

Conformal transformations in general relativistic elasticity

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Abstract

Conformal transformations are applied in the context of general relativistic elasticity. Expressions relating relativistic elastic quantities and tensors are obtained for two conformal spacetimes, whose material metrics are also conformally related. Non-static shear-free spherically symmetric elastic solutions of the Einstein field equations are constructed by performing conformal transformations of spacetime and material metrics of a known non-static shear-free spherically symmetric elastic solution.

Keywords: conformal transformations; relativistic elasticity; Einstein field equations; spherically symmetric spacetimes.

1 Introduction

The theory of relativistic elasticity is relevant for many astrophysical applications, e.g. for the study of deformations of massive stellar objects [1]-[3]. Different formulations of relativistic elasticity have appeared in the literature [4]-[7]. Thereby, spherically symmetric solutions of the Einstein field equations for elastic fluids were obtained and discussed in [6], [8]-[18], where in [10],[11] and [16] non-static configurations were considered.

Conformal transformations have been used in general relativity as a method for generating solutions of the Einstein field equations. Carot and Mas [19] and Tupper [20] obtained viscous fluid solutions by performing conformal transformations of vacuum solutions. Perfect fluid solutions were constructed by conformal transformations in [21], [22], [23] and [24]. In [25]

and in an extension of that work, [26], scalar and electromagnetic fields of uniformly accelerated charges in de Sitter spacetimes were constructed by employing the conformal relation between Minkowski and de Sitter spacetimes. In particular, the character of the fields were investigated in various coordinate systems, and the relation of the different types of coordinates in the de Sitter space to the corresponding coordinates on conformally related Minkowski spaces were obtained in [26].

The purpose of this paper is to study conformal transformations in the context of relativistic elasticity, where the matter is described by an elastic fluid. Assuming that both spacetime metrics and material metrics of two spacetimes are conformally related, expressions for quantities and tensors in the context of relativistic elasticity are derived. Starting from a known spacetime and carrying out conformal transformations, these expressions allow then to obtain results for the new conformally related spacetime. Also, considering non-static spherically symmetric elastic fluids, the problem of generating a solution of the Einstein field equations from a given shear-free solution by conformal transformations of spacetime and material metrics is addressed.

This paper is organized as follows. In the present section, a brief introduction to the theory of elasticity in general relativity is given (the reader is referred to the references [5], [6] and [16] for more detailed aspects). In section 2, formulas for relativistic elastic quantities and tensors of conformally related spacetimes are derived, assuming that the material metrics are also conformally related. In section 3 we will consider a known non-static shear-free spherically symmetric solution of the Einstein field equations for elastic matter and show how a new solution can be generated by applying conformal transformations to the spacetime and pulled-back material metrics. Some physical properties of the new solution are analysed and the matching problem with the exterior Schwarzschild solution is investigated.

Let (M, g) be a space-time, where M is a four-dimensional manifold with Lorentz metric g of signature $(-, +, +, +)$. Coordinates in M will be denoted by x^a , $a = 0, 1, 2, 3$. Suppose that the space-time is filled with a continuous material. The material space X is a three-dimensional manifold, whose points represent the particles of the material. The material space is endowed with a Riemannian metric γ , the material metric, which measures the distance between particles in the locally relaxed state of the material. Let y^A , $A = 1, 2, 3$, denote the material coordinates in X . The space-time configuration of the material is described by the configuration mapping $\psi : M \rightarrow X$ through the fields $y^A = y^A(x^a)$, $A = 1, 2, 3$. The differential map $\psi_* : T_p M \rightarrow T_{\psi(p)} X$ gives rise to a rank 3 matrix with

entries

$$y_b^A = \frac{\partial y^A}{\partial x^a}, \quad A = 1, 2, 3, \quad a = 0, 1, 2, 3, \quad (1)$$

which is called the relativistic deformation gradient. The velocity field of the matter u^a , a future oriented, timelike unit vector field, which spans the one-dimensional Kernel of the relativistic deformation gradient, is defined by the conditions:

$$u^a y_a^A = 0, \quad u^a u_a = -1, \quad u^0 > 0. \quad (2)$$

The pulled-back material metric $k_{ab} = (\psi^* \gamma)_{ab} = y_a^A y_b^B \gamma_{AB}$ is used to construct the strain tensor

$$s_{ab} = \frac{1}{2}(h_{ab} - k_{ab}) = \frac{1}{2}(g_{ab} - K_{ab}), \quad (3)$$

where $h_{ab} = g_{ab} + u_a u_b$ and $K_{ab} = -u_a u_b + k_{ab}$. The material is said to be locally relaxed if the strain tensor vanishes. In the locally relaxed state, the operator $K^a_b = k^a_b - u^a u_b$ becomes the identity operator. Notice that since $K^a_b u^b = u^a$, one of its eigenvalue is 1 and the other three positive eigenvalues are the eigenvalues of k^a_b . One can use the orthonormal tetrad $\{u, e_1, e_2, e_3\}$, where e_1, e_2, e_3 are the spatial eigenvectors of k^a_b and u is the velocity field of the matter, for which the metric g_{ab} can be written as

$$g_{ab} = -u_a u_b + e_{1a} e_{1b} + e_{2a} e_{2b} + e_{3a} e_{3b} \quad (4)$$

and the pulled-back material metric k_{ab} as

$$k_{ab} = n_1^2 e_{1a} e_{1b} + n_2^2 e_{2a} e_{2b} + n_3^2 e_{3a} e_{3b}, \quad (5)$$

where n_1^2, n_2^2, n_3^2 are the eigenvalues of k^a_b .

Assuming that the internal energy v of an elastic deformation, accumulated in an infinitesimal portion of the material, is invariant with respect to its space-time orientation, v depends on three invariants of the strain tensor. Since the strain tensor is a linear function of K and the identity operator, any three invariants I_1, I_2, I_3 of K can be chosen, so that $v = v(I_1, I_2, I_3)$, which is the constitutive function of the material. Following [10] the invariants can be established as

$$I_1 = \frac{1}{2}(\text{Tr}K - 4), \quad I_2 = \frac{1}{4}[\text{Tr}K^2 - (\text{Tr}K)^2] + 3, \quad I_3 = \frac{1}{2}(\det K - 1), \quad (6)$$

where

$$\text{Tr}K = 1 + n_1^2 + n_2^2 + n_3^2, \quad \text{Tr}K^2 = 1 + n_1^4 + n_2^4 + n_3^4, \quad \det K = n_1^2 n_2^2 n_3^2. \quad (7)$$

The energy density ρ is defined by

$$\rho = \epsilon v(I_1, I_2, I_3) = \epsilon_0 \sqrt{\det K} v(I_1, I_2, I_3), \quad (8)$$

where ϵ_0 is the particle number density as measured in the material space, or rather, with respect to the volume form associated with $k_{ab} = (\psi^* \gamma)_{ab}$, and ϵ is that with respect to h_{ab} (see e.g. [5],[10]). Since the invariants can be written in terms of the eigenvalues of K using (7), the constitutive function can be viewed as a function of the eigenvalues, $v = v(n_1^2, n_2^2, n_3^2)$. The energy-momentum tensor for elastic matter, when decomposed with respect to the matter velocity u , takes the form

$$T_{ab} = \rho u_a u_b + p_{ab} = \rho u_a u_b + p h_{ab} + \pi_{ab}, \quad (9)$$

where

$$p_{ab} = p h_{ab} + \pi_{ab} \quad (10)$$

is the pressure tensor, p being the isotropic pressure and π_{ab} the tracefree anisotropic pressure tensor. The pressure tensor p_{ab} can be expressed as

$$p_{ab} = p_1 e_{1a} e_{1b} + p_2 e_{2a} e_{2b} + p_3 e_{3a} e_{3b} \quad (11)$$

and the principal pressures p_1, p_2, p_3 satisfy $p_1 + p_2 + p_3 = 3p$.

Following [10] (see also [16]) the energy-momentum tensor for elastic matter can also be written as

$$T^a_b = -\rho \delta^a_b + \frac{\partial \rho}{\partial I_3} \det K h^a_b - \left(\text{Tr} K \frac{\partial \rho}{\partial I_2} - \frac{\partial \rho}{\partial I_1} \right) k^a_b + \frac{\partial \rho}{\partial I_2} k^a_c k^c_b. \quad (12)$$

This tensor is obtained from the Lagrangian $\mathcal{L} = \sqrt{-g} \rho$, which is a function of x^a , y^A and y^A_a . The corresponding Euler-Lagrange equations are given by $\frac{\partial \mathcal{L}}{\partial y^A} - \partial_a \left(\frac{\partial \mathcal{L}}{\partial y^A_a} \right) = 0$, where $\frac{\partial \mathcal{L}}{\partial y^A_a}$ is sometimes called relativistic Piola-Kirchhoff momentum tensor. Using Noether's theorem one constructs the canonical energy-momentum tensor

$$\tilde{T}^a_b = \frac{1}{\sqrt{-g}} \frac{\partial \mathcal{L}}{\partial y^A_a} y^A_b - \delta^a_b \mathcal{L}, \quad (13)$$

which satisfies the energy-momentum conservation law $\nabla_a \tilde{T}^{ab} = 0$, and the symmetric energy-momentum tensor is the negative of the canonical energy-momentum tensor (see for instance [17]): $T_{ab} = -\tilde{T}_{ab}$. Note that (9) can be obtained from (13) (see [17]), where $p h_{ab} + \pi_{ab} = 2 \frac{\partial \rho}{\partial G^{AB}} y^A_a y^B_b - \rho h_{ab}$, with $G^{AB} = y^A_a y^B_b g^{ab}$.

The interaction of the elastic material with the gravitational field is then described by the Einstein field equations $G^a_b = 8\pi T^a_b$.

2 Conformally related metrics in relativistic elasticity

Consider two conformally related spacetime metrics

$$g_{ab} = f^2 \bar{g}_{ab}, \quad (14)$$

where f^2 is the conformal factor, f being a smooth, strictly positive function depending on the spacetime coordinates: $f = f(x^a)$. Suppose that the pulled-back material metrics k_{ab} , belonging to (M, g) , and $\bar{k}_{ab}$, belonging to $(M, \bar{g})$, are also conformally related,

$$k_{ab} = h^2 \bar{k}_{ab}, \quad (15)$$

where $h = h(x^a)$ is a smooth, strictly positive function.

In $(M, \bar{g})$, using the orthonormal tetrad $\{\bar{u}, \bar{e}_1, \bar{e}_2, \bar{e}_3\}$, where $\bar{e}_1, \bar{e}_2, \bar{e}_3$ are the spatial eigenvectors of $\bar{k}^a_b$ and $\bar{u}$ is a timelike vector, the velocity field of the matter, the metric $\bar{g}_{ab}$ and the pulled-back material metric $\bar{k}_{ab}$ can be expressed as

$$\bar{g}_{ab} = -\bar{u}_a \bar{u}_b + \bar{e}_{1a} \bar{e}_{1b} + \bar{e}_{2a} \bar{e}_{2b} + \bar{e}_{3a} \bar{e}_{3b} \quad (16)$$

and

$$\bar{k}_{ab} = \bar{n}_1^2 \bar{e}_{1a} \bar{e}_{1b} + \bar{n}_2^2 \bar{e}_{2a} \bar{e}_{2b} + \bar{n}_3^2 \bar{e}_{3a} \bar{e}_{3b}, \quad (17)$$

respectively, $\bar{n}_1^2, \bar{n}_2^2, \bar{n}_3^2$ being the eigenvalues of $\bar{k}^a_b$.

The following relations between quantities associated with g and k and quantities associated with $\bar{g}$ and $\bar{k}$ can be established. These formulas allow then to determine quantities for (M, g) once the corresponding quantities of $(M, \bar{g})$ are known.

The tetrad vectors are related as follows

$$u_a = f \bar{u}_a, \quad e_{1a} = f \bar{e}_{1a}, \quad e_{2a} = f \bar{e}_{2a}, \quad e_{3a} = f \bar{e}_{3a}. \quad (18)$$

Since $k_{ab} = h^2 \bar{k}_{ab}$, one concludes from (5), (17) and (18) that one can relate the eigenvalues of k with those of $\bar{k}$ in the following way:

$$n_A^2 = \frac{h^2}{f^2} \bar{n}_A^2, \quad A = 1, 2, 3. \quad (19)$$

The strain tensors satisfy

$$s_{ab} = f^2 \bar{s}_{ab} + \frac{1}{2} \bar{k}_{ab} (f^2 - h^2). \quad (20)$$

The expressions

$$\text{Tr} K = \frac{h^2}{f^2} \text{Tr} \bar{K} - \frac{h^2}{f^2} + 1, \quad (21)$$

$$\text{Tr} K^2 = \frac{h^4}{f^4} \text{Tr} \bar{K}^2 - \frac{h^4}{f^4} + 1, \quad (22)$$

$$\det K = \frac{h^6}{f^6} \det \bar{K}, \quad (23)$$

which are obtained from (7) taking into account (19), allow to write the invariants (6) in terms of $\bar{I}_1, \bar{I}_2, \bar{I}_3$:

$$I_1 = \frac{h^2}{f^2} \bar{I}_1 + \frac{3}{2} \left(\frac{h^2}{f^2} - 1 \right), \quad (24)$$

$$I_2 = \frac{h^4}{f^4} \bar{I}_2 + \left(\frac{h^4}{f^4} - \frac{h^2}{f^2} \right) \bar{I}_1 - \frac{3}{2} \left(\frac{h^4}{f^4} + \frac{h^2}{f^2} \right) + 3, \quad (25)$$

$$I_3 = \frac{h^6}{f^6} \bar{I}_3 + \frac{1}{2} \left(\frac{h^6}{f^6} - 1 \right). \quad (26)$$

The energy density (8) becomes

$$\rho = \frac{h^3}{f^3} \frac{\epsilon_0 v}{\bar{\epsilon}_0 \bar{v}}, \quad (27)$$

where $\bar{\epsilon}_0$ and $\bar{v}$ are the particle number density and the constitutive function $\bar{v} = \bar{v}(\bar{I}_1, \bar{I}_2, \bar{I}_3)$ associated with $\bar{k}$. Then, using (21)-(27) and since $\frac{\partial \rho}{\partial I_A} = \frac{\partial \rho}{\partial \bar{\rho}} \frac{\partial \bar{\rho}}{\partial \bar{I}_B} \frac{\partial \bar{I}_B}{\partial I_A}$, B being a summation index (running from 1 to 3), one can establish the following relation between the energy-momentum tensors, using the definition (12),

$$T^a_b = \frac{h^3}{f^3} \frac{\epsilon_0 v}{\bar{\epsilon}_0 \bar{v}} \bar{T}^a_b. \quad (28)$$

Ahead in this section, a formula will be derived, which enables to determine $\epsilon_0 v$ from $\bar{\epsilon}_0 \bar{v}$ and from other quantities in $(M, \bar{g})$.

It is a known result, see for instance [27], that given two conformally related spacetime metrics (14), the corresponding Riemann tensors satisfy

$$\begin{aligned} R_{abc}{}^d &= \bar{R}_{abc}{}^d + 2\delta_{[a}^d \bar{\nabla}_{b]} \bar{\nabla}_c \ln f - 2\bar{g}^{de} \bar{g}_{c[a} \bar{\nabla}_{b]} \bar{\nabla}_e \ln f + 2(\bar{\nabla}_{[a} \ln f) \delta_{b]}^d \bar{\nabla}_c \ln f \\ &\quad - 2(\bar{\nabla}_{[a} \ln f) \bar{g}_{b]c} \bar{g}^{dh} \bar{\nabla}_h \ln f - 2\bar{g}_{c[a} \delta_{b]}^d \bar{g}^{eh} (\bar{\nabla}_e \ln f) \bar{\nabla}_h \ln f \end{aligned} \quad (29)$$

and the Ricci tensors,

$$R_{ac} = \bar{R}_{ac} - 2 \bar{\nabla}_a \bar{\nabla}_c \ln f - \bar{g}_{ac} \bar{g}^{de} \bar{\nabla}_d \bar{\nabla}_e \ln f + 2 \bar{\nabla}_a \ln f \bar{\nabla}_c \ln f - 2 \bar{g}_{ac} \bar{g}^{de} \bar{\nabla}_d \ln f \bar{\nabla}_e \ln f, \quad (30)$$

where $\bar{\nabla}$ denotes the connection associated with $\bar{g}$. The Ricci scalars are related by

$$R = f^{-2} (\bar{R} - 6 \bar{g}^{ac} \bar{\nabla}_a \bar{\nabla}_c \ln f - 6 \bar{g}^{ac} \bar{\nabla}_a \ln f \bar{\nabla}_c \ln f). \quad (31)$$

Note that the covariant derivative of a scalar field is equal to the partial derivative, so that $\bar{\nabla}_a \ln f = \nabla_a \ln f = \partial_a \ln f$.

From the definition of the Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$ one gets

$$G_{ab} = \bar{G}_{ab} - 2 \bar{\nabla}_b \bar{\nabla}_a \ln f + 2 \bar{\nabla}_a \ln f \bar{\nabla}_b \ln f + \bar{g}_{ab} (2 \bar{g}^{lm} \bar{\nabla}_m \bar{\nabla}_l \ln f + \bar{g}^{lm} \bar{\nabla}_l \ln f \bar{\nabla}_m \ln f), \quad (32)$$

which can be written in a useful form (cf. [24], [28]) as

$$G_{ab} = \bar{G}_{ab} - 2\sigma_{ab} + \bar{g}_{ab}(2\Delta_2\sigma + \Delta_1\sigma), \quad (33)$$

where

$$\begin{aligned} \sigma_{ab} &= \bar{\nabla}_b \bar{\nabla}_a \ln f - \bar{\nabla}_a \ln f \bar{\nabla}_b \ln f, \\ \Delta_2\sigma &= \bar{g}^{lm} \bar{\nabla}_m \bar{\nabla}_l \ln f, \\ \Delta_1\sigma &= \bar{g}^{lm} \bar{\nabla}_l \ln f \bar{\nabla}_m \ln f. \end{aligned} \quad (34)$$

Then, the Einstein field equations $G_{ab} = 8\pi T_{ab}$, $\bar{G}_{ab} = 8\pi \bar{T}_{ab}$ and (33) imply

$$\begin{aligned} 8\pi T^a_b &= G^a_b = \frac{1}{f^2} [\bar{G}^a_b - 2\bar{g}^{am} \sigma_{mb} + \bar{g}^a_b (2\Delta_2\sigma + \Delta_1\sigma)] \\ &= \frac{1}{f^2} [8\pi \bar{T}^a_b - 2\bar{g}^{am} \sigma_{mb} + \bar{g}^a_b (2\Delta_2\sigma + \Delta_1\sigma)]. \end{aligned} \quad (35)$$

This expression allows to obtain a formula for the energy densities

$$\rho = \frac{1}{f^2} \bar{\rho} - \frac{1}{4\pi f^2} \left(\sigma_{ml} \bar{u}^m \bar{u}^l + \Delta_2\sigma + \frac{\Delta_1\sigma}{2} \right), \quad (36)$$

and for the principal pressures

$$p_A = \frac{1}{f^2} \bar{p}_A - \frac{1}{4\pi f^2} \left(\sigma_{ml} \bar{e}_A^m \bar{e}_A^l - \Delta_2\sigma - \frac{\Delta_1\sigma}{2} \right), \quad A = 1, 2, 3. \quad (37)$$

Hence, from (27) and (36) one obtains

$$\epsilon_0 v = \frac{f}{h^3} \bar{\epsilon}_0 \bar{v} - \frac{f}{4\pi h^3 \sqrt{\det \bar{K}}} \left(\sigma_{ml} \bar{u}^m \bar{u}^l + \Delta_2 \sigma + \frac{\Delta_1 \sigma}{2} \right), \quad (38)$$

relating both constitutive functions v and $\bar{v}$.

The pressure tensors satisfy

$$p_{ab} = \bar{p}_{ab} + \frac{\bar{h}_{ab}}{8\pi} (2\Delta_2 \sigma + \Delta_1 \sigma) - \frac{\sigma_{ml}}{4\pi} (\bar{e}_1^m \bar{e}_1^l \bar{e}_{1a} \bar{e}_{1b} + \bar{e}_2^m \bar{e}_2^l \bar{e}_{2a} \bar{e}_{2b} + \bar{e}_3^m \bar{e}_3^l \bar{e}_{3a} \bar{e}_{3b}). \quad (39)$$

The isotropic pressures are related by

$$p = \frac{1}{f^2} \bar{p} - \frac{1}{8\pi f^2} \left[\frac{2}{3} \sigma_{mb} \bar{h}^{mb} - 2\Delta_2 \sigma - \Delta_1 \sigma \right] \quad (40)$$

and the anisotropic pressure tensors by

$$\begin{aligned} \pi_{ab} = & \bar{\pi}_{ab} + \frac{\sigma_{lm}}{12\pi} \left[\bar{e}_1^m \bar{e}_1^l (-2\bar{e}_{1a} \bar{e}_{1b} + \bar{e}_{2a} \bar{e}_{2b} + \bar{e}_{3a} \bar{e}_{3b}) \right. \\ & \left. + \bar{e}_2^m \bar{e}_2^l (\bar{e}_{1a} \bar{e}_{1b} - 2\bar{e}_{2a} \bar{e}_{2b} + \bar{e}_{3a} \bar{e}_{3b}) + \bar{e}_3^m \bar{e}_3^l (\bar{e}_{1a} \bar{e}_{1b} + \bar{e}_{2a} \bar{e}_{2b} - 2\bar{e}_{3a} \bar{e}_{3b}) \right]. \end{aligned} \quad (41)$$

In order to describe the kinematics of the fluid flow with velocity field $u^a = \frac{1}{f} \bar{u}^a$, we present the expressions for the kinematical quantities, which can for example also be found in [29]. Recall that the covariant derivative of a timelike unit vector field can be decomposed as

$$u_{a;b} = -\dot{u}_a u_b + \omega_{ab} + \sigma_{ab} + \frac{1}{3} \Theta h_{ab}, \quad (42)$$

where

$$\Theta = u^a_{;a}, \quad (43)$$

$$\dot{u}_a = u_{a;b} u^b, \quad (44)$$

$$\omega_{ab} = u_{[a;b]} + \dot{u}_{[a} u_{b]}, \quad (45)$$

$$\sigma_{ab} = u_{(a;b)} + \dot{u}_{(a} u_{b)} - \frac{1}{3} \Theta h_{ab}, \quad (46)$$

are the expansion, the acceleration, the vorticity tensor and the shear tensor, respectively. Since

$$\Gamma_{bc}^a = \bar{\Gamma}_{bc}^a + \frac{1}{f} \left(f_{,c} \bar{g}^a_b + f_{,b} \bar{g}^a_c - f_{,d} \bar{g}^{ad} \bar{g}_{bc} \right) \quad (47)$$

and

$$u_{a;b} = f\bar{u}_{a;b} - f_{,a}\bar{u}_b + f_{,d}\bar{u}^d\bar{g}_{ab}, \quad (48)$$

a straightforward calculation shows that (cf. [29])

$$\Theta = \frac{1}{f}\bar{\Theta} + \frac{3}{f^2}f_{,a}\bar{u}^a, \quad (49)$$

$$\dot{u}_a = \dot{\bar{u}}_a + \frac{1}{f}\left(f_{,a} + f_{,d}\bar{u}^d\bar{u}_a\right), \quad (50)$$

$$\omega_{ab} = f\bar{\omega}_{ab}, \quad (51)$$

$$\sigma_{ab} = f\bar{\sigma}_{ab}. \quad (52)$$

3 Applications to spherical symmetry

We will consider a known non-static shear-free solution of the Einstein field equations for elastic matter in spherical symmetry and show through an example that new solutions can be generated by applying the conformal transformations (14) and (15) and using (35). Thereby, the conformal factor of (14) must obey a condition imposed by (34) and the Einstein field equations (35).

In [16], spherically symmetric spacetimes with elastic matter were studied and static and non-static shear-free solutions of the Einstein field equations were obtained. The line-element of the corresponding spacetime metric, here denoted by $\bar{g}$, in the non-static shear-free case for particular solutions takes the form

$$d\bar{s}^2 = -dt^2 + t^2 B^2(r)(dr^2 + d\theta^2 + \sin^2\theta d\phi^2). \quad (53)$$

The line-element of the material metric is given by

$$d\bar{\sigma}^2 = q^2(r) [dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2)] \quad (54)$$

and the pulled-back material metric $\bar{k}^a_b$ reads

$$\bar{k}^a_b = \bar{g}^{ac}\bar{k}_{cb} = \bar{g}^{ac}\bar{\gamma}_{CB}y_c^C y_b^B = \frac{q^2(r)}{t^2 B^2} [\delta_1^a \delta_b^1 + r^2 \delta_1^a \delta_b^1 + r^2 \sin^2\theta \delta_1^a \delta_b^1], \quad (55)$$

which has the eigenvalues

$$\bar{\eta} = \frac{q^2}{t^2 B^2}, \quad \bar{s} = \frac{q^2 r^2}{t^2 B^2}, \quad (56)$$

$\bar{s}$ having algebraic multiplicity two (note that $\bar{n}_1^2 = \bar{\eta}$ and $\bar{n}_2^2 = \bar{n}_3^2 = \bar{s}$, cf. (17)).

Consider a non-static spherically symmetric metric $\bar{g}$ with line-element given by (53) and a conformal transformation of this metric $g = f^2 \bar{g}$ with $f = f(t, r)$. Then, σ_{ab} defined in (34) has the following non-zero components

$$\sigma_{00} = \frac{\ddot{f}}{f} - 2 \left(\frac{\dot{f}}{f} \right)^2, \quad (57)$$

$$\sigma_{01} = \sigma_{10} = \frac{\dot{f}'}{f} - 2 \frac{f' \dot{f}}{f^2} - \frac{f'}{t f}, \quad (58)$$

$$\sigma_{11} = \frac{f''}{f} - 2 \left(\frac{f'}{f} \right)^2 - \frac{t B^2 \dot{f}}{f} - \frac{f' B'}{f B}, \quad (59)$$

$$\sigma_{22} = -\frac{t B^2 \dot{f}}{f} + \frac{f' B'}{f B}, \quad (60)$$

$$\sigma_{33} = \sin^2 \theta \sigma_{22}, \quad (61)$$

where a dot indicates partial differentiation with respect to t and a prime, with respect to r .

In the case of elastic matter, the spacetime represented by the metric (53) is associated with the energy-momentum tensor (12), which has $\bar{T}_0^0, \bar{T}_1^1, \bar{T}_2^2 = \bar{T}_3^3$ as non-zero components (see [16]). The non-zero components of $\bar{G}^a_b$ are $\bar{G}_0^0, \bar{G}_1^1, \bar{G}_2^2 = \bar{G}_3^3$.

Assuming that $g = f^2 \bar{g}$ also corresponds to a spacetime with elastic matter, then the non-zero components of the associated energy-momentum tensor are $T_0^0, T_1^1, T_2^2 = T_3^3$, which can be calculated from (12) or (35), and the non-zero components of G^a_b are $G_0^0, G_1^1, G_0^1, G_1^0, G_2^2 = G_3^3$. These components can be related with the corresponding components of $\bar{G}^a_b$ by (33). Note that G_1^0 and G_0^1 are non-zero due to the dependence on σ_{10} :

$$G_1^0 = -\frac{2\sigma_{10}}{f^2}, \quad G_0^1 = -\frac{2\sigma_{10}}{f^2 t^2 B^2}. \quad (62)$$

From the Einstein field equations $G^a_b = 8\pi T^a_b$ and $\bar{G}^a_b = 8\pi \bar{T}^a_b$ and from (32) it follows that $\sigma_{10} = 0$, which is equivalent to the condition

$$\frac{\dot{f}'}{f} - 2 \frac{f' \dot{f}}{f^2} - \frac{f'}{t f} = 0. \quad (63)$$

Solving this equation, assuming that f is in a separated form a product of functions of t and r , gives

$$f(t, r) = f_1(t) f_2(r), \text{ where } -\frac{f_2'}{f_2} (c_1 + 1) = 0 \text{ and } \frac{\dot{f}_1}{f_1} = \frac{c_1}{t}, \quad (64)$$

c_1 being a constant.

Thus, choosing f to be of this form and taking for $\bar{g}$ the elastic solution (53) with one of the functions $B(r)$ presented in [16], one obtains a new conformal metric g . We will analyse the conformal metric, if it is possible to obtain a new solution of the Einstein field equations for elastic matter and study the matching problem of this solution with the exterior Schwarzschild metric.

Let $c_1 = -1$ in (64), then one obtains $f(t, r) = c \frac{l(r)}{t}$, where $c > 0$ and $l(r)$ is a smooth, strictly positive function. Setting $c = 1$, then the line-element of the metric $g = \frac{l^2(r)}{t^2} \bar{g}$ is given by

$$ds^2 = -\frac{l^2(r)}{t^2} dt^2 + l^2(r) B^2(r) (dr^2 + d\theta^2 + \sin^2 \theta d\phi^2). \quad (65)$$

We can choose $l(r) = r^k$, with $k > 1$, so that the line-element of the metric g reads

$$ds^2 = -\frac{r^{2k}}{t^2} dt^2 + r^{2k} B^2(r) (dr^2 + d\theta^2 + \sin^2 \theta d\phi^2). \quad (66)$$

The velocity field of the matter is given by $u^a = (\frac{t}{r^k}, 0, 0, 0)$. The line element of the material metric is

$$d\sigma^2 = h^2(r) q^2(r) [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)] \quad (67)$$

and the pulled-back material metric reads

$$k^a_b = g^{ac} k_{cb} = g^{ac} \gamma_{CB} y^C_c y^B_b = \frac{q^2 h^2}{r^{2k} B^2} [\delta^a_1 \delta^1_b + r^2 \delta^a_1 \delta^1_b + r^2 \sin^2 \theta \delta^a_1 \delta^1_b], \quad (68)$$

which has the eigenvalues

$$\eta = \frac{q^2 h^2}{r^{2k} B^2} \quad \text{and} \quad s = \frac{q^2 h^2}{r^{2k-2} B^2}, \quad (69)$$

s having algebraic multiplicity two. Note that the previous quantities can easily be determined from the quantities associated with $\bar{g}$ and $\bar{k}$, using the formulas developed in section 2: due to (19), multiplying (56) by $\frac{h^2}{f^2} = \frac{h^2}{r^{2k}/t^2}$ gives (69).

The Einstein field equations $G^a_b = 8\pi T^a_b$ for the conformal metric g can be written as follows

$$G^0_0 = 8\pi T^0_0:$$

$$\frac{2B''}{r^{2k} B^3} - \frac{B'^2}{r^{2k} B^4} + \frac{2kB'}{r^{2k+1} B^3} + \frac{k^2}{r^{2k+2} B^2} - \frac{2k}{r^{2k+2} B^2} - \frac{1}{r^{2k} B^2} = -\epsilon v 8\pi \quad (70)$$

$$G^1_1 = 8\pi T^1_1:$$

$$\frac{B'^2}{r^{2k}B^4} + \frac{4kB'}{r^{2k+1}B^3} + \frac{3k^2}{r^{2k+2}B^2} - \frac{1}{r^{2k}B^2} = 2\epsilon\eta \frac{\partial v}{\partial \eta} 8\pi, \quad (71)$$

$$G^2_2 = 8\pi T^2_2:$$

$$\frac{B''}{r^{2k}B^3} - \frac{B'^2}{r^{2k}B^4} + \frac{k^2}{r^{2k+2}B^2} - \frac{2k}{r^{2k+2}B^2} = \epsilon s \frac{\partial v}{\partial s} 8\pi. \quad (72)$$

The energy density, the radial and tangential pressures are expressed by $\rho = \epsilon v$, $p_1 = 2\epsilon\eta \frac{\partial v}{\partial \eta}$ and $p_2 = \epsilon s \frac{\partial v}{\partial s}$.

Following the process presented in [30], dividing (71) and (72) through by (70) and setting $E \equiv \ln \eta$, $\Sigma \equiv \ln s$, one can calculate expressions for $\frac{\partial \ln v}{\partial E}$ and $\frac{\partial \ln v}{\partial \Sigma}$:

$$\frac{\partial \ln v}{\partial E} = -\frac{1}{2} \frac{r^2 B'^2 + 4rkBB' + 3k^2 B^2 - r^2 B^2}{2r^2 BB'' - r^2 B'^2 + 2krBB' - r^2 B^2 + k^2 B^2 - 2kB^2} \quad (73)$$

$$\frac{\partial \ln v}{\partial \Sigma} = -\frac{r^2 BB'' - r^2 B'^2 + k^2 B^2 - 2kB^2}{2r^2 BB'' - r^2 B'^2 + 2krBB' - r^2 B^2 + k^2 B^2 - 2kB^2}, \quad (74)$$

which using

$$\frac{\partial \Sigma}{\partial E} = \frac{\frac{\partial \ln v}{\partial E}}{\frac{\partial \ln v}{\partial \Sigma}}, \quad (75)$$

lead to

$$\frac{\partial \Sigma}{\partial E} = \frac{1}{2} \frac{r^2 B'^2 + 4rkBB' + 3k^2 B^2 - r^2 B^2}{r^2 BB'' - r^2 B'^2 + k^2 B^2 - 2kB^2}. \quad (76)$$

From (69) one gets

$$E = 2 \ln q + 2 \ln h - 2k \ln r - 2 \ln B, \quad \Sigma = E + 2 \ln r, \quad (77)$$

and setting $F = \ln v$, so that $F = F(E, \Sigma)$, $\frac{\partial F}{\partial E} = A(r)$, $\frac{\partial F}{\partial \Sigma} = C(r)$, allows to write

$$A(r) = \frac{F'}{2 \left(\frac{q'}{q} + \frac{h'}{h} - \frac{k}{r} - \frac{B'}{B} \right)}, \quad (78)$$

$$C(r) = \frac{F'}{2 \left(\frac{1}{r} + \frac{q'}{q} + \frac{h'}{h} - \frac{k}{r} - \frac{B'}{B} \right)}. \quad (79)$$

Since the condition

$$\frac{\partial^2 F}{\partial E \partial \Sigma} = \frac{\partial^2 F}{\partial \Sigma \partial E} \quad (80)$$

must be fulfilled in order for a constitutive equation $v = v(\eta, s)$, equivalently $F(E, \Sigma) = \ln v(E, \Sigma)$, to exist, it follows that

$$A' \left(\frac{q'}{q} + \frac{h'}{h} - \frac{k}{r} - \frac{B'}{B} \right) = C' \left(\frac{1}{r} + \frac{q'}{q} + \frac{h'}{h} - \frac{k}{r} - \frac{B'}{B} \right). \quad (81)$$

On the other hand, as a consequence of (78) and (79), one has

$$A \left(\frac{q'}{q} + \frac{h'}{h} - \frac{k}{r} - \frac{B'}{B} \right) = C \left(\frac{1}{r} + \frac{q'}{q} + \frac{h'}{h} - \frac{k}{r} - \frac{B'}{B} \right), \quad (82)$$

so that

$$A = k_0 C, \quad k_0 > 0. \quad (83)$$

Substituting (83) into (82), and noting that for the given elastic solution, the function q is such that (see [16])

$$\frac{q'}{q} = \frac{B'}{B^2} - \frac{2}{3rB}, \quad (84)$$

one obtains

$$\frac{h'}{h} = -\frac{B'}{B^2} + \frac{B'}{B} + \frac{2}{3rB} + \frac{k}{r} + \frac{1}{r(k_0 - 1)}, \quad (85)$$

where, taking into account (73)-(75), $k_0 \neq 1$ is subject to

$$r^2 B'^2 + 4rkBB' + 3k^2 B^2 - r^2 B^2 = 2k_0 (r^2 BB'' - r^2 B'^2 + k^2 B^2 - 2kB^2). \quad (86)$$

Thus, given $k > 1$, the conformal metric g in (66) is a solution of the Einstein field equations for elastic matter, where the material metric is given by (67) and the conformal factor h^2 satisfies (85). This solution is regular everywhere, except at $r = 0$.

We present the following example and study a particular solution.

Example

Consider the solution obtained in [16], where the metric $\bar{g}$ is given by the line-element (53) with

$$B(r) = \frac{\sqrt{3}}{9} (2 + 3(r - r_0)^2)^{\frac{3}{2}}, \quad (87)$$

where $r_0 \neq 0$. Here we choose $r_0 = 1$. Then, taking $k = 2$, the conformal metric $g = f^2 \bar{g}$, given by the line-element (66), with conformal factor $f^2 = \frac{r^4}{t^2}$, takes the form

$$ds^2 = -\frac{r^4}{t^2} dt^2 + r^4 B^2(r) (dr^2 + d\theta^2 + \sin^2 \theta d\phi^2), \quad (88)$$

with

$$B(r) = \frac{\sqrt{3}}{9} (2 + 3(r-1)^2)^{\frac{3}{2}}. \quad (89)$$

In this case, (85) is given by

$$\frac{h'}{h} = -\frac{B'}{B^2} + \frac{B'}{B} + \frac{2}{3rB} + \frac{2}{r} + \frac{1}{r(k_0 - 1)}, \quad (90)$$

together with

$$r^2 B'^2 + 8rBB' + 12B^2 - r^2 B^2 = 2k_0 (r^2 BB'' - r^2 B'^2). \quad (91)$$

Although it is not possible to find an explicit expression for h by integrating (85), it is possible to obtain a numerical solution for h by providing an initial condition (see an example in Figure 1).

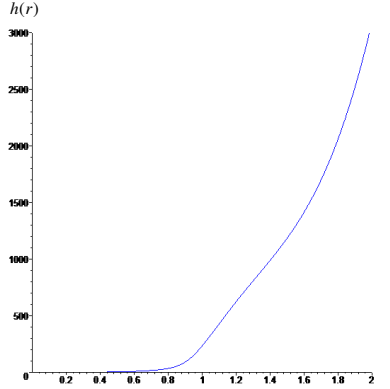


Figure 1: Graph of $h(r)$ with initial condition $h(0.1) = 0.1$.

From (70)-(72) one obtains the following expressions for the energy density ρ and the pressures p_1 and p_2 :

$$\rho = \epsilon v = \frac{1}{8\pi} \left(-\frac{4B'}{r^5 B^3} + \frac{B'^2}{r^4 B^4} - \frac{2B''}{r^4 B^3} + \frac{1}{r^4 B^2} \right), \quad (92)$$

$$p_1 = 2\epsilon\eta \frac{\partial v}{\partial \eta} = \frac{1}{8\pi} \left(\frac{B'^2}{r^4 B^4} + \frac{8B'}{r^5 B^3} + \frac{12}{r^6 B^2} - \frac{1}{r^4 B^2} \right), \quad (93)$$

$$p_2 = \epsilon s \frac{\partial v}{\partial s} = \frac{1}{8\pi} \left(\frac{B''}{r^4 B^3} - \frac{B'^2}{r^4 B^4} \right). \quad (94)$$

The energy density satisfies $\rho \geq 0$ and the Dominant Energy Condition ($\rho \geq 0$, $|p_1| \leq \rho$, $|p_2| \leq \rho$) is fulfilled for certain ranges of $r \in (0, \infty)$. Calculating the kinematic quantities of the fluid flow $u^a = (\frac{t}{r^2}, 0, 0, 0)$ gives

$$\Theta = \sigma_{ab} = \omega_{ab} = 0, \quad \dot{u}_a = \left(0, \frac{2}{r}, 0, 0 \right), \quad (95)$$

so that the solution is non-expanding and shear-free, it has zero vorticity, but non-zero acceleration.

In contrast, the fluid flow $\bar{u}^a = (1, 0, 0, 0)$, corresponding to the given metric $\bar{g}$ (53), is shear-free and vorticity-free, as expected (see (52) and (51)), but it has non-zero expansion and zero acceleration:

$$\bar{\sigma}_{ab} = \bar{\omega}_{ab} = \dot{\bar{u}}_a = 0, \quad \bar{\Theta} = \frac{3}{t}. \quad (96)$$

Concerning the acceleration, it is interesting to observe that also in [25],[26] fields of accelerated charges in de Sitter spacetimes were obtained by conformal transformations of fields of unaccelerated charges in Minkowski spacetime. As for the zero expansion, note that there exist non-static spherically symmetric anisotropic fluid models without expansion, see e.g. [31], [32]. Next, the matching problem of the elastic conformal metric (88) with the Schwarzschild exterior metric will be investigated. Consider the vacuum Schwarzschild exterior metric given by the line-element

$$ds_+^2 = - \left(1 - \frac{2M}{R} \right) dt^2 + \frac{1}{1 - \frac{2M}{R}} dR^2 + R^2 d\Omega^2 \quad (97)$$

and let both spacetimes be matched across a spherically symmetric surface Σ , where $r = r_\Sigma$. The signs $-$ and $+$ are used to denote, respectively, interior and exterior quantities. The tangent space to Σ can be generated by the following orthonormal tangent vectors

$$T\Sigma^\pm = \langle e_1^\pm = \partial_t, e_2^\pm = \partial_\theta, e_3^\pm = \partial_\phi \rangle \quad (98)$$

and the normal vectors to Σ^+ and Σ^- are respectively given by

$$n^{a-} = r^{-2} B^{-1} \partial_r, \quad n^{a+} = \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} \partial_R. \quad (99)$$

The first fundamental forms $q_{AB}^\pm = e_A^{\pm a} e_B^{\pm b} g_{ab}^\pm$ at Σ for both spacetimes are

$$d\sigma_-^2 = -\frac{r^4}{t^2} dt^2 + r^4 B(r)^2 d\Omega^2, \quad (100)$$

$$d\sigma_+^2 = -\left(1 - \frac{2M}{R}\right) dt^2 + R^2 d\Omega^2, \quad (101)$$

where $B(r) = \frac{\sqrt{3}}{9} (2 + 3(r-1)^2)^{\frac{3}{2}}$ and all quantities must be evaluated on Σ . The first matching condition $q_{AB}^+ = q_{AB}^-$ leads to

$$1 - \frac{2M}{R} \stackrel{\Sigma}{=} \frac{r^4}{t^2}, \quad (102)$$

$$R^2 \stackrel{\Sigma}{=} r^4 B^2, \quad (103)$$

where $\stackrel{\Sigma}{=}$ means that the equality holds only on the boundary Σ . The non-zero, independent components of the second fundamental forms $H_{AB}^\pm = -n_a^\pm e_A^{\pm b} \nabla_b^\pm e_B^{\pm a}$ at Σ are

$$H_{tt}^- = -\frac{2r}{t^2 B}, \quad H_{\theta\theta}^- = r(2B + rB'), \quad (104)$$

$$H_{tt}^+ = -\frac{M}{R^2} \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}}, \quad H_{\theta\theta}^+ = R \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}}, \quad (105)$$

where all quantities must be evaluated on Σ . The equality of the second fundamental forms $H_{AB}^+ = H_{AB}^-$ imply

$$\frac{M}{R^2} \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} \stackrel{\Sigma}{=} \frac{2r}{t^2 B}, \quad (106)$$

$$R \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} \stackrel{\Sigma}{=} r(2B + rB'). \quad (107)$$

As a consequence of the matching, using (102), (103), (106) and (107), it follows from (93) that

$$p_1 \stackrel{\Sigma}{=} 0, \quad (108)$$

and calculating the mass $m = 4\pi \int_0^r \rho r^2 B(r^2 B)' dr$ for the interior solution, with ρ defined in (92), yields

$$m \stackrel{\Sigma}{=} M. \quad (109)$$

4 Conclusion

Conformally transformed spacetime and material metrics were considered in general relativistic elasticity and expressions relating tensors and quantities of both conformal spacetimes were derived. Applications to non-static shear-free spherically symmetric spacetimes with elastic matter were presented, where conditions for the conformal factors were obtained allowing to generate new solutions of the Einstein field equations with elastic matter. It was shown through a particular example how it is possible to construct new solutions. From a known non-static shear-free spherically symmetric solution, which is expanding, vorticity-free and has zero acceleration, a new conformal shear-free solution was constructed, which, in contrast to the given one, has zero expansion and non-zero acceleration. The new solution can be matched to the Schwarzschild exterior spacetime.

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