b_s\} \dot{\cup} \{d_k\}$ for V. Therefore, $|S| \leq |S| + |K| = \operatorname{codim} \langle c_\ell, e_r\rangle = d(\alpha) = n$ and $|R| \leq |R| + |K| = \operatorname{codim} \langle c_\ell, b_s\rangle = g(\beta) = n$. Hence, $g(\alpha\beta) = |R| + g(\alpha) = n$ and $d(\alpha\beta) = |S| + d(\beta) = n$. Thus, $\alpha\beta \in G^{-1}G$ and so $G^{-1}G$ is a semigroup. Since $g(\alpha^{-1}) = d(\alpha)$ and $d(\alpha^{-1}) = g(\alpha)$, it follows that $G^{-1}G$ is an inverse semigroup. As in the proof of Theorem 5.4.2, $d(\alpha) < m$ for each $\alpha \in G^{-1}G$ implies $r(\alpha) = m$ and thus $\alpha \neq 0$: that is, $G^{-1}G$ has no nilpotents. □

As observed by Sullivan (personal communication) there are analogues of Theorems 5.4.2 and 5.4.4 for sets. That is, if X is a set and $\aleph_0 \leq q < p = |X|$, we can show, in a manner similar to the above proofs, that

$$BB^{-1} = \{\alpha \in I(X) : g(\alpha) \leq q,\ d(\alpha) \leq q\},$$

$$B^{-1}B = \{\alpha \in I(X) : g(\alpha) = d(\alpha) = q\}$$

and that these are inverse subsemigroups of $I(X)$ without nilpotents (indeed, the above proofs hold provided we omit all references to bases and their extension). Since these semigroups look so much like GG^{-1} and $G^{-1}G$, an obvious problem is to decide whether the corresponding pairs are isomorphic.

In Theorem 5.1.2, we showed that $I(X)$ and $I(V)$ are almost never isomorphic. Sullivan (personal communication) used a similar idea to show the above corresponding pairs are never isomorphic.

The idempotents of $I(X)$ have the form id_Y where $Y \subseteq X$; and they are partially ordered by

$$\mathrm{id}_A \leq \mathrm{id}_B \iff \mathrm{id}_A = \mathrm{id}_A \circ \mathrm{id}_B \iff A \subseteq B.$$

We say id_B *covers* id_A, and write $\mathrm{id}_A < \mathrm{id}_B$, if there is no idempotent in $I(X)$ strictly between id_A and id_B under this partial order: in other words, when this occurs, $B = A \cup \{x\}$ for some $x \notin A$. In addition, if $M \subseteq X$ and there are distinct $a, b \notin M$, we say

$$\mathrm{id}_M < \mathrm{id}_{M\cup\{a\}} < \mathrm{id}_{M\cup\{a,b\}} \tag{5.4.3}$$

is an *idempotent chain of length* 2; and that $\alpha \in I(X)$ *preserves* this chain if

$$\mathrm{id}_M \circ \alpha = \mathrm{id}_M \quad \text{and} \quad \mathrm{id}_{M\cup\{a,b\}} = \alpha\alpha^{-1}.$$

Clearly, when this happens, $a\alpha = x$ and $b\alpha = y$ for some $x \neq y$, and hence

$$\mathrm{id}_M < \mathrm{id}_{M\cup\{x\}} < \mathrm{id}_{M\cup\{x,y\}}$$

where $\mathrm{id}_{M\cup\{a\}} \circ \alpha = \mathrm{id}_{M\cup\{x\}}$ and $\mathrm{id}_{M\cup\{a,b\}} \circ \alpha = \mathrm{id}_{M\cup\{x,y\}}$ (this explains the terminology).

Let S denote either BB^{-1} and $B^{-1}B$, and let $D_2(S)$ equal the set of all $\alpha \in S$ of the form:

$$\alpha = \begin{pmatrix} a & b \\ x & y \end{pmatrix} \cup \mathrm{id}_M, \tag{5.4.4}$$

where $a \neq b$, $x \neq y$, $M \subseteq X$, and none of a, b, x, y belong to M. Clearly, $D_2(S)$ is non-empty, and we assert that its elements are characterised by the statement:

$$\alpha \in S \text{ and } \alpha \text{ preserves an idempotent chain in } S \text{ of length 2.} \tag{5.4.5}$$

Clearly, if $\alpha \in S$ has the form in (5.4.4) then $|X \setminus M| \leq q$, hence both id_M and $\mathrm{id}_{M\cup\{a\}}$ belong to S, and so α preserves the idempotent chain of length 2 in (5.4.3). Conversely, suppose the chain in (5.4.3) belongs to S and assume $\alpha \in S$ preserves

it. Then, by definition, $\operatorname{dom}\alpha = M \cup \{a, b\}$ and $\mathrm{id}_M \subseteq \alpha$; and moreover, since α is injective, we have $a\alpha = x$ and $b\alpha = y$ for some $x \neq y$, so α has the form in (5.4.4).

Next we assert that S satisfies:

$$\begin{aligned} &\text{if } \alpha \in S \text{ preserves a chain } \mathrm{id}_M < \mathrm{id}_{M\cup\{a\}} < \mathrm{id}_{M\cup\{a,b\}} \\ &\text{and } \alpha\gamma, \alpha^2\gamma \neq \mathrm{id}_M \text{ for some idempotent } \gamma \in S \text{ that covers } \mathrm{id}_M, \\ &\text{and if } \alpha^2 \text{ is not idempotent, then } \gamma\alpha = \mathrm{id}_M\,. \end{aligned} \tag{5.4.6}$$

To see this, suppose $\alpha \in S$ satisfies the initial condition, hence it has the form in (5.4.4), and assume an idempotent $\gamma \in S$ covers id_M and $\alpha\gamma \neq \mathrm{id}_M$. Then $\operatorname{ran}\alpha \cap \operatorname{dom}\gamma \neq M$ and so, without loss of generality,

$$\gamma = \begin{pmatrix} x \\ x \end{pmatrix} \cup\ \mathrm{id}_M\,.$$

Suppose $x = a$: in this case, if $b = y$ then $\alpha = \mathrm{id}_{\{a,b\}} \cup \mathrm{id}_M$ is idempotent; and if $b \neq y$ then $\alpha^2 = a_a \cup\ \mathrm{id}_M$ is idempotent, contradicting the supposition in both cases. On the other hand, if $x = b$ then $y \neq b$ (since α is injective) and $y \neq a$ (since α^2 is not idempotent). Hence, in this case, $\alpha^2 = a_y \cup \mathrm{id}_M$, so $\alpha^2\gamma \neq \mathrm{id}_M$ implies $y \in \operatorname{dom}\gamma$, which is impossible since $y \notin M \cup \{x\}$. Therefore, $x \notin \{a, b\}$ and it follows that $\gamma\alpha = \mathrm{id}_M$.

Clearly, property (5.4.6) will be preserved under an isomorphism φ from BB^{-1} onto GG^{-1} (and likewise for the other pair of analogues). This is because the notions of 'cover' and 'idempotent chain of length 2' can be described algebraically, and also because BB^{-1} is an inverse semigroup (hence, both α^{-1} and $\alpha\alpha^{-1}$ belong to BB^{-1} if $\alpha \in BB^{-1}$). We assert that the image of S under φ does not satisfy (5.4.6).

To see this, choose linearly independent $a, b \in V$, then $\{b, a + b\}$ is also linearly independent. Suppose N is a subspace of V such that $N \cap \langle a, b\rangle = \{0\}$ and assume

$$\beta = \begin{pmatrix} a & b \\ b & a+b \end{pmatrix} \cup\ \mathrm{id}_N \in S\varphi$$

(this containment simply restricts the codimension of N as required). Now β preserves the idempotent chain of length 2 in $S\varphi$ given by

$$\mathrm{id}_N < \mathrm{id}_{N+\langle a\rangle} < \mathrm{id}_{N+\langle a,b\rangle}$$

since $\mathrm{id}_N \circ \beta = \mathrm{id}_N$ and $\mathrm{id}_{N+\langle a,b\rangle} = \beta\beta^{-1}$. Hence β is the image of some element of $D_2(S)$. However, β does not satisfy (5.4.6). For, clearly β^2 is not idempotent. Also, if $c = a + b$ then $\gamma = \mathrm{id}_{N+\langle c\rangle}$ is an idempotent in $S\varphi$ which covers id_N. In addition, $\beta\gamma \neq \mathrm{id}_N$ and $\beta^2\gamma \neq \mathrm{id}_N$. However, also $\gamma\beta \neq \mathrm{id}_N$, and the last assertion follows.

The above argument shows that, despite their similarities, these corresponding pairs of semigroups are essentially different, something which is not apparent when they are regarded as semigroups of endomorphisms of a strong independence algebra.

Chapter 6

An isomorphism problem

As observed by Sullivan in 1975, many authors had previously described the automorphisms of the non-zero ideals of $P(X)$, $T(X)$ and $I(X)$ for an arbitrary set X: in each case, the automorphisms are "inner", and each ideal has the symmetric group $G(X)$ as its automorphism group. To unify these results, Sullivan introduced two simple concepts: a semigroup S of partial transformations of X is said to *cover* X if for every $x \in X$ there exists an idempotent map in S with range $\{x\}$; and S is $G(X)$-*normal* if $g^{-1}\alpha g \in S$ for every $\alpha \in S$ and every $g \in G(X)$. Using these ideas, he showed that all automorphisms of a subsemigroup S of $P(X)$ which covers X are inner; and, if S is also $G(X)$-normal, then its automorphism group is isomorphic to $G(X)$. Clearly, this leaves open the possibility of a semigroup with no constants having all of its automorphisms inner. In effect, in the same year, Fitzpatrick and Symons showed that if X is infinite and S is a subsemigroup of $T(X)$ containing $G(X)$, then every automorphism of S is inner. And a decade later, Levi, Wood, Sullivan and Schein together showed that every Baer-Levi semigroup $BL(p, q)$ has the same property (note that although $BL(p, q)$ is $G(X)$-normal, it contains neither constants nor permutations). In 1987, Levi proved that all automorphisms of a $G(X)$-normal subsemigroup of $T(X)$ are inner: this result was first conjectured by Sullivan in the early 1980s.

In 1997, Schwachhöfer and Stroppel proved that if V and W are vector spaces over the fields F and K, respectively, with $\dim V \geq 2$, then every isomorphism between subsemigroups of $T(V)$ and $T(W)$ which contain all linear transformations with rank 1 is induced by a semilinear transformation from V onto W. To do this, they used Baer's First Fundamental Theorem of Projective Geometry which determines the order-isomorphisms between the posets consisting of all subspaces of two given vector spaces. The result by Schwachhöfer and Stroppel is an analogue for vector spaces of Sullivan's result on the automorphisms of a semigroup of transformations covering the underlying set. Hence it is natural to try to extend their work to any semigroup of linear transformations closed under conjugation by semilinear transformations:

thus, the linear Baer-Levi semigroups become a test case. And, as happened for the set version, it might be worthwhile to first describe the automorphism group of a subsemigroup of $T(V)$ which contains $G(V)$ but does not contain elements with one-dimensional range.

In this chapter, we consider the afore-mentioned problems concerning isomorphisms between certain semigroups of linear transformations. In section 6.1, we extend Sullivan's result on the automorphisms of subsemigroups of $T(X)$ covering X to isomorphisms of semigroups of transformations covering the underlying sets, and we compare this result with the work by Schwachhöfer and Stroppel. Also, we contrast the work by Fitzpatrick and Symons with two simple examples given by Araújo in his Ph. D. thesis concerning a linear version of their work. Although the problem of describing all isomorphisms between linear Baer-Levi semigroups is as yet unsolved, in section 6.2 we present some results in that direction. And finally, in section 6.3, we extend Baer's First Fundamental Theorem of Projective Geometry to certain posets of subspaces which may be important for a complete solution of this problem.

6.1. Some conjectures

According to Sullivan [40], in the previous three decades many papers were written describing the automorphism group of several semigroups of transformations defined on a set; and, in all cases studied, the automorphisms were shown to be "inner". In [40] Theorems 1 and 2, the author gives some general results from which all previous work can be derived (as described in [40] section 3). Here, we extend some of his results to isomorphisms between semigroups of partial transformations defined on sets. To do this, we closely follow the arguments in [40].

We let X and Y be arbitrary non-empty sets throughout this section. As in [40], we let A_x denote the element of $P(X)$ with domain A and range $\{x\}$ and call it a *constant based on* x; a subsemigroup S of $P(X)$ is said to *cover* X if for every $x \in X$, there exists some constant idempotent in S based on x, and S is said to be $G(X)$-*normal* if for every $\alpha \in S$ and every $g \in G(X)$, $g^{-1}\alpha g \in S$. In [40] Theorem 1, Sullivan shows that every automorphism of a semigroup S of partial transformations on X is inner if S covers X; and in [40] Theorem 2, he proves that if S is also $G(X)$-

normal then the automorphism group of S is isomorphic to $G(X)$. We now extend [40] Theorem 1 to isomorphisms. To do this, as in [40], we let $K(S)$ denote the set of all constant mappings in S with $\emptyset$ adjoined. Clearly, $K(S)$ is a semigroup and the following holds.

Lemma 6.1.1. Let S be a subsemigroup of $P(X)$ covering X and S' a subsemigroup of $P(Y)$ covering Y. If $\phi : S \to S'$ is an isomorphism, then $\phi|K(S)$ is an isomorphism from $K(S)$ onto $K(S')$.

Proof. First we show that any constant idempotent of S is mapped to a constant idempotent of S' by ϕ. To do this, let $A_x \in K(S)$ be an idempotent. Then $A_x^2 = A_x \neq \emptyset$ implies $x \in A$. Let $\beta = A_x\phi$. Since A_x is an idempotent, so is β. Given $y \in \text{ran } \beta$, there exists some $z \in Y$ such that $y = z\beta$ and so $y\beta = (z\beta)\beta = z\beta^2 = z\beta = y$. Since S' covers Y, there exists some constant idempotent B_y in S' based on y. Note that, as before, $y \in B$. Since ϕ is onto, there exists $\gamma \in S$ such that $\gamma\phi = B_y$. We have $\gamma^2\phi = B_y^2 = B_y = \gamma\phi$ and the injectivity of ϕ implies $\gamma^2 = \gamma$. Also, $(\gamma A_x)\phi = (\gamma\phi)(A_x\phi) = B_y\beta$ and since $y\beta = y$, it follows that $(\gamma A_x)\phi = B_y = \gamma\phi$. Therefore, $\gamma A_x = \gamma$. Hence $\gamma = C_x$ for some $C \subseteq X$ with $x \in C$. Thus $\beta = A_x\phi = (A_xC_x)\phi = (A_x\phi)(C_x\phi) = \beta B_y$, so $\beta = D_y$ for some $D \subseteq Y$ with $y \in D$. In other words, we have shown that ϕ maps a constant idempotent of S to a constant idempotent of S'.

Since ϕ is an isomorphism of S onto S' we have $\emptyset\phi = \emptyset$. Now consider $A_x \in K(S)$ with $x \notin A$. Since S covers X, there exists some constant idempotent in S based on x and so there exists $B_x \in K(S)$ with $x \in B$. From above, $B_x\phi = C_y$ for some $C \subseteq Y$ with $y \in C$. Let $A_x\phi = \beta$. Then $\beta^2 = A_x^2\phi = \emptyset\phi = \emptyset$. We also have $\beta = A_x\phi = (A_xB_x)\phi = \beta C_y$. Therefore, $\beta = D_y$ for some $D \subseteq Y$ where $y \notin D$. Hence $A_x\phi = D_y$ with $y \notin D$.

Therefore ϕ maps $K(S)$ into $K(S')$. Since ϕ is an isomorphism of S onto S', ϕ^{-1} is also an isomorphism (of S' onto S). Hence ϕ^{-1} maps $K(S')$ into $K(S)$, so $\phi|K(S)$ is onto and the result follows. □

Next we show that every isomorphism between semigroups of partial transformations covering the underlying sets is induced by a bijection between such sets. To do this,

we need a preliminary Lemma.

Lemma 6.1.2. If S is a subsemigroup of $P(X)$ covering X and S' is a subsemigroup of $P(Y)$ covering Y, then every isomorphism of $K(S)$ onto $K(S')$ is induced by a bijection $f : X \to Y$.

Proof. Let $\varphi : K(S) \to K(S')$ be an isomorphism. For every $x \in X$, there exists a constant idempotent in S based on x (since S covers X). Define $f : X \to Y$ by

$$xf = y \text{ if and only if } A_x\varphi = B_y,$$

where A_x and B_y are constant idempotents based on x and y, respectively. We assert that f is a bijection from X onto Y. For suppose $A_x\varphi = B_y$. If there exists $C_x \in K(S)$ with $x \in C$ and $C_x\varphi = D_z$, then $B_y = A_x\varphi = (A_xC_x)\varphi = B_yD_z$, hence $y = z$. Therefore f is a well defined mapping from X to Y. Also, if $xf = y = zf$ then $A_x\varphi = B_y$ and $C_z\varphi = D_y$ where $x \in A$ and $z \in C$. Thus, $B_y^2 = A_x^2\varphi = A_x\varphi = B_y$ and so $y \in B$. Similarly we have $D_y^2 = D_y$, hence $y \in D$. Therefore $A_x\varphi = B_y = B_yD_y = (A_xC_z)\varphi$ and this implies $A_x = A_xC_z$. Thus $x = z$ and f is one-to-one. Finally, let $y \in Y$ and $B_y \in K(S')$ where $y \in B$. Since φ is onto, there exists $D_z \in K(S)$ such that $D_z\varphi = B_y$. Then $D_z^2\varphi = B_y^2 = B_y = D_z\varphi$ and so $D_z^2 = D_z$. Hence $z \in D$ and $zf = y$. Therefore f is onto.

Next, we show that f induces φ: that is, $A_x\varphi = f^{-1}A_xf$ for every $A_x \in K(S)$. If $A_x\varphi = B_y$ and $a \in A$, then there exists $C_a \in K(S)$ with $a \in C$ and we have $C_a\varphi = D_{af}$ where $af \in D$. Hence $\emptyset \neq (C_aA_x)\varphi = D_{af}B_y$ and this implies $af \in B$. Thus $Af \subseteq B$. On the other hand, if $b \in B$ then there exists $C_b \in K(S')$ with $b \in C$ and, since φ is onto, there exists $D_z \in K(S)$ with $z \in D$ such that $D_z\varphi = C_b$. Therefore $zf = b$. Since $b \in B$, we have $C_bB_y = C_y$ and so $(D_zA_x)\varphi = D_z\varphi A_x\varphi = C_bB_y = C_y \neq \emptyset$. Hence $D_zA_x \neq \emptyset$ and $z \in A$. Thus $B \subseteq Af$. Then $A_x\varphi = B_y = (Af)_{xf} = f^{-1}A_xf$. In other words, we have shown that the isomorphism φ is induced by the bijection f. □

We note that, from the two Lemmas above, it follows that if S is a subsemigroup of $P(X)$ covering X and S' is a subsemigroup of $P(Y)$ covering Y and, if S and S' are isomorphic, then $|X| = |Y|$. The next result describes all isomorphisms between S and S'.

Theorem 6.1.3. Let S be a subsemigroup of $P(X)$ covering X and S' a subsemigroup of $P(Y)$ covering Y. If $\phi : S \to S'$ is an isomorphism from S onto S', then ϕ is induced by a bijection $f : X \to Y$.

Proof. By Lemma 6.1.1, we know $\phi|K(S)$ is an isomorphism from $K(S)$ onto $K(S')$; and by Lemma 6.1.2, $\phi|K(S)$ is induced by a bijection $f : X \to Y$: that is, $A_x\phi = f^{-1}A_x f$ for every $A_x \in K(S)$.

Now let $\alpha \in S$. We assert that $\mathrm{dom}\,(\alpha\phi) = \mathrm{dom}\,(f^{-1}\alpha f)$. To see this, let $y \in \mathrm{dom}\,(\alpha\phi)$. Since S' covers Y, there exists $B_y \in K(S')$ with $y \in B$. Also, there exists $A_x \in K(S)$ such that $A_x\phi = B_y$ (since $\phi|K(S)$ is onto). By the injectivity of ϕ we know A_x is an idempotent, so $x \in A$. Therefore, $B_y = A_x\phi = f^{-1}A_x f$ and

$$\begin{aligned} B_y(\alpha\phi) =& (A_x\phi)(\alpha\phi) = (A_x\alpha)\phi = A_{x\alpha}\phi \\ &= f^{-1}A_{x\alpha}f = (f^{-1}A_x f)(f^{-1}\alpha f) = B_y(f^{-1}\alpha f). \end{aligned} \tag{6.1.1}$$

Hence $y \in \mathrm{dom}\,(f^{-1}\alpha f)$ and $\mathrm{dom}\,(\alpha\phi) \subseteq \mathrm{dom}\,(f^{-1}\alpha f)$. Conversely, suppose $y \in \mathrm{dom}\,(f^{-1}\alpha f)$. Then, $yf^{-1}\alpha f = z$ for some $z \in Y$, so $yf^{-1}\alpha = zf^{-1}$. Since S covers X, there exists $A_{yf^{-1}} \in K(S)$ with $yf^{-1} \in A$. We then have $A_{yf^{-1}}\alpha \in S$ and, since $yf^{-1} \in \mathrm{dom}\,\alpha$, this mapping is non-empty. Moreover, $A_{yf^{-1}}\alpha = A_{yf^{-1}\alpha} = A_{zf^{-1}}$. We then obtain

$$A_{zf^{-1}}\phi = (A_{yf^{-1}}\alpha)\phi = (A_{yf^{-1}}\phi)(\alpha\phi) = (f^{-1}A_{yf^{-1}}f)(\alpha\phi) = f^{-1}A_y(\alpha\phi).$$

Therefore $f^{-1}A_y(\alpha\phi) \neq \emptyset$ and $y \in \mathrm{dom}\,(\alpha\phi)$. Hence $\mathrm{dom}\,(f^{-1}\alpha f) \subseteq \mathrm{dom}\,(\alpha\phi)$ and the equality follows. From (6.1.1), for every $y \in \mathrm{dom}\,(\alpha\phi)$, we have $B_y\alpha\phi = B_y f^{-1}\alpha f$, that is, $B_{y\alpha\phi} = B_{yf^{-1}\alpha f}$. Hence $y\alpha\phi = yf^{-1}\alpha f$ for every $y \in \mathrm{dom}\,(\alpha\phi)$, and $\alpha\phi = f^{-1}\alpha f$. Thus, $\alpha\phi = f^{-1}\alpha f$ for every $\alpha \in S$: that is, the isomorphism ϕ is induced by f. □

Although it seems the authors did not know the earlier work by Sullivan, in [38] Schwachhöfer and Stroppel considered a linear version of this question; and they proved an analogue of Theorem 6.1.3 for vector spaces in [38] Theorem 6.2, which we quote below for convenience. To do this, they used a very different approach and more complicated argument, where Baer's First Fundamental Theorem of Projective Geometry (see Theorem 6.3.1 in section 6.3) was a crucial tool.

Let V and W be vector spaces over fields F and K, respectively. A *semilinear transformation from V onto W* is a bijection $f : V \to W$ such that for some isomorphism of fields $f^* : F \to K$ and for every $u, v \in V$ and $x \in F$, we have

$$(u+v)f = uf + vf, \qquad (xu)f = (xf^*)(uf).$$

Clearly, if $F = K$ then every bijective linear transformation from V onto W is a semilinear transformation where $f^* = \mathrm{id}_F$.

Theorem 6.1.4. Let V and W be vector spaces over fields F and K, respectively, with $\dim V \geq 2$. If S and S' are subsemigroups of $T(V)$ and $T(W)$, respectively, containing all linear transformations with rank 1, then every isomorphism from S onto S' is induced by a semilinear transformation from V onto W.

Despite the unquestionable interest of this result, a more general one had already been proved four decades before: in [8], Gluskin considered the problem of describing all isomorphisms between certain semigroups of linear transformations, and he proved a result (that we quote below for convenience) from which Schwachhöfer and Stroppel's Theorem follows directly.

Given a vector space V over a field F, let $K(V)$ denote the subsemigroup of $T(V)$ which consists of zero and all linear transformations of V into itself with rank 1. In [8] section 2, Gluskin studied the structure of $K(V)$ and some of its subsemigroups: namely, he considered certain right ideals of $K(V)$ as we proceed to describe.

Let V^* denote the *conjugate space of V*: that is, V^* consists of all linear transformations of V into the one-dimensional vector space F. We associate an index k to each one-dimensional subspace V_k of V and we let Λ_V denote the set of all indices k. Likewise, we associate an index i to each one-dimensional subspace V_i^* of V^* and denote the set of all indices i by I_V. For every $k \in \Lambda_V$, fix a non-zero element a_k in V_k. Similarly, fix $f_i \in V_i^* \setminus \{0\}$ for every $i \in I_V$. In [8] Theorem 2.2, Gluskin showed that $K(V)$ is isomorphic to the Rees matrix semigroup $\mathcal{M}^0(G; I_V, \Lambda_V; [a_k f_i])$, where G is the group $(F, \cdot)$ with zero adjoined. Thus, for every non-empty subset I of I_V, the semigroup $R(I) = \mathcal{M}^0(G; I, \Lambda_V; [a_k f_i])$ is a non-zero right ideal of $K(V)$ and conversely, every non-zero right ideal of $K(V)$ is of this form for some $I \subseteq I_V$.

Now, let IF be the set of all linear forms $f_i x$, where $i \in I$ and $x \in F$, and let $V^*(I)$ denote the subspace of V^* generated by IF. Each right ideal $R(I)$ of $K(V)$ which is completely 0-simple and such that $V^*(I) = IF$ is denoted by $K(V, V^*(I))$. It is not difficult to see that $K(V)$ is of this type and, as stated by Gluskin in [8] p. 114, $K(V)$ is the unique semigroup of type $K(V, V^*(I))$ if and only if V is finite-dimensional. In [8] Theorem 3.5, the author proved the following.

Theorem 6.1.5. Let V and W be vector spaces over the fields F and K, respectively, with $\dim V \geq 2$. If $I \subseteq I_V \cap I_W$ and if S and S' are subsemigroups of $T(V)$ and $T(W)$ containing $K(V, V^*(I))$ and $K(W, W^*(I))$, respectively, as two-sided ideals, then every isomorphism ϕ of S onto S' is induced by a semilinear transformation of V onto W; moreover, $K(V, V^*(I))\phi = K(W, W^*(I))$.

According to Sullivan [44], the aim of [40] Theorems 1 and 2 was to "provide a simple test for a transformation semigroup to have all its automorphisms inner and to have the symmetric group as its automorphism group: namely, it must contain 'sufficient' constant maps and be 'normal' in the usual group-theoretic sense". However, as observed by Sullivan in [44] p. 213, it may be possible for some transformation semigroups without constant maps (and which are not permutation groups) to have all its automorphisms inner; and in fact, in [6], Fitzpatrick and Symons proved that if X is infinite and S is a subsemigroup of $T(X)$ containing $G(X)$, then every automorphism of S is inner. In fact, they first showed a weaker result: they proved that every automorphism of $G(X)$ can be extended to at most one automorphism of S.

The simple analogue of this result for vector spaces would be as follows: if V is an infinite-dimensional vector space over a field F and S is a subsemigroup of $T(V)$ which contains $G(V)$, then every automorphism of $G(V)$ can be extended to at most one automorphism of S. However, as observed by Araújo in [1] pp. 57-58, this result does not hold: in [1] Lemmas 21 and 39, Araújo gives two simple examples of subsemigroups A_1 and A_2 of $T(V)$ which contain $G(V)$, but where the identity in $G(V)$ can be extended in infinitely many ways to automorphisms of A_1 and A_2, respectively. Below, we quote [1] Lemmas 21 and 39 for convenience and we include brief proofs for completeness.

Lemma 6.1.6. Let V be an infinite-dimensional vector space over the field of real numbers and let

$$A_1 = \{\alpha \in T(V) : n(\alpha) < \aleph_0,\ d(\alpha) = 0\}.$$

For every non-zero real number r, define $\phi_r : A_1 \to A_1$ as follows: given $\alpha \in A_1$, let $\alpha\phi_r = r^{n(\alpha)}\alpha$. Then ϕ_r is an automorphism of A_1 and $\phi_r|G(V) = \mathrm{id}_{G(V)}$.

Proof. First, we note that if $\alpha \in T(V)$ and k is a non-zero real number, then $\ker(k\alpha) = \ker\alpha$ and $\mathrm{ran}(k\alpha) = \mathrm{ran}\,\alpha$. Thus, ϕ_r is a well-defined mapping from A_1 to A_1. By chapter 3, Lemma 3.1.1(b), we know $n(\alpha\beta) = n(\alpha) + n(\beta)$ for every $\alpha, \beta \in A_1$. It is now easy to see that ϕ_r is a morphism: given $\alpha, \beta \in A_1$, we have

$$(\alpha\beta)\phi_r = r^{n(\alpha\beta)}(\alpha\beta) = r^{n(\alpha)}r^{n(\beta)}(\alpha\beta) = (r^{n(\alpha)}\alpha)(r^{n(\beta)}\beta) = \alpha\phi_r\beta\phi_r.$$

To see that ϕ_r is injective, let $\alpha, \beta \in A_1$ be such that $\alpha\phi_r = \beta\phi_r$. Since $\ker\gamma = \ker(\gamma\phi_r)$, for every $\gamma \in A_1$, it follows that $\ker\alpha = \ker(\alpha\phi_r) = \ker(\beta\phi_r) = \ker\beta$. In particular $n(\alpha) = n(\beta)$, and $r^{n(\alpha)}\alpha = \alpha\phi_r = \beta\phi_r = r^{n(\beta)}\beta = r^{n(\alpha)}\beta$ implies $\alpha = \beta$. Therefore, ϕ_r is one-to-one. Finally, we show ϕ_r is onto. Given $\alpha \in A_1$, we know that $n(\alpha) < \aleph_0$ and so $r^{n(\alpha)}$ is a non-zero real number. Thus, so is $r^{-n(\alpha)}$ and hence $r^{-n(\alpha)}\alpha \in A_1$. Since $\ker(r^{-n(\alpha)}\alpha) = \ker\alpha$, it follows that

$$[r^{-n(\alpha)}\alpha]\phi_r = r^{n(r^{-n(\alpha)}\alpha)}r^{-n(\alpha)}\alpha = r^{n(\alpha)}r^{-n(\alpha)}\alpha = \alpha.$$

Hence, ϕ_r is an automorphism of A_1 for every non-zero real number r and clearly, $\phi_r|G(V) = \mathrm{id}_{G(V)}$. □

Lemma 6.1.7. Let V be an infinite-dimensional vector space over the field of real numbers and let

$$A_2 = \{\alpha \in T(V) : n(\alpha) = 0,\ d(\alpha) < \aleph_0\}.$$

For every non-zero real number r, define $\theta_r : A_2 \to A_2$ as follows: given $\alpha \in A_2$, let $\alpha\theta_r = r^{d(\alpha)}\alpha$. Then θ_r is an automorphism of A_2 and $\theta_r|G(V) = \mathrm{id}_{G(V)}$.

Proof. As before, it is easy to see that θ_r is well-defined. Now suppose $\alpha, \beta \in A_2$. From chapter 3, Lemma 3.1.1(a), it follows that $d(\alpha\beta) = d(\alpha) + d(\beta)$. Therefore,

$$(\alpha\beta)\theta_r = r^{d(\alpha\beta)}(\alpha\beta) = r^{d(\alpha)}r^{d(\beta)}(\alpha\beta) = (r^{d(\alpha)}\alpha)(r^{d(\beta)}\beta) = \alpha\theta_r\beta\theta_r.$$

Hence, θ_r is a morphism. Since $\operatorname{ran}\gamma = \operatorname{ran}(k\gamma)$ for every $\gamma \in A_2$ and every non-zero real number k, we have $\operatorname{ran}\gamma = \operatorname{ran}(\gamma\theta_r)$ for every $\gamma \in A_2$, so $d(\gamma) = d(\gamma\theta_r)$. Therefore, if $\alpha, \beta \in A_2$ are such that $\alpha\theta_r = \beta\theta_r$, then $d(\alpha) = d(\alpha\theta_r) = d(\beta\theta_r) = d(\beta)$, so $r^{d(\alpha)}\alpha = r^{d(\beta)}\beta$ implies $\alpha = \beta$. Hence, θ_r is injective. To show that θ_r is onto, let $\alpha \in A_2$. Since $d(\alpha)$ is finite, it follows that $r^{d(\alpha)}$ is a non-zero real number and so is $r^{-d(\alpha)}$. Thus, $r^{-d(\alpha)}\alpha \in A_2$ and

$$(r^{-d(\alpha)}\alpha)\theta_r = r^{d(r^{-d(\alpha)}\alpha)}r^{-d(\alpha)}\alpha = r^{d(\alpha)}r^{-d(\alpha)}\alpha = \alpha.$$

Therefore, θ_r is onto. Hence, θ_r is an automorphism of A_2 and it is not difficult to see that $\theta_r|G(V) = \mathrm{id}_{G(V)}$. □

As observed by Sullivan in [44] p. 213, the Baer-Levi semigroups $BL(p,q)$ are $G(X)$-normal and they contain no constants or permutations, hence "they became a test case for Sullivan's Conjecture in the early 1980s" that every automorphism of a $G(X)$-normal transformation semigroup is inner. In effect, the authors of [24] showed that the automorphisms of a Baer-Levi semigroup are inner; and a few years later, Levi proved Sullivan's Conjecture in [22]. As before, one can ask if the linear version of this question holds: the following results were conjectured by Sullivan (personal communication).

Conjecture 6.1.1. Every automorphism of a linear Baer-Levi semigroup $GS(m,n)$ defined on V is induced by a semilinear transformation of V.

Conjecture 6.1.2. Let S be a subsemigroup of $T(V)$ closed under conjugation by all semilinear transformations of V. Then, every automorphism of S is induced by a semilinear transformation of V.

The problem of determining all automorphisms of a linear Baer-Levi semigroup is still unsolved, but we have several results in that direction: the next section is devoted to them.

6.2. Some results

In [24], the authors proved that every automorphism of a Baer-Levi semigroup is inner; and chapter 2, Theorem 2.1.11 extends this result to isomorphisms between

Baer-Levi semigroups: it is shown that the semigroups $BL(X,p,q)$ and $BL(Y,m,n)$ are isomorphic if and only if $p = m$ and $q = n$, and in this case, every isomorphism $\theta : BL(X,p,q) \to BL(X,m,n)$ is induced by a bijection from X onto Y. To prove this result, we used the same method of proof as in [24]: namely, every isomorphism $\theta : BL(X,p,q) \to BL(Y,m,n)$ gives rise, in a natural way, to an order-isomorphism from $\mathrm{Ran}(BL(X,p,q))$ onto $\mathrm{Ran}(BL(Y,m,n))$. Moreover, each order-isomorphism from $\mathrm{Ran}(BL(X,p,q))$ onto $\mathrm{Ran}(BL(Y,m,n))$ is induced by a bijection from X onto Y, and it is this bijection which induces θ.

By analogy with the work by the authors of [24], the description of all automorphisms of a linear Baer-Levi semigroup $GS(p,q)$, and in a more general setting, the description of all isomorphisms between linear Baer-Levi semigroups $GS(V,p,q)$ and $GS(W,m,n)$, might depend on a variation of Baer's First Fundamental Theorem of Projective Geometry: namely, a description of the isomorphisms between lattices of subspaces of V and W. Next we describe our attempts to determine all isomorphisms between linear Baer-Levi semigroups following such an approach.

Suppose $\phi : GS(V,p,q) \to GS(W,m,n)$ is an isomorphism. From chapter 2, Lemma 2.3.1, it follows that for every $\alpha, \beta \in GS(V,p,q)$,

$$\mathrm{ran}\ \alpha \subseteq \mathrm{ran}\ \beta \quad \text{if and only if} \quad \mathrm{ran}\,(\alpha\phi) \subseteq \mathrm{ran}\,(\beta\phi). \tag{6.2.1}$$

Let $\mathcal{G}(V,p,q)$ denote the family of all subspaces A of V such that $\dim A = p$ and $\mathrm{codim}\ A = q$; similarly, let $\mathcal{G}(W,m,n)$ denote the family of all subspaces B of W such that $\dim B = m$ and $\mathrm{codim}\ B = n$. We observe that, as in chapter 2, section 2.1, (6.2.1) can be used to show that ϕ determines (in a natural way) an order-isomorphism ψ from $\mathcal{G}(V,p,q)$ onto $\mathcal{G}(W,m,n)$: namely, for each $A \in \mathcal{G}(V,p,q)$, let $A\psi = \mathrm{ran}\,(\alpha\phi)$ for some $\alpha \in GS(V,p,q)$ such that $\mathrm{ran}\ \alpha = A$. It is easy to see that ψ preserves meets and joins when they exist in $\mathcal{G}(V,p,q)$. However, we are not able to show that ψ is induced by a semilinear transformation; and we suspect this is due to the fact that the lattices of subspaces of V and W being considered do not contain the one-dimensional subspaces of V and W, respectively (for the reason why we think one-dimensional subspaces are needed, see our extension of Baer's Theorem in section 6.3).

Now let $\mathcal{C}(V,q)$ be the family of all subspaces A of V such that codim $A \geq q$ and let $\mathcal{C}(W,n)$ be the family of all subspaces B of W such that codim $B \geq n$ (note that both families contain all one-dimensional subspaces of the corresponding vector space).

From chapter 2, Lemma 2.3.5, if $p > q$, we know 'Fix $\alpha \subseteq$ Fix β' can be characterised algebraically for every $\alpha, \beta \in GS(p,q)$ such that $s(\alpha) = q = s(\beta)$. If the same is true for arbitrary α and β in $GS(p,q)$, with $p \geq q$, then any isomorphism $\phi : GS(V,p,q) \to GS(W,m,n)$ gives rise in a natural way to an order-isomorphism from $\mathcal{C}(V,q)$ onto $\mathcal{C}(W,n)$. However, it is unclear what the desired algebraic conditions may be.

Example 6.2.1. For instance, consider the following condition:

$$\text{for all } \lambda \in GS(p,q), \quad \alpha^2\lambda = \alpha \text{ implies } \lambda\beta = \lambda^2. \tag{6.2.2}$$

It is easy to see that if $\alpha, \beta \in GS(p,q)$ satisfy (6.2.2), then Fix $\alpha \subseteq$ Fix β; but the converse does not hold. To see this, let $\alpha \in GS(p,q)$ and suppose $\{a_k\}$ is a basis for Fix α. Expand $\{a_k\}$ to a basis $\{a_k\} \dot\cup \{a_i\}$ for V. Then, $\{a_k\} \dot\cup \{a_i\alpha\}$ is a basis for ran α and it can be extended to another basis for V, say $\{a_k\} \dot\cup \{a_i\alpha\} \dot\cup \{a_j\}$. Note that $|K| + |I| = p$ and $|J| = d(\alpha) = q$. Write $\{a_j\} = \{b_j\} \dot\cup \{c_j\}$ (this is possible since q is infinite). We have

$$\alpha = \begin{pmatrix} a_k & a_i \\ a_k & a_i\alpha \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} a_k & a_i \\ a_k & a_i\alpha^2 \end{pmatrix}.$$

Expand $\{a_k\} \dot\cup \{a_i\alpha^2\}$ to another basis $\{a_k\} \dot\cup \{a_i\alpha^2\} \dot\cup \{d_j\}$ for V (note that $d(\alpha^2) = q = |J|$). Now define $\lambda \in T(V)$ by

$$\lambda = \begin{pmatrix} a_k & a_i\alpha^2 & d_j \\ a_k & a_i\alpha & b_j \end{pmatrix}.$$

Then $\lambda \in GS(V,p,q)$ and $\alpha^2\lambda = \alpha$. Also

$$\lambda^2 = \begin{pmatrix} a_k & a_i\alpha^2 & d_j \\ a_k & a_i\alpha\lambda & b_j\lambda \end{pmatrix}.$$

Expand $\{a_k\} \dot\cup \{a_i\alpha\lambda\} \dot\cup \{b_j\lambda\}$ to a basis for V, say $\{a_k\} \dot\cup \{a_i\alpha\lambda\} \dot\cup \{b_j\lambda\} \dot\cup \{e_j\}$, and write $\{e_j\} = \{f_j\} \dot\cup \{g_j\}$. Define $\beta \in T(V)$ by

$$\beta = \begin{pmatrix} a_k & a_i\alpha & b_j & c_j \\ a_k & a_i\alpha\lambda & f_j & g_j \end{pmatrix}.$$

Then $\beta \in GS(V,p,q)$ and Fix $\alpha \subseteq$ Fix β. However,

$$\lambda\beta = \begin{pmatrix} a_k & a_i\alpha^2 & d_j \\ a_k & a_i\alpha\lambda & f_j \end{pmatrix} \neq \lambda^2.$$

Although the problem of determining all isomorphisms between linear Baer-Levi semigroups is unsolved, it can be reduced to two cases. First, we adapt the argument used for the proof of Lemma 2.3.6 in chapter 2 to obtain the following result.

Lemma 6.2.2. The semigroups $GS(A,p,p)$ and $GS(B,m,n)$ are not isomorphic for any infinite cardinals p,m,n with $m > n$.

Proof. Suppose $m > n$ and $\phi : GS(B,m,n) \to GS(A,p,p)$ is an isomorphism. Since $m > n$ and as before (see chapter 2, before Lemma 2.3.5), we have a subsemigroup of $GS(m,n)$ defined by:

$$\Sigma(n) = \{\alpha \in GS(B,m,n) : s(\alpha) = n\}.$$

Furthermore, we can characterise $\Sigma(n)$ algebraically as follows: given $\alpha \in GS(B,m,n)$,

$$s(\alpha) = n \quad \text{if and only if} \quad \lambda\alpha = \lambda \text{ for some } \lambda \in GS(B,m,n).$$

Let $S = \{\alpha \in GS(A,p,p) : \mu\alpha = \mu \text{ for some } \mu \in GS(A,p,p)\}$. Clearly ϕ induces a one-to-one mapping from $\Sigma(n)$ onto S. Since $\Sigma(n)$ is a subsemigroup of $GS(m,n)$ and ϕ is an isomorphism, it follows that $S = \Sigma(n)\phi$ must be a subsemigroup of $GS(A,p,p)$. We assert that this is not true.

Consider a basis for A of the form $\{a_{1k}\} \cup \{a_{2k}\} \cup \ldots$: that is, $A = \langle a_{ik} \rangle$ where $|I| = \aleph_0$ and $|K| = p$. Let A_1 be the subspace of A generated by $\{a_{2k}\} \cup \{a_{3k}\} \cup \ldots$. Define $\gamma \in T(A_1)$ by

$$\gamma = \begin{pmatrix} a_{2k} & a_{3k} & \ldots & a_{ik} & \ldots \\ a_{3k} & a_{4k} & \ldots & a_{i+1,k} & \ldots \end{pmatrix}.$$

Then $d(\gamma) = \dim\langle a_{2k} \rangle = p$ and so $\gamma \in GS(A_1,p,p)$. Moreover, Fix $\gamma = \{0\}$. Now define $\alpha \in T(A)$ by

$$\alpha = \begin{pmatrix} a_{1k} & a_{2k} & a_{3k} & \ldots & a_{ik} & \ldots \\ a_{1k} & a_{3k} & a_{4k} & \ldots & a_{i+1,k} & \ldots \end{pmatrix}.$$

We have $\alpha \in GS(A,p,p)$ and Fix $\alpha = \langle a_{1k} \rangle$.

Write $\{a_{2k}\}$ as $\{b_{1k}\}\cup\{b_{2k}\}\cup\ldots$: that is, $A_2 = \langle a_{2k}\rangle = \langle b_{ik}\rangle$. Let A_3 be the subspace of A generated by $\{a_{4k}\}\cup\{a_{5k}\}\cup\ldots$. Define $\lambda \in T(A_2)$ and $\mu \in T(A_3)$ by

$$\lambda = \begin{pmatrix} b_{1k} & b_{2k} & \ldots & b_{jk} & \ldots \\ b_{2k} & b_{3k} & \ldots & b_{j+1,k} & \ldots \end{pmatrix} \text{ and } \mu = \begin{pmatrix} a_{4k} & a_{5k} & \ldots & a_{ik} & \ldots \\ a_{5k} & a_{6k} & \ldots & a_{i+1,k} & \ldots \end{pmatrix}.$$

Then $\lambda \in GS(A_2, p, p)$ and Fix $\lambda = \{0\}$; also $\mu \in GS(A_3, p, p)$ and Fix $\mu = \{0\}$. Now define $\beta \in T(A)$ by

$$\beta = \begin{pmatrix} a_{1k} & b_{1k} & b_{2k} & \ldots & b_{jk} & \ldots & a_{3k} & a_{4k} & a_{5k} & \ldots & a_{ik} & \ldots \\ b_{1k} & b_{2k} & b_{3k} & \ldots & b_{j+1,k} & \ldots & a_{3k} & a_{5k} & a_{6k} & \ldots & a_{i+1,k} & \ldots \end{pmatrix}.$$

Since $d(\beta) = \dim\langle a_{1k}, a_{4k}\rangle = p$, we know $\beta \in GS(A, p, p)$. Each $b \in A$ can be written as

$$\begin{aligned} b = &\sum_k z_{1k}a_{1k} + \sum_k y_{j_1,k}b_{j_1,k} + \ldots + \sum_k x_{j_s,k}b_{j_s,k} \\ &+ \sum_k z_{3k}a_{3k} + \sum_k x_{i_1,k}a_{i_1,k} + \ldots + \sum_k x_{i_r,k}a_{i_r,k} \end{aligned} \tag{6.2.3}$$

where $z_{1k}, z_{3k}, y_{j_\ell,k}, x_{i_t,k}$ are scalars, each sum is over a finite (and possibly different) index set and we can assume $j_1 < j_2 < \ldots < j_s$ and $4 \le i_1 < i_2 < \ldots < i_r$. Therefore, if $b\beta = b$, we have:

$$\begin{aligned} &\sum_k z_{1k}a_{1k} + \sum_k y_{j_1,k}b_{j_1,k} + \ldots + \sum_k y_{j_s,k}b_{j_s,k} \\ &\quad + \sum_k z_{3k}a_{3k} + \sum_k x_{i_1,k}a_{i_1,k} + \ldots + \sum_k x_{i_r,k}a_{i_r,k} \\ &\qquad = \sum_k z_{1k}b_{1k} + \sum_k y_{j_1,k}b_{j_1+1,k} + \ldots + \sum_k y_{j_s,k}b_{j_s+1,k} \\ &\qquad\quad + \sum_k z_{3k}a_{3k} + \sum_k x_{i_1,k}a_{i_1+1,k} + \ldots + \sum_k x_{i_r,k}a_{i_r+1,k}. \end{aligned} \tag{6.2.4}$$

Clearly we can cancel $\sum_k z_{3k}a_{3k}$ on each side of this equation. Also, since $a_{1k}, b_{j_\ell,k}, a_{i_t,k}$ are all linearly independent, and a_{1k} does not appear on the right of equation (6.2.4), we may conclude that $z_{1k} = 0$ for all k. Then (6.2.4) reduces to

$$\begin{aligned} &\sum_k y_{j_1,k}b_{j_1,k} + \ldots + \sum_k y_{j_s,k}b_{j_s,k} \\ &\quad + \sum_k x_{i_1,k}a_{i_1,k} + \ldots + \sum_k x_{i_r,k}a_{i_r,k} \\ &\qquad = \sum_k y_{j_1,k}b_{j_1+1,k} + \ldots + \sum_k y_{j_s,k}b_{j_s+1,k} \\ &\qquad\quad + \sum_k x_{i_1,k}a_{i_1+1,k} + \ldots + \sum_k x_{i_r,k}a_{i_r+1,k}. \end{aligned} \tag{6.2.5}$$

Now $b_{j_1,k}$ does not appear on the right of equation (6.2.5) and so $y_{j_1,k} = 0$ for all k. In a similar way we deduce that all coefficients $y_{j_\ell,k}$ equal 0 and so (6.2.5) reduces to

$$\sum_k x_{i_1,k}a_{i_1,k} + \ldots + \sum_k x_{i_r,k}a_{i_r,k} = \sum_k x_{i_1,k}a_{i_1+1,k} + \ldots + \sum_k x_{i_r,k}a_{i_r+1,k}. \quad (6.2.6)$$

Therefore $\sum_k x_{i_1,k}a_{i_1,k} + \ldots + \sum_k x_{i_r,k}a_{i_r,k} \in \text{Fix } \mu = \{0\}$ and it follows that $b = \sum_k z_{3k}a_{3k}$. Hence Fix $\beta = \langle a_{3k} \rangle$.

We have

$$\alpha\beta = \begin{pmatrix} a_{1k} & a_{2k} & a_{3k} & a_{4k} & \ldots & a_{ik} & \ldots \\ b_{1k} & a_{3k} & a_{5k} & a_{6k} & \ldots & a_{i+2,k} & \ldots \end{pmatrix}$$

in $GS(A, p, p)$. Moreover, each $a \in A$ can be written as

$$a = \sum_k x_{1k}a_{1k} + \sum_k x_{2k}a_{2k} + \sum_k y_{i_1,k}a_{i_1,k} + \ldots + \sum_k y_{i_t,k}a_{i_t,k} \quad (6.2.7)$$

where the $x_{1k}, x_{2k}, y_{i_j,k}$ are scalars, each sum is over a finite (and possibly different) index set and we can assume $3 \leq i_1 < i_2 < \ldots < i_t$. If $a(\alpha\beta) = a$, we have:

$$\begin{aligned} &\sum_k x_{1k}a_{1k} + \sum_k x_{2k}a_{2k} + \sum_k y_{i_1,k}a_{i_1,k} + \ldots + \sum_k y_{i_t,k}a_{i_t,k} \\ &\quad = \sum_k x_{1k}b_{1k} + \sum_k x_{2k}a_{3k} + \sum_k y_{i_1,k}a_{i_1+2,k} + \ldots + \sum_k y_{i_t,k}a_{i_t+2,k}. \end{aligned} \quad (6.2.8)$$

Since $a_{1k}, a_{2k}, a_{3k}, a_{i_j,k}$ (with $i_j \neq 3$) are all linearly independent, and a_{1k} does not appear on the right of this equation, we deduce that $x_{1k} = 0$ for all k. Then (6.2.8) reduces to

$$\begin{aligned} &\sum_k x_{2k}a_{2k} + \sum_k y_{i_1,k}a_{i_1,k} + \ldots + \sum_k y_{i_t,k}a_{i_t,k} \\ &\quad = \sum_k x_{2k}a_{3k} + \sum_k y_{i_1,k}a_{i_1+2,k} \ldots + \sum_k y_{i_t,k}a_{i_t+2,k}. \end{aligned} \quad (6.2.9)$$

Again, a_{2k} appears nowhere on the right of this new equation, so $x_{2k} = 0$ for all k. Hence (6.2.8) reduces to

$$\sum_k y_{i_1,k}a_{i_1,k} + \ldots + \sum_k y_{i_t,k}a_{i_t,k} = \sum_k y_{i_1,k}a_{i_1+2,k} \ldots + \sum_k y_{i_t,k}a_{i_t+2,k}. \quad (6.2.10)$$

Since $a_{i_1,k}$ does not appear on the right of (6.2.10), it follows that $y_{i_1,k} = 0$ for all k and we then have

$$\sum_k y_{i_2,k}a_{i_2,k} + \ldots + \sum_k y_{i_t,k}a_{i_t,k} = \sum_k y_{i_2,k}a_{i_2+2,k} \ldots + \sum_k y_{i_t,k}a_{i_t+2,k}. \quad (6.2.11)$$

In like manner, all coefficients $y_{i_j,k}$ equal 0, hence $a = 0$. Therefore $\mathrm{Fix}\,(\alpha\beta) = \{0\}$.

Let $\{e_k\}$ be a basis for A and define $\delta, \varepsilon \in GS(A, p, p)$ by

$$\delta = \begin{pmatrix} e_k \\ a_{1k} \end{pmatrix} \quad \text{and} \quad \varepsilon = \begin{pmatrix} e_k \\ a_{3k} \end{pmatrix}.$$

Then $\delta\alpha = \delta$ and $\varepsilon\beta = \varepsilon$. Hence both α and β are elements of the set S.

Suppose $\alpha\beta \in S$. Then there exists $\pi \in GS(A, p, p)$ such that $\pi(\alpha\beta) = \pi$ and so $\mathrm{ran}\,\pi \subseteq \mathrm{Fix}\,(\alpha\beta) = \{0\}$, a contradiction. Hence S is not a subsemigroup of $GS(A, p, p)$ and the result follows. □

The next result is due to Sullivan (personal communication).

Lemma 6.2.3. Suppose $\phi : GS(V, p, q) \to GS(W, m, n)$ is an isomorphism where $p \geq q$ and $m \geq n$. Then $q = n$.

Proof. As before, ϕ gives rise in a natural way to a mapping ψ from $\mathcal{G}(V, p, q)$ into $\mathcal{G}(W, m, n)$ (where $\mathcal{G}(V, p, q)$ and $\mathcal{G}(W, m, n)$ are as defined before Example 6.2.1): for each $A \in \mathcal{G}(V, p, q)$, let $A\psi = \mathrm{ran}\,(\alpha\phi)$ for some $\alpha \in GS(V, p, q)$ such that $\mathrm{ran}\,\alpha = A$. Now choose $\alpha \in GS(V, p, q)$ and write $V = \mathrm{ran}\,\alpha \oplus \langle\{u_i\}\rangle$ where $|I| = q$ (that is, in what follows, $\langle u_i\rangle$ will denote the subspace generated by the single vector u_i). Now, for each $v \notin \mathrm{ran}\,\alpha$, there exists $w \notin \mathrm{ran}\,(\alpha\phi)$ such that

$$(\mathrm{ran}\,\alpha \oplus \langle v\rangle)\psi = \mathrm{ran}\,(\alpha\phi) \oplus \langle w\rangle.$$

In particular, this is true for $v = u_i$ and we let w_i be the corresponding element of W. We assert that $\{w_i\}$ is linearly independent. For, suppose $w_1 = x_2 w_2 + \ldots + x_r w_r$ for some $1, \ldots, r \in I$ and scalars x_j. Note that, in general, since ψ is an order-isomorphism, $\mathrm{ran}\,\alpha + \langle u\rangle \subseteq \mathrm{ran}\,\alpha + \langle u\rangle + \langle v\rangle$ implies

$$(\mathrm{ran}\,\alpha + \langle u\rangle)\psi + (\mathrm{ran}\,\alpha + \langle v\rangle)\psi \subseteq (\mathrm{ran}\,\alpha + \langle u\rangle + \langle v\rangle)\psi.$$

Therefore, since $\mathrm{ran}\,\alpha + \langle u_2\rangle + \ldots + \langle u_r\rangle \in \mathcal{G}(V, p, q)$, we have:

$$\begin{aligned}
(\mathrm{ran}\,\alpha + \langle u_1\rangle)\psi &= \mathrm{ran}\,(\alpha\phi) + \langle w_1\rangle \\
&\subseteq \mathrm{ran}\,(\alpha\phi) + \langle w_2\rangle + \ldots + \langle w_r\rangle \\
&= \mathrm{ran}\,(\alpha\phi) + \langle w_2\rangle + \ldots + \mathrm{ran}\,(\alpha\phi) + \langle w_r\rangle \\
&= (\mathrm{ran}\,\alpha + \langle u_2\rangle)\psi + \ldots + (\mathrm{ran}\,\alpha + \langle u_r\rangle)\psi \\
&\subseteq (\mathrm{ran}\,\alpha + \langle u_2\rangle + \ldots + \langle u_r\rangle)\psi.
\end{aligned}$$

It follows that $\langle u_1 \rangle \subseteq \text{ran } \alpha + \langle u_2 \rangle + \ldots + \langle u_r \rangle$, contradicting the choice of the u_i, and this proves the assertion. In other words, $\dim\langle\{w_i\}\rangle = q$.

Now we show that $\text{ran}\,(\alpha\phi) \cap \langle\{w_i\}\rangle = \{0\}$. Suppose $1, \ldots, r \in I$ and $x_1 w_1 + \ldots + x_r w_r \in \text{ran}\,(\alpha\phi)$ for some scalars x_j where $x_1 \neq 0$. Then we have $w_1 \in \text{ran}\,(\alpha\phi) + \langle w_2 \rangle + \ldots + \text{ran}\,(\alpha\phi) + \langle w_r \rangle$, and hence

$$(\text{ran } \alpha + \langle u_1 \rangle)\psi \subseteq (\text{ran } \alpha + \langle u_2 \rangle + \ldots + \langle u_r \rangle)\psi,$$

which is a contradiction as before. From this, we deduce that $d(\alpha\phi) = n \geq q$ and, using ϕ^{-1}, we obtain $n \leq q$. □

As observed before, by Lemmas 6.2.2 and 6.2.3 we have reduced the isomorphism problem to two cases:

(1) $\phi : GS(V, p, p) \to GS(V, p, p)$ and

(2) $\phi : GS(V, p, q) \to GS(W, m, q)$ where $p > q$ and $m > q$.

6.3. Extension of Baer's Theorem

In [3] chapter 3, section 1, Baer proves the First Fundamental Theorem of Projective Geometry, which we quote below for convenience. Throughout this section, V and W are vector spaces over fields F and K, respectively, and $\dim V \geq 3$ over F.

Theorem 6.3.1. Every order-isomorphism between the posets consisting of all subspaces of V and W, respectively, is induced by a semilinear transformation from V onto W.

As before, for every $q \leq \dim V$, let $\mathcal{C}(V, q)$ be the family of all subspaces A of V such that $\text{codim } A \geq q$ and, for every $n \leq \dim W$, let $\mathcal{C}(W, n)$ be the family of all subspaces B of W such that $\text{codim } B \geq n$. In this section, we extend Baer's result to order-isomorphisms between $\mathcal{C}(V, q)$ and $\mathcal{C}(W, n)$. To do this, we closely follow the arguments in [3] chapter 3, section 1.

In essence, the next result is [3] chapter 3, section 1, pp. 40–41, properties (e)–(f) and Proposition 1, but in a more general setting (since Baer assumes an order-isomorphism between the poset of *all* subspaces of V and W, respectively, and we assume less).

Lemma 6.3.2. Suppose $\theta : \mathcal{C}(V,q) \to \mathcal{C}(W,n)$ is an order-isomorphism. Then

(a) $\{0_V\}\theta = \{0_W\}$;

(b) θ maps one-dimensional subspaces of V onto one-dimensional subspaces of W;

(c) θ preserves intersection in $\mathcal{C}(V,q)$;

(d) θ preserves the sum of subspaces of V in $\mathcal{C}(V,q)$ when it belongs to $\mathcal{C}(V,q)$;

(e) θ preserves rank.

Proof. Since $\operatorname{codim}\{0_V\} = p \geq q$, we have $\{0_V\}$ in $\mathcal{C}(V,q)$ and so there exists B in $\mathcal{C}(W,n)$ such that $\{0_V\}\theta = B$. On the other hand, since $\{0_W\} \in \mathcal{C}(W,n)$ and θ is onto, there exists A in $\mathcal{C}(V,q)$ such that $A\theta = \{0_W\}$. We then have $\{0_V\} \subseteq A$ and $\{0_W\} \subseteq B$. Since θ preserves order, it follows that $\{0_V\}\theta \subseteq A\theta$. Hence $A\theta = \{0_W\} \subseteq B = \{0_V\}\theta \subseteq A\theta$ and equality follows.

To see if condition (b) holds, we recall that a subspace A of V is one-dimensional if and only if $A \neq \{0\}$ and for every subspace B of V, $\{0\} \subsetneq B \subseteq A$ implies $A = B$. Let A be a subspace of V with dimension one. Then $\operatorname{codim} A = p \geq q$ and so $A \in \mathcal{C}(V,q)$. Thus, $A\theta \in \mathcal{C}(W,n)$ and, since $A \neq \{0\}$, we have $A\theta \neq \{0\}$. Suppose C is a subspace of W such that $\{0\} \subsetneq C \subseteq A\theta$. Since $A\theta \in \mathcal{C}(W,n)$, we have $\operatorname{codim} C \geq \operatorname{codim} A\theta \geq n$ and so $C \in \mathcal{C}(W,n)$. Therefore there exists B in $\mathcal{C}(V,q)$ such that $B\theta = C$. Since θ is an order-isomorphism and $\{0\}\theta = \{0\} \subsetneq B\theta \subseteq A\theta$, it follows that $\{0\} \subsetneq B \subseteq A$ and so $B = A$. Hence, $C = B\theta = A\theta$. Thus $A\theta$ has dimension one.

We now let A, B be subspaces of V with codimension $\geq q$. Then $\operatorname{codim}(A \cap B) \geq \operatorname{codim} A \geq q$, so $A \cap B \in \mathcal{C}(V,q)$. Since θ preserves order and $A \cap B \subseteq A, B$, we then have $(A \cap B)\theta \subseteq A\theta, B\theta$, hence $(A \cap B)\theta \subseteq A\theta \cap B\theta$. On the other hand, since $A\theta \cap B\theta \in \mathcal{C}(W,n)$, there exists $C \in \mathcal{C}(V,q)$ such that $C\theta = A\theta \cap B\theta$. Therefore $C\theta \subseteq A\theta$ and $C\theta \subseteq B\theta$, and these imply $C \subseteq A$ and $C \subseteq B$. Hence $C \subseteq A \cap B$ and so $A\theta \cap B\theta = C\theta \subseteq (A \cap B)\theta$. Thus $(A \cap B)\theta = A\theta \cap B\theta$.

If $\{A_i\}$ is a family of subspaces of V, then the sum $\sum A_i$ is the subspace of V generated by $\cup A_i$ (see [21] chapter 2, section 7, p. 54). Hence, if $A_i \in \mathcal{C}(V,q)$ for each i and $\sum A_i \in \mathcal{C}(V,q)$, then $A_i \subseteq \sum A_i$ implies $A_i\theta \subseteq (\sum A_i)\theta$ for every i. Therefore,

$\sum(A_i\theta) \subseteq (\sum A_i)\theta$. On the other hand, since codim $\sum(A_i\theta) \geq$ codim $(\sum A_i)\theta \geq n$, we have $\sum(A_i\theta) \in \mathcal{C}(W,n)$, so there exists some D in $\mathcal{C}(V,q)$ such that $D\theta = \sum(A_i\theta)$. Moreover, $A_i\theta \subseteq D\theta$ implies $A_i \subseteq D$ for every i, so $\sum A_i \subseteq D$. Thus, $(\sum A_i)\theta \subseteq D\theta = \sum(A_i\theta)$ and equality follows.

Next let A be a subspace of V with codimension at least q and suppose $\{a_i\}$ is a basis for A. Then $A \in \mathcal{C}(V,q)$, and also $\langle a_i\rangle \in \mathcal{C}(V,q)$ for every i (note that here, and in what follows, $\langle a_i\rangle$ denotes the subspace generated by the single vector a_i). From (b), $\langle a_i\rangle\theta = \langle b_i\rangle$ for some $b_i \in W \setminus \{0\}$. We assert that $\{b_i\}$ is a basis for $A\theta$. Since θ preserves the sum of subspaces of V in $\mathcal{C}(V,q)$ (if the sum belongs to $\mathcal{C}(V,q)$) and $A = \langle\{a_i\}\rangle = \langle\cup\langle a_i\rangle\rangle = \sum\langle a_i\rangle$, we have

$$A\theta = (\sum\langle a_i\rangle)\theta = \sum(\langle a_i\rangle\theta) = \sum\langle b_i\rangle = \langle\{b_i\}\rangle.$$

Suppose $b_j = x_{i_1}b_{i_1} + \cdots + x_{i_r}b_{i_r}$ for some $i_1,\ldots,i_r \in I \setminus \{j\}$ and some scalars $x_{i_1},\ldots,x_{i_r}$. Then $\langle b_j\rangle \subseteq \langle b_{i_1}\rangle + \cdots + \langle b_{i_r}\rangle$, that is, $\langle a_j\rangle\theta \subseteq \langle a_{i_1}\rangle\theta + \cdots + \langle a_{i_r}\rangle\theta$. Clearly, every finite subspace of V is in $\mathcal{C}(V,q)$, hence from (d) we have $\langle a_j\rangle\theta \subseteq (\langle a_{i_1}\rangle + \cdots + \langle a_{i_r}\rangle)\theta$. Since θ is an order-isomorphism, it follows that

$$\langle a_j\rangle \subseteq \langle a_{i_1}\rangle + \cdots + \langle a_{i_r}\rangle,$$

so there exist $y_{i_1},\ldots,y_{i_r} \in F$ such that $a_j = y_{i_1}a_{i_1} + \cdots + y_{i_r}a_{i_r}$, a contradiction. Thus, the set $\{b_i\}$ is linearly independent and so $\{b_i\}$ is a basis for $A\theta$. Hence, dim $A\theta = |I| =$ dim A. □

In passing, we observe that if $A, B \in \mathcal{C}(V,q)$ and dim $B < p$, then $A + B \in \mathcal{C}(V,q)$: for, if $A \cap B = \langle u_i\rangle$, $A = \langle u_i, v_j\rangle$ and $B = \langle u_i, w_k\rangle$ then $A + B = \langle u_i, v_j, w_k\rangle$; hence $|K| \leq$ dim $B < q$, and codim $A \geq q$ implies codim $(A + B) =$ codim $A \geq q$.

Now suppose $\theta : \mathcal{C}(V,q) \to \mathcal{C}(W,n)$ is an order-isomorphism. We claim that θ is induced by a semilinear transformation. To see this, we need four preliminary Lemmas: in essence, they are [3] chapter 3, section 1, pp. 44-47, properties (1), (2), (4) and (6), respectively, but in a more general setting (as before, Baer assumes an order-isomorphism between the poset of all subspaces of V and W, respectively, and we assume less). In fact, Baer's arguments hold in this case but we include proofs for completeness.

Lemma 6.3.3. Let u, v be two linearly independent elements of V and let $u' \in W$ be such that $\langle u\rangle\theta = \langle u'\rangle$. Then there exists one and only one element v' in W such that $\langle v\rangle\theta = \langle v'\rangle$ and $\langle u - v\rangle\theta = \langle u' - v'\rangle$.

Proof. Since $\langle u-v\rangle \subseteq \langle u\rangle + \langle v\rangle$, we have $\langle u-v\rangle\theta \subseteq (\langle u\rangle + \langle v\rangle)\theta = \langle u\rangle\theta + \langle v\rangle\theta$. From Lemma 6.3.2, and since $\dim\langle u - v\rangle = 1$, there exists some $w \in W \setminus \{0\}$ such that $\langle u - v\rangle\theta = \langle w\rangle$. We then have

$$\langle w\rangle \subseteq \langle u\rangle\theta + \langle v\rangle\theta = \langle u'\rangle + \langle v\rangle\theta$$

and so there exist some $x \in K$ and $a \in \langle v\rangle\theta$ such that $w = xu' + a$. If $x = 0$ then $w = a$, so $\langle u - v\rangle\theta = \langle w\rangle \subseteq \langle v\rangle\theta$. Since θ preserves order, it follows that $\langle u - v\rangle \subseteq \langle v\rangle$ and so $u = u - v + v \in \langle v\rangle$, a contradiction. Therefore $x \neq 0$. Now suppose $a = 0$. Then $w = xu' \in \langle u'\rangle$, hence $\langle u - v\rangle\theta \subseteq \langle u'\rangle$. Thus, $\langle u - v\rangle \subseteq \langle u\rangle$ and so $v = u - (u - v) \in \langle u\rangle$, which contradicts the linear independence of u and v. Therefore $a \neq 0$.

Let $v' = -x^{-1}a$. Then $v' \in \langle v\rangle\theta \setminus \{0\}$, and so $\langle v'\rangle \subseteq \langle v\rangle\theta$. Since $\langle v\rangle\theta$ has dimension one, it follows that $\langle v'\rangle = \langle v\rangle\theta$. Moreover, $\langle u - v\rangle\theta = \langle w\rangle = \langle x^{-1}w\rangle = \langle u' - v'\rangle$.

Assume now that v^* is another element of W such that $\langle v\rangle\theta = \langle v^*\rangle$ and $\langle u - v\rangle\theta = \langle u' - v^*\rangle$. Then

$$\langle v^*\rangle = \langle v'\rangle \quad \text{and} \quad \langle u' - v^*\rangle = \langle u' - v'\rangle.$$

Since $v' \in \langle v^*\rangle \setminus \{0\}$ and $u' - v' \in \langle u' - v^*\rangle \setminus \{0\}$, there exist some $y, z \in K \setminus \{0\}$ such that $v' = yv^*$ and $u' - v' = z(u' - v^*) = zu' - zv^*$. Thus $u' - yv^* = u' - v' = zu' - zv^*$. Since $\langle u\rangle\theta = \langle u'\rangle$, $\langle v\rangle\theta = \langle v^*\rangle$ and u, v are linearly independent, it follows that u', v^* are also linearly independent, hence $u' - yv^* = zu' - zv^*$ implies $z = 1$ and $y = z$. Therefore, $y = 1$ and $v' = v^*$. □

From the Lemma above, for every pair of linearly independent elements u and v of V and every u' in W such that $\langle u\rangle\theta = \langle u'\rangle$, there exists a unique v' in W such that $\langle v\rangle\theta = \langle v'\rangle$ and $\langle u - v\rangle\theta = \langle u' - v'\rangle$. In this case, we write $h(u, u', v) = v'$. For convenience, we let $h(u, u', 0) = 0$ since $\{0\}\theta = \{0\}$.

Lemma 6.3.4. Let u and v be two linearly independent elements in V. Then,

$$h(u, u', v) = v' \quad \text{if and only if} \quad h(v, v', u) = u'.$$

Proof. Suppose $h(u,u',v)=v'$. Then $\langle u\rangle\theta=\langle u'\rangle$, $\langle v\rangle\theta=\langle v'\rangle$ and $\langle u-v\rangle\theta=\langle u'-v'\rangle$. Therefore, we have $\langle v-u\rangle\theta=\langle u-v\rangle\theta=\langle u'-v'\rangle=\langle v'-u'\rangle$ and so $h(v,v',u)=u'$. Similarly, we conclude that the converse is also true. □

As observed by Baer in [3] chapter 3, section 1, p. 45, (3), if u,v,w are three linearly independent elements in V, then

$$\langle v-w\rangle=(\langle v\rangle+\langle w\rangle)\cap(\langle u-v\rangle+\langle u-w\rangle). \tag{6.3.1}$$

In fact, since $v-w=-(u-v)+(u-w)$, we have $v-w\in(\langle v\rangle+\langle w\rangle)\cap(\langle u-v\rangle+\langle u-w\rangle)$. On the other hand, if $a=x_1v+x_2w=x_3(u-v)+x_4(u-w)$ with $x_1,x_2,x_3,x_4\in F$, then $x_1v+x_2w=(x_3+x_4)u-x_3v-x_4w$ and, from the linear independence of u,v,w, it follows that $x_3+x_4=0$, $x_1=-x_3$ and $x_2=-x_4$. Therefore, $x_1=-x_3=x_4=-x_2$ and so $a=x_1(v-w)\in\langle v-w\rangle$.

Lemma 6.3.5. Let u,v,w be linearly independent elements in V. If $h(u,u',v)=v'$ and $h(u,u',w)=w'$, then $h(v,v',w)=w'$.

Proof. Suppose $h(u,u',v)=v'$ and $h(u,u',w)=w'$. Then $\langle u\rangle\theta=\langle u'\rangle$, $\langle v\rangle\theta=\langle v'\rangle$, $\langle u-v\rangle\theta=\langle u'-v'\rangle$, $\langle w\rangle\theta=\langle w'\rangle$ and $\langle u-w\rangle\theta=\langle u'-w'\rangle$. Since u,v,w are linearly independent and $\langle u\rangle\theta=\langle u'\rangle$, $\langle v\rangle\theta=\langle v'\rangle$ and $\langle w\rangle\theta=\langle w'\rangle$, we have $\langle u,v,w\rangle\theta=\langle u',v',w'\rangle$ and u',v',w' are linearly independent. Since θ is an order-isomorphism, from (6.3.1) we have

$$\begin{aligned}\langle v-w\rangle\theta&=[(\langle v\rangle+\langle w\rangle)\cap(\langle u-v\rangle+\langle u-w\rangle)]\theta\\&=(\langle v\rangle\theta+\langle w\rangle\theta)\cap(\langle u-v\rangle\theta+\langle u-w\rangle\theta)\\&=(\langle v'\rangle+\langle w'\rangle)\cap(\langle u'-v'\rangle+\langle u'-w'\rangle)=\langle v'-w'\rangle.\end{aligned}$$

Thus, $\langle v\rangle\theta=\langle v'\rangle$, $\langle w\rangle\theta=\langle w'\rangle$ and $\langle v-w\rangle\theta=\langle v'-w'\rangle$. Hence, $h(v,v',w)=w'$. □

For the proof of the next Lemma, we will need the following equalities (see [3] chapter 3, section 1, p. 46, (5)): for all linearly independent elements $u,v,w\in V$,

$$\langle u-v-w\rangle=(\langle u-v\rangle+\langle w\rangle)\cap(\langle u-w\rangle+\langle v\rangle) \tag{6.3.2}$$

$$\text{and}\quad\langle v+w\rangle=(\langle v\rangle+\langle w\rangle)\cap(\langle u-v-w\rangle+\langle u\rangle). \tag{6.3.3}$$

Lemma 6.3.6. Let $u, v, w \in V$ with $u \neq 0$ and let $u' \in W \setminus \{0\}$ be such that $\langle u\rangle\theta = \langle u'\rangle$. If $u \notin \langle v\rangle + \langle w\rangle$, then $h(u, u', v + w) = h(u, u', v) + h(u, u', w)$.

Proof. Suppose u, v, w are linearly independent. Let $v' = h(u, u', v)$ and $w' = h(u, u', w)$. Then, $\langle u\rangle\theta = \langle u'\rangle$, $\langle v\rangle\theta = \langle v'\rangle$, $\langle w\rangle\theta = \langle w'\rangle$ and $\langle u - v\rangle\theta = \langle u' - v'\rangle$, $\langle u - w\rangle\theta = \langle u' - w'\rangle$. Since u, v, w are linearly independent in V, we know u', v', w' are linearly independent in W and, from (6.3.2), it follows that

$$\begin{aligned}\langle u - (v + w)\rangle\theta &= \langle u - v - w\rangle\theta \\ &= (\langle u - v\rangle + \langle w\rangle)\theta \cap (\langle u - w\rangle + \langle v\rangle)\theta \\ &= (\langle u - v\rangle\theta + \langle w\rangle\theta) \cap (\langle u - w\rangle\theta + \langle v\rangle\theta) \\ &= (\langle u' - v'\rangle + \langle w'\rangle) \cap (\langle u' - w'\rangle + \langle v'\rangle) \\ &= \langle u' - v' - w'\rangle = \langle u' - (v' + w')\rangle.\end{aligned}$$

Also, (6.3.3) implies

$$\begin{aligned}\langle v + w\rangle\theta &= (\langle v\rangle + \langle w\rangle)\theta \cap (\langle u - v - w\rangle + \langle u\rangle)\theta \\ &= (\langle v\rangle\theta + \langle w\rangle\theta) \cap (\langle u - v - w\rangle\theta + \langle u\rangle\theta) \\ &= (\langle v'\rangle + \langle w'\rangle) \cap (\langle u' - v' - w'\rangle + \langle u'\rangle) = \langle v' + w'\rangle.\end{aligned}$$

Therefore, since u and $v + w$ are linearly independent and $\langle u\rangle\theta = \langle u'\rangle$, $\langle v + w\rangle\theta = \langle v' + w'\rangle$ and $\langle u - (v + w)\rangle\theta = \langle u' - (v' + w')\rangle$, we have $h(u, u', v + w) = h(u, u', v) + h(u, u', w)$. Now suppose u, v, w are not linearly independent. If $v = 0$, then

$$\begin{aligned}h(u, u', v + w) &= h(u, u', w) = 0 + h(u, u', w) \\ &= h(u, u', 0) + h(u, u', w) = h(u, u', v) + h(u, u', w).\end{aligned}$$

Similarly we conclude that $h(u, u', v+w) = h(u, u', v)+h(u, u', w)$ if $w = 0$. Hence we can suppose $v \neq 0$ and $w \neq 0$. Since $u \notin \langle v\rangle + \langle w\rangle$, we have the linear independence of u and v and of u and w. By supposition, $xu + yv + zw = 0$ for some $x, y, z \in F$, where $y \neq 0$ and $z \neq 0$, hence $x = 0$, and so $\langle v\rangle = \langle w\rangle$. Therefore, $\langle u\rangle \neq \langle v\rangle = \langle w\rangle$.

Let $a \in V$ be such that u, v, a are linearly independent (recall that $\dim V \geq 3$). Then, $u, a + v, w$ are linearly independent (since $\langle v\rangle = \langle w\rangle$); and if $v + w \neq 0$, then $u, v + w, a$ are also linearly independent. Hence, from the above,

$$\begin{aligned}&h(u, u', a) + h(u, u', v + w) \\ &= h(u, u', a + (v + w)) = h(u, u', (a + v) + w) \\ &= h(u, u', a + v) + h(u, u', w) \\ &= (h(u, u', a) + h(u, u', v)) + h(u, u', w)\end{aligned}$$

and this implies $h(u, u', v+w) = h(u, u', v)+h(u, u', w)$. If $v+w = 0$ then $h(u, u', v) = h(u, u', -w)$, hence the equality follows. □

Finally, our next result extends Baer's Theorem, in the sense that Baer assumes an order-isomorphism between the posets consisting of *all* subspaces, whereas we assume less (I was unable to find a similar result in the literature). Again, we closely follow Baer's arguments in [3] chapter 3, section 1, pp. 48-50.

Theorem 6.3.7. Given $q \leq \dim V$ and $n \leq \dim W$, every order-isomorphism θ from $\mathcal{C}(V, q)$ onto $\mathcal{C}(W, n)$ is induced by a semilinear transformation.

Proof. Fix three linearly independent elements $u, v, w \in V$. Since $\langle u\rangle\theta$ has dimension one, there exists non-zero $u' \in W$ such that $\langle u\rangle\theta = \langle u'\rangle$. We let $v' = h(u, u', v)$ and $w' = h(u, u', w)$ and we note that, from Lemmas 6.3.4 and 6.3.5, $h(e, e', f) = f'$ for any two distinct elements e and f of u, v, w. If $e \in \{u, v, w\}$, then $h(e, e', 0) = 0$, by convention.

Let $a \in V \setminus \{0\}$. Then $\langle a\rangle$ is a one-dimensional subspace of V and it is different from at least two of the subspaces $\langle u\rangle$, $\langle v\rangle$ and $\langle w\rangle$. Suppose $\langle a\rangle$ is different from $\langle u\rangle$ and $\langle v\rangle$. We assert that $h(u, u', a) = h(v, v', a)$. If $a \notin \langle u, v\rangle$, then u, v, a are linearly independent in V. Since $h(u, u', v) = v'$, then $h(u, u', a) = a'$ implies $h(v, v', a) = a'$ (from Lemma 6.3.5), and the desired equality follows. On the other hand, if $a \in \langle u, v\rangle$, then $a \notin \langle u, w\rangle$ and $a \notin \langle w, v\rangle$. From what we have just seen in the previous case, $h(u, u', a) = h(w, w', a)$ and $h(w, w', a) = h(v, v', a)$. Hence $h(u, u', a) = h(v, v', a)$. Thus, for every $a \in V$, at least two of the vectors $h(u, u', a)$, $h(v, v', a)$ and $h(w, w', a)$ are defined; and those which are defined have the same value. We denote it by af and so we have a mapping f from V to W. Moreover, f is such that $\langle a\rangle\theta = \langle af\rangle$.

Next we see that f is an isomorphism between the additive groups V and W. Let $a, b \in V$ and suppose $\langle u\rangle, \langle v\rangle, \langle w\rangle \subseteq \langle a, b\rangle$. Then $\langle u, v, w\rangle \subseteq \langle a, b\rangle$, a contradiction since $\langle u, v, w\rangle$ has dimension 3 and $\langle a, b\rangle$ has dimension at most 2. Therefore, at least one of the elements u, v, w is not in $\langle a, b\rangle$. Assume $u \notin \langle a, b\rangle$. Then, from Lemma 6.3.6, we have

$$(a + b)f = h(u, u', a + b) = h(u, u', a) + h(u, u', b) = af + bf$$

and so f preserves sum. If $a, b \in V$ are such that $af = bf$, then $(a-b)f = 0$ and so $\langle a-b\rangle\theta = \{0\}$. Thus, $a - b = 0$ and f is one-to-one. Now let $c \in W$. If $c = 0$, then $c = 0 = 0f$ with $0 \in V$. If not, then $\langle c\rangle$ is a subspace of W with dimension one and so there exists $a \in V\setminus\{0\}$ such that $\langle a\rangle\theta = \langle c\rangle$. Since u, v, w are linearly independent, $\langle a\rangle$ is different from at least one of the subspaces $\langle u\rangle$, $\langle v\rangle$, $\langle w\rangle$. Suppose $\langle a\rangle \neq \langle u\rangle$. Then $\langle c\rangle \neq \langle u'\rangle$. Consider the subspace $\langle u' + c\rangle$ of W and let $\langle b\rangle$ be the one-dimensional subspace of V such that $\langle b\rangle\theta = \langle u'+c\rangle$. Since $\langle b\rangle\theta = \langle u'+c\rangle \subseteq \langle u'\rangle+\langle c\rangle = (\langle u\rangle+\langle a\rangle)\theta$ and $\langle b\rangle\theta = \langle u' + c\rangle \neq \langle c\rangle = \langle a\rangle\theta$, it follows that $\langle b\rangle \subseteq \langle u\rangle + \langle a\rangle$ and $\langle b\rangle \neq \langle a\rangle$. Therefore, $b = x_1u + x_2a$ with $x_1, x_2 \in F$, where $x_1 \neq 0$ (since $\langle b\rangle \neq \langle a\rangle$) and $x_2 \neq 0$ (otherwise, $\langle b\rangle = \langle u\rangle$, hence $\langle u'\rangle = \langle u\rangle\theta = \langle u' + c\rangle$ and so $\langle c\rangle = \langle u'\rangle$, a contradiction). Consequently, $\langle b\rangle = \langle u + x_1^{-1}x_2a\rangle$. Let $d = x_1^{-1}x_2a$. We then have $\langle b\rangle = \langle u + d\rangle$ and $\langle a\rangle = \langle d\rangle$, since $x_2 \neq 0$.

Thus, $\langle u' + c\rangle = \langle b\rangle\theta = \langle u + d\rangle\theta = \langle (u + d)f\rangle = \langle uf + df\rangle = \langle u' + df\rangle$ and so

$$c - df = (u' + c) - (u' + df) \in \langle b\rangle\theta.$$

On the other hand, $\langle c\rangle = \langle a\rangle\theta = \langle d\rangle\theta = \langle df\rangle$ and so

$$c - df \in \langle a\rangle\theta.$$

Since $\langle u' + c\rangle \neq \langle c\rangle$, we know $\langle b\rangle \neq \langle a\rangle$, hence $\langle b\rangle \cap \langle a\rangle = \{0\}$ and so $c - df \in (\langle b\rangle \cap \langle a\rangle)\theta = \{0\}$. Hence $c = df$. In other words, we have shown that f is a (group) isomorphism from V onto W.

We assert that $A\theta = Af$ for every subspace $A \in \mathcal{C}(V, q)$. To see this, let $b \in A\theta$. Then, $b = af$ for some $a \in V$. Since $\langle a\rangle\theta = \langle af\rangle = \langle b\rangle \subseteq A\theta$ and θ is an order-isomorphism, we have $\langle a\rangle \subseteq A$. Therefore, $a \in A$ and so $b \in Af$. Conversely, if $b = af \in Af$ then $\langle b\rangle = \langle af\rangle = \langle a\rangle\theta \subseteq A\theta$, so $b \in A\theta$.

If $a \in V \setminus \{0\}$ and $x \in F \setminus \{0\}$, then $\langle af\rangle = \langle a\rangle\theta = \langle xa\rangle\theta = \langle (xa)f\rangle$. Therefore, there exists $y \in K \setminus \{0\}$ such that $(xa)f = y(af)$. In this case, we write $g(x, a) = y$. If we let $g(0, a) = 0$, then

$$(xa)f = g(x, a)(af)$$

holds for every $x \in F$ and every $a \in V \setminus \{0\}$. To see that $g(x, a) = g(x, b)$ for every $x \in F$ and every $a, b \in V \setminus \{0\}$, we first suppose $a \notin \langle b\rangle$. Then, $\langle a\rangle \neq \langle b\rangle$ and so

$\langle af\rangle \neq \langle bf\rangle$. Hence af, bf are linearly independent and

$$\begin{aligned} g(x,a+b)(af)+g(x,a+b)(bf) &= g(x,a+b)(af+bf) = g(x,a+b)(a+b)f \\ &= [x(a+b)]f = (xa+xb)f = (xa)f+(xb)f = g(x,a)(af)+g(x,b)(bf) \end{aligned},$$

and these imply

$$g(x,a) = g(x,a+b) = g(x,b).$$

If $a \in \langle b\rangle$, then there exists $c \notin \langle a\rangle = \langle b\rangle$ and from what we have just seen,

$$g(x,a) = g(x,c) = g(x,b).$$

We denote by xf^* the value $g(x,a)$, for $a \in V \setminus \{0\}$. Then, f^* is a mapping from F into K such that

$$(xa)f = (xf^*)(af) \quad \text{for every} \quad x \in F \quad \text{and} \quad a \in V.$$

Next we show that $f^* : F \to K$ is an isomorphism of fields. Let $x, y \in F$ and let $a \in V \setminus \{0\}$. Then,

$$\begin{aligned} (x+y)f^*(af) &= [(x+y)a]f = (xa+ya)f \\ &= (xa)f+(ya)f = (xf^*)(af)+(yf^*)(af) = (xf^*+yf^*)(af) \end{aligned}$$

and, since $af \neq 0$, it follows that $(x+y)f^* = xf^*+yf^*$. Likewise we see that

$$\begin{aligned} (xy)f^*(af) &= [(xy)a]f = [x(ya)]f \\ &= (xf^*)(ya)f = (xf^*)[(yf^*)(af)] = [(xf^*)(yf^*)](af), \end{aligned}$$

and so $(xy)f^* = (xf^*)(yf^*)$. If x, y are such that $xf^* = yf^*$, then for some $a \in V \setminus \{0\}$, $(xa)f = (xf^*)(af) = (yf^*)(af) = (ya)f$ and since $f : V \to W$ is an isomorphism, it follows that $xa = ya$. Thus, $x = y$. Now let $z \in K$ and let $a \in V \setminus \{0\}$. Since $f : V \to W$ is onto, there exists $b \in V$ such that $bf = z(af)$. We then have $\langle b\rangle\theta = \langle bf\rangle = \langle af\rangle = \langle a\rangle\theta$ and so $\langle b\rangle = \langle a\rangle$. Therefore, there exists $x \in F$ such that $b = xa$ and $z(af) = bf = (xa)f = (xf^*)(af)$. Since $af \neq 0$, we have $z = xf^*$ with $x \in F$. We have shown that f is a semilinear transformation such that $A\theta = Af$ for every $A \in \mathcal{C}(V,q)$: that is, the order-isomorphism θ is induced by f. $\square$

References

In this thesis, I often rely on comments made by other authors for historical background, hence there are some itens in the list of references below which I have not seen but that I consider relevant for the work in the thesis. These items are marked with a dagger (†).

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