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# On Appell-Vietoris Polynomials

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**Abstract**

In this paper we introduce a polynomial sequence $(\mathcal{V}_n(x))_{n\geq 0}$ involving the Vietoris' number sequence and satisfying the Appell property. We derive several equivalent representations of $\mathcal{V}_n(x)$ and we deduce important relations closely related to the aforementioned Appell property. Finally, we address the problem of the zeros distribution of $\mathcal{V}_n(x)$ and conclude that, when $n$ is even, the polynomials $\mathcal{V}_n(x)$ have no real zeros, whereas for odd $n$, the unique real zero of $\mathcal{V}_n(x)$ lies within the interval $(-1, 0)$.

## 1 Introduction

Some years ago, authors of this paper constructed sequences of multivariate generalized Appell polynomials [10, 13] which involve the sequence of rational numbers

$$c_k := \frac{1}{2^k}\binom{k}{\lfloor \frac{k}{2} \rfloor}, \ k = 0, 1, \ldots, \tag{1}$$

where $\lfloor\cdot\rfloor$ is the floor function. In the context of positive trigonometric sums, in the celebrated paper [17], Vietoris considered a sequence that coincides with (1) from the second term onwards. For this reason, the sequence (1) came to be known as the *Vietoris sequence*.

In this work we focus on a sequence of real polynomials $(\mathcal{V}_n(x))_{n\geq 0}$ of the form

$$\mathcal{V}_n(x) = \sum_{k=0}^{n} \binom{n}{k} c_{n-k} x^k,$$

and study some of its properties. Since this sequence is constructed to fulfill the Appell property [3]

$$\mathcal{V}_n'(x) = n\mathcal{V}_{n-1}(x),$$

Table 1: The first 10 rows of $\mathcal{T}_k^n$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $1$ | | | | | | | | | |
| $\frac{1}{2}$ | $1$ | | | | | | | | |
| $\frac{1}{2}$ | $1$ | $1$ | | | | | | | |
| $\frac{3}{8}$ | $\frac{3}{2}$ | $\frac{3}{2}$ | $1$ | | | | | | |
| $\frac{3}{8}$ | $\frac{3}{2}$ | $3$ | $2$ | $1$ | | | | | |
| $\frac{5}{16}$ | $\frac{15}{8}$ | $\frac{15}{4}$ | $5$ | $\frac{5}{2}$ | $1$ | | | | |
| $\frac{5}{16}$ | $\frac{15}{8}$ | $\frac{45}{8}$ | $\frac{15}{2}$ | $\frac{15}{2}$ | $3$ | $1$ | | | |
| $\frac{35}{128}$ | $\frac{35}{16}$ | $\frac{105}{16}$ | $\frac{105}{8}$ | $\frac{105}{8}$ | $\frac{21}{2}$ | $\frac{7}{2}$ | $1$ | | |
| $\frac{35}{128}$ | $\frac{35}{16}$ | $\frac{35}{4}$ | $\frac{35}{2}$ | $\frac{105}{4}$ | $21$ | $14$ | $4$ | $1$ | |
| $\frac{63}{256}$ | $\frac{315}{128}$ | $\frac{315}{32}$ | $\frac{105}{4}$ | $\frac{315}{8}$ | $\frac{189}{4}$ | $\frac{63}{2}$ | $18$ | $\frac{9}{2}$ | $1$ |

we designate it as the *Appell-Vietoris polynomial sequence*.

This paper is organized as follows: in Sect. 2 we recall some properties of the Vietoris sequence $(c_n)_{n\geq 0}$ and prove some new identities, needed for the sequel, including those related to its binomial inverse. In Sect. 3 we present important properties of Appell polynomials and derive the corresponding results for the Appell-Vietoris polynomials. Finally, the paper ends with some new results on the zeros of the polynomials under consideration.

# 2 Vietoris Sequence and its Inverse

## 2.1 Combinatorial Identities Involving the Vietoris Sequence

One can find in the literature (see e.g. [11, 12]) several well-known combinatorial identities involving the even-order terms $c_{2k}$ of the Vietoris sequence (see also [6] and https://w3.math.uminho.pt/VietorisSequence). For example, identities (3.90), (3.91) and (3.85) of [11] can be rewritten, respectively, as

$$\sum_{s=0}^{k} c_{2s}c_{2k-2s} = 1,$$

$$\sum_{s=0}^{k} (-1)^s c_{2s}c_{2k-2s} = \begin{cases} 0, & \text{if } k \text{ odd} \\ c_k, & \text{if } k \text{ even} \end{cases},$$

and

$$\sum_{s=0}^{k} (-1)^s \binom{k}{s} c_{2k-2s} = (-1)^k c_{2k}.$$

For the purpose of this study, we now derive additional convolution identities involving all the terms of the Vietoris sequence. These identities emerge as we explore the sequence of coefficients associated with the introduced Appell-Vietoris polynomials. Within this framework, we examine an infinite array of numbers defined by:

$$\mathcal{T}_k^n = \binom{n}{k} c_{n-k}, \ n = 0, 1, \ldots; \ k = 0, 1, \ldots, n. \tag{2}$$

These numbers form a triangular array with $n+1$ rows, ordered from $k = 0$ to $k = n$. In Table 1, we present the first 10 rows of this triangular arrangement.

**Theorem 1.** *Let $\mathcal{T}_k^n$ denote the triangle (2). Then the following identities hold.*

**Id**$_1$ *Relations between adjacent elements:*

$$\mathcal{T}_k^n = \frac{n}{2\lfloor\frac{n-k+1}{2}\rfloor}\mathcal{T}_k^{n-1} = \frac{2\lfloor\frac{n-k+2}{2}\rfloor}{k}\mathcal{T}_{k-1}^n = \frac{n}{k}\mathcal{T}_{k-1}^{n-1}.$$

**Id**$_2$ *Relation of elements in the $k$-column with the first column:*

$$c_k\mathcal{T}_k^n = \binom{\lfloor\frac{n+1}{2}\rfloor}{\lfloor\frac{k+1}{2}\rfloor}\binom{\lfloor\frac{n}{2}\rfloor}{\lfloor\frac{k}{2}\rfloor}\mathcal{T}_0^n.$$

**Id**$_3$ *Row sum:*

$$\sum_{k=0}^{n}\mathcal{T}_k^n = \sum_{k=0}^{n}\binom{n}{k}c_k = 2^{n+1}c_{2n+1}.$$

**Id**$_4$ *Alternating row sum:*

$$\sum_{k=0}^{n}(-1)^k\mathcal{T}_k^n = (-1)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}c_k = \frac{(-2)^n}{n+1}c_{2n}.$$

*Proof.* Identity **Id**$_1$ follows easily from the well-known properties of the binomial coefficients

$$\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k} = \frac{n+1-k}{k}\binom{n}{k-1} = \frac{n}{k}\binom{n-1}{k-1}.$$

Concerning Identity **Id**$_2$, we only consider here the case where $n$ and $k$ are both even; the proof in the other cases is similar. Simple calculations allow to write

$$\begin{aligned}\binom{2s}{s}\mathcal{T}_{2s}^{2m} &= \binom{2s}{s}\binom{2m}{2s}c_{2m-2s} = \binom{2s}{s}\binom{2m}{2s}\binom{2m-2s}{m-s}2^{2s-2m}\\ &= \frac{\binom{2s}{s}\binom{2m}{2s}\binom{2m-2s}{m-s}}{\binom{2m}{m}}2^{2s}c_{2m} = 4^s\binom{m}{s}^2c_{2m} = 4^s\binom{m}{s}^2\mathcal{T}_0^{2m}\end{aligned}$$

and the result is proved.

To prove Identity **Id**$_3$, we observe that the triangle $\widetilde{\mathcal{T}}_k^n = 2^n\mathcal{T}_{n-k}^n$ is the OEIS A051289 sequence [16], corresponding to the triangular array $\widetilde{\mathcal{T}}_k^n$ read by rows and whose row sum is

$$\sum_{k=0}^{n}\widetilde{\mathcal{T}}_k^n = \binom{2n+1}{n+1}.$$

Therefore

$$\sum_{k=0}^{n}\mathcal{T}_k^n = \sum_{k=0}^{n}\mathcal{T}_{n-k}^n = 2^{-n}\binom{2n+1}{n+1} = 2^{-n}\binom{2n+1}{n} = 2^{n+1}\frac{\binom{2n+1}{n}}{2^{2n+1}} = 2^{n+1}c_{2n+1}$$

and the second identity is proved. We now write

$$2^{n+1}c_{2n+1} = \sum_{k=0}^{n}\binom{n}{k}c_k = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}c_{2k} + \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2k+1}c_{2k+1}.$$

Identity (3.99) of [11] can be rewritten as

$$\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}c_{2k} = 2^nc_{2n} \tag{3}$$

and this allows to conclude that

$$\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2k+1}c_{2k+1}=2^{n+1}c_{2n+1}-2^{n}c_{2n}=2^{n}\frac{n}{n+1}c_{2n}, \tag{4}$$

since $c_{2n+1}=\frac{2n+1}{2n+2}c_{2n}$. Identity $\mathbf{Id}_4$ follows at once from (3)-(4). □

**Remark 1.** If $\mathcal{C}_n$ denotes the $n$-th Catalan number [7, 12], then Identity $\mathbf{Id}_4$ can be written as

$$\sum_{k=0}^{n}(-1)^k\mathcal{T}_k^n=\sum_{k=0}^{n}(-1)^k\binom{n}{k}c_k=(-1)^n\frac{\mathcal{C}_n}{2^n}.$$

**Theorem 2.** *For each fixed $n$, the finite sequence $\mathcal{T}_0^n,\mathcal{T}_1^n,\ldots,\mathcal{T}_n^n$ is*

$(i)$ *log-concave, i.e.,*

$$(\mathcal{T}_k^n)^2\geq\mathcal{T}_{k-1}^n\mathcal{T}_{k+1}^n,\ k=1,\ldots,n-1;$$

$(ii)$ *unimodal, i.e.,*

$$\mathcal{T}_0^n\leq\mathcal{T}_1^n\leq\cdots\leq\mathcal{T}_{s-1}^n\leq\mathcal{T}_s^n\geq\mathcal{T}_{s+1}^n\geq\cdots\geq\mathcal{T}_n^n, \tag{5}$$

*where $s=\lfloor\frac{n+1}{2}\rfloor$.*

*Proof.* Inequality $(i)$ follows easily by using one of the identities in $\mathbf{Id}_1$, i.e.,

$$(\mathcal{T}_k^n)^2-\mathcal{T}_{k-1}^n\mathcal{T}_{k+1}^n=\mathcal{T}_{k-1}^n\mathcal{T}_k^n\left(\frac{2}{k}\lfloor\frac{n-k+2}{2}\rfloor-\frac{2}{k+1}\lfloor\frac{n-k+1}{2}\rfloor\right)\geq 0.$$

It is well known that a log-concave sequence of positive numbers is unimodal [18], i.e., there exists $s$, $0\leq s\leq n$, such that (5) holds. Therefore, it is enough to prove that for $s=\lfloor\frac{n+1}{2}\rfloor$,

$$\mathcal{T}_{s-1}^n\leq\mathcal{T}_s^n\geq\mathcal{T}_{s+1}^n.$$

The repeated use of Identity $\mathbf{Id}_1$ allows to write

$$\begin{array}{llll}
\mathcal{T}_{2m}^{4m}=\frac{m+1}{m}\mathcal{T}_{2m-1}^{4m}>\mathcal{T}_{2m-1}^{4m} & \text{and} & \mathcal{T}_{2m+1}^{4m}=\frac{2m}{2m+1}\mathcal{T}_{2m}^{4m}<\mathcal{T}_{2m}^{4m}\\
\mathcal{T}_{2m+1}^{4m+1}=\frac{2m+2}{2m+1}\mathcal{T}_{2m}^{4m+1}>\mathcal{T}_{2m}^{4m+1} & \text{and} & \mathcal{T}_{2m+2}^{4m+1}=\frac{m}{m+1}\mathcal{T}_{2m+1}^{4m+1}<\mathcal{T}_{2m+1}^{4m+1}\\
\mathcal{T}_{2m+2}^{4m+2}=\frac{m+1}{m+1}\mathcal{T}_{2m+1}^{4m+1}=\mathcal{T}_{2m+1}^{4m+1} & \text{and} & \mathcal{T}_{2m+3}^{4m+2}=\frac{2m}{2m+3}\mathcal{T}_{2m+2}^{4m+2}<\mathcal{T}_{2m+2}^{4m+2}\\
\mathcal{T}_{2m+2}^{4m+3}=\frac{m+1}{m+1}\mathcal{T}_{2m+1}^{4m+3}=\mathcal{T}_{2m+1}^{4m+3} & \text{and} & \mathcal{T}_{2m+3}^{4m+3}=\frac{2m+2}{2m+3}\mathcal{T}_{2m+2}^{4m+3}<\mathcal{T}_{2m+2}^{4m+3}
\end{array}$$

and the result $(ii)$ follows at once. □

## 2.2 The Vietoris Binomial Inverse Sequence

Given two sequences $\mathbf{u}=(u_n)_{n\geq 0}$ and $\mathbf{v}=(v_n)_{n\geq 0}$, we recall that the binomial convolution of $\mathbf{u}$ and $\mathbf{v}$, here denoted by $\mathbf{u}\otimes\mathbf{v}=((u\otimes v)_n)_{n\geq 0}$, is the sequence defined as

$$(u\otimes v)_n=\sum_{k=0}^{n}\binom{n}{k}u_kv_{n-k}.$$

Moreover, as it is well-known, if $f_{\mathbf{u}}(t)$ and $f_{\mathbf{v}}(t)$ are the exponential generating functions of the sequences $\mathbf{u}$ and $\mathbf{v}$, respectively, then the exponential generating function of the binomial convolution of the two sequences, $f_{\mathbf{u}\otimes\mathbf{v}}(t)$, is given by

$$f_{\mathbf{u}\otimes\mathbf{v}}(t)=f_{\mathbf{u}}(t)f_{\mathbf{v}}(t),$$

i.e., the multiplication of two exponential generating functions corresponds to the binomial convolution of sequences (see, for instance, [2]). We call $f_{\mathbf{u}}$ and $f_{\mathbf{v}}$ reciprocals if

$$f_{\mathbf{u}}(t)f_{\mathbf{v}}(t)=f_{\mathbf{v}}(t)f_{\mathbf{u}}(t)=1.$$

Table 2: The first terms of the Vietoris sequence and its binomial inverse

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c_n$ | $1$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{5}{16}$ | $\frac{5}{16}$ | $\frac{35}{128}$ | $\frac{35}{128}$ | $\frac{63}{256}$ | $\frac{63}{256}$ | $\frac{231}{1024}$ | $\frac{231}{1024}$ |
| $d_n$ | $1$ | $-\frac{1}{2}$ | $0$ | $\frac{3}{8}$ | $-\frac{3}{8}$ | $-\frac{5}{16}$ | $\frac{25}{16}$ | $-\frac{175}{128}$ | $-\frac{735}{128}$ | $\frac{5607}{256}$ | $-\frac{63}{16}$ | $-\frac{223839}{1024}$ | $\frac{703395}{1024}$ |

We recall that the exponential generating function of the Vietoris sequence $(c_n)_{n\geq 0}$ is [4, 6],

$$\boldsymbol{F}(t) = I_0(t) + I_1(t), \tag{6}$$

where $I_0$ and $I_1$ are the modified Bessel functions of the first kind, i.e.,

$$\boldsymbol{F}(t) = I_0(t) + I_1(t) = \sum_{k=0}^{\infty} c_k \frac{t^k}{k!}.$$

Therefore, the reciprocal $\boldsymbol{G}$ of $\boldsymbol{F}$ is such that

$$\boldsymbol{G}(t) = \frac{1}{I_0(t) + I_1(t)} = \sum_{k=0}^{\infty} d_k \frac{t^k}{k!}, \tag{7}$$

with

$$(c \otimes d)_n = \sum_{k=0}^{n} \binom{n}{k} c_k d_{n-k} = \delta_{n,0}, \tag{8}$$

where $\delta_{i,j}$ is the Kronecker delta function. We will call the sequence $(d_n)_{n\geq 0}$ the binomial inverse of the Vietoris sequence $(c_n)_{n\geq 0}$.

From (1) and (8), it follows immediately that the binomial inverse of the Vietoris sequence is defined recursively by

$$\begin{cases} d_0 = \dfrac{1}{c_0} = 1 \\ d_n = -\dfrac{1}{c_0} \displaystyle\sum_{k=1}^{n} \binom{n}{k} c_k d_{n-k} = -\sum_{k=1}^{n} \binom{n}{k} c_k d_{n-k}, \ n = 1, 2, \ldots. \end{cases} \tag{9}$$

**Remark 2.** Relations (1) and (8) also imply that

$$c_n = -\frac{1}{d_0} \sum_{k=1}^{n} \binom{n}{k} d_k c_{n-k} = -\sum_{k=1}^{n} \binom{n}{k} d_k c_{n-k}, \ n = 1, 2, \ldots,$$

with $c_0 = \frac{1}{d_0} = 1$.

Table 2 contains the first terms of the Vietoris sequence and its binomial inverse.

Observe that

$$d_k = \boldsymbol{G}^{(k)}(0), \ k = 0, 1, \ldots$$

(cf.(7)). The $k$-th derivative of the function $\boldsymbol{G}$ can be obtained by using the Faà di Bruno's formula in the form

$$\frac{d^n}{dx^n} f(g(x)) = \sum_{k=0}^{n} f^{(k)}(g(x)) \mathcal{B}_{n,k}\left(g'(x), g''(x), \ldots, g^{(n)}(x)\right), \tag{10}$$

where $\mathcal{B}_{n,k}$ are the incomplete exponential Bell polynomials [7, Th11.4]

$$\mathcal{B}_{n,k}(x_1, x_2, \ldots, x_n) = \sum_{\sigma(n,k)} \frac{n!}{k_1! k_2! \cdots k_n!} \left(\frac{x_1}{1!}\right)^{k_1} \left(\frac{x_2}{2!}\right)^{k_2} \cdots \left(\frac{x_n}{n!}\right)^{k_n},$$

$\sigma(n,k)$ denoting the set of all partitions of $n$ into $k$ parts.

Writing $g(x) = I_0(x) + I_1(x)$ and $f(x) = \frac{1}{x}$, and observing that $g^{(k)}(0) = c_k$ and $f^{(k)}(1) = (-1)^k k!$, we obtain from (10)

$$d_n = \sum_{k=0}^{n} (-1)^k k! \mathcal{B}_{n,k}(c_1, c_2, \ldots, c_n),\ n = 0, 1, \ldots. \tag{11}$$

**Remark 3.** The same reasoning leads to

$$c_n = \sum_{k=0}^{n} (-1)^k k! \mathcal{B}_{n,k}(d_1, d_2, \ldots, d_n),\ n = 0, 1, \ldots.$$

# 3 Appell-Vietoris Polynomials

## 3.1 Appell Polynomials Revisited

In this section we consider polynomials in $x$ forming a sequence

$$\mathcal{A}_0(x), \mathcal{A}_1(x), \ldots, \mathcal{A}_n(x), \ldots,$$

whose term $\mathcal{A}_n(x)$ is a polynomial of degree $n$, satisfying the Appell property,

$$\mathcal{A}_n'(x) = n\mathcal{A}_{n-1}(x),\ n = 1, 2, \ldots, \tag{12}$$

with $\mathcal{A}_0(x) \neq 0$. The sequence $(\mathcal{A}_n(x))_{n\geq 0}$ is commonly referred to as Appell sequence and its terms as Appell polynomials. The following theorem summarizes some properties that characterize Appell polynomials.

**Theorem 3.** *Appell sequence of polynomials $\mathcal{A}_n(x)$ (of degree $n$) may be defined by either of the following equivalent conditions:*

1. *There exists a formal power series*

$$f(t) = \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{n!},\ (\alpha_0 \neq 0), \tag{13}$$

*such that (again formally)*

$$f(t)e^{tx} = \sum_{n=0}^{\infty} \mathcal{A}_n(x) \frac{t^n}{n!}, \tag{14}$$

*i.e., (14) is the exponential generating function of the sequence $(\mathcal{A}_n(x))_{n\geq 0}$.*

2. *There exists a sequence $(\alpha_n)_{n\geq 0}$, with $\alpha_0 \neq 0$, such that*

$$\mathcal{A}_n(x) = \sum_{k=0}^{n} \alpha_k \binom{n}{k} x^{n-k},\ n = 0, 1, \ldots. \tag{15}$$

3. *The sequence $(\mathcal{A}_n(x))_{n\geq 0}$ satisfies the binomial identity*

$$\mathcal{A}_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{A}_{n-k}(x) y^k,\ n = 0, 1, 2 \ldots. \tag{16}$$

*4. There exists a sequence* $(\beta_n)_{n\geq 0}$, *with* $\beta_0 \neq 0$, *such that*

$$\mathcal{A}_0(x) = \frac{1}{\beta_0} \tag{17}$$

$$\mathcal{A}_n(x) = \frac{(-1)^n}{\beta_0^{n+1}} \begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} & x^n \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}, \ n = 1, 2, \ldots \tag{18}$$

The equivalence of (12), (14) and (15) was proved by Appell in [3], while the representation (17)-(18) was derived by Costabile and Longo [8], using a determinantal approach and also by Adell and Lekuona [2] using binomial convolutions. In both works we can find the proof that the sequence $(\beta_n)_{n\geq 0}$ in (17)-(18) is the binomial inverse of $(\alpha_n)_{n\geq 0}$ in (15), i.e.,

$$\begin{cases} \beta_0 = \dfrac{1}{\alpha_0} \\ \beta_k = -\dfrac{1}{\alpha_0} \displaystyle\sum_{s=0}^{k-1} \binom{k}{s} \alpha_{k-s}\beta_s, \quad k = 1, \ldots, n \end{cases} \tag{19}$$

(cf. (9)).

A proof that polynomials of exact degree $n$ are Appell polynomials iff they satisfy the binomial identity (16) can be seen, for example, in [5, Th1].

Beyond Theorem 3, there exists a lesser-known unifying matrix approach, introduced in [1], for representing Appell polynomials. Considering the generating function (14) as well as the formal power series (13), the authors arrived at the representation of the Appell vector $\mathbf{a}(x) = [\mathcal{A}_0(x)\ \mathcal{A}_1(x)\ \cdots\ \mathcal{A}_n(x)]^T$ in the form

$$\mathbf{a}(x) = f(H)\xi(x),$$

where $\xi(x) = [1\ x\ x^2 \cdots x_n]^T$, $H = (H_{ij})$ is the nilpotent matrix of order $n$ defined by

$$H_{ij} = \begin{cases} i, & i = j+1 \\ 0, & \text{otherwise}, \quad i, j = 0, \ldots, n, \end{cases}$$

and

$$f(H) = \sum_{m=0}^{n} \alpha_m \frac{H^m}{m!}.$$

Since the matrix $M = f(H)$, called the transfer matrix for the Appell vector $\mathbf{a}(x)$, is nonsingular, the use of its inverse

$$M^{-1} = \sum_{m=0}^{n} \beta_m \frac{H^m}{m!}$$

enables the derivation of (19). Nevertheless, determining a closed form of the coefficients $\beta_m$ in the matrix $M^{-1}$ proves challenging. So far, such determination has been successful only in selected cases of Appell polynomials (cf. [1, p. 439]).

## 3.2 The Appell-Vietoris polynomials

We consider now the sequence of polynomials $(\mathcal{V}_n(x))_{n\geq 0}$ of the form

$$\mathcal{V}_n(x) = \sum_{k=0}^{n} \mathcal{T}_k^n x^k = \sum_{k=0}^{n} \binom{n}{k} c_{n-k} x^k, \ n = 0, 1, 2 \ldots, \tag{20}$$

Table 3: Appell-Vietoris Polynomials $\mathcal{V}_n,\ n=0,1,\ldots,7$

| $n$ | $\mathcal{V}_n(x)$ |
|---|---|
| 0 | $1$ |
| 1 | $x+\frac{1}{2}$ |
| 2 | $x^2+x+\frac{1}{2}$ |
| 3 | $x^3+\frac{3x^2}{2}+\frac{3x}{2}+\frac{3}{8}$ |
| 4 | $x^4+2x^3+3x^2+\frac{3x}{2}+\frac{3}{8}$ |
| 5 | $x^5+\frac{5x^4}{2}+5x^3+\frac{15x^2}{4}+\frac{15x}{8}+\frac{5}{16}$ |
| 6 | $x^6+3x^5+\frac{15x^4}{2}+\frac{15x^3}{2}+\frac{45x^2}{8}+\frac{15x}{8}+\frac{5}{16}$ |
| 7 | $x^7+\frac{7x^6}{2}+\frac{21x^5}{2}+\frac{105x^4}{8}+\frac{105x^3}{8}+\frac{105x^2}{16}+\frac{35x}{16}+\frac{35}{128}$ |

where $\mathcal{T}_k^n$ and $c_k$ are respectively, the triangle (2) and the $k$-th term of the Vietoris sequence (1). This sequence inherently fulfills the Appell property stated in (12),(cf. (15)). This validation supports our choice of naming it the Appell-Vietoris sequence of polynomials. Table 3 and Fig. 1 present the initial polynomials of the Appell-Vietoris sequence.

Since $\mathcal{V}_n(0)=c_n,\ n=0,1,\ldots$, the exponential generating function of $(\mathcal{V}_n(x))_{n\geq 0}$ is (see (14) and (6))

$$(I_0(t)+I_1(t))e^{tx}=\sum_{n=0}^{\infty}\mathcal{V}_n(x)\frac{t^n}{n!}.$$

The determinantal form of $\mathcal{V}_n(x)$ is (cf. (17)-(18))

$$\mathcal{V}_0(x)=1$$

$$\mathcal{V}_n(x)=(-1)^n\begin{vmatrix} 1 & x & x^2 & \ldots & x^{n-1} & x^n \\ 1 & d_1 & d_2 & \ldots & d_{n-1} & d_n \\ 0 & 1 & \binom{2}{1}d_1 & \ldots & \binom{n-1}{1}d_{n-2} & \binom{n}{1}d_{n-1} \\ 0 & 0 & 1 & \ldots & \binom{n-1}{2}d_{n-3} & \binom{n}{2}d_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \binom{n}{n-1}d_1 \end{vmatrix},\ n=1,2,\ldots,$$

where $(d_n)_{n\geq 0}$ is the binomial inversion of the Vietoris sequence, given by (9) or (11). For example, when $n=7$, we have

$$\mathcal{V}_7(x)=\begin{vmatrix} 1 & x & x^2 & x^3 & x^4 & x^5 & x^6 & x^7 \\ 1 & \frac{1}{2} & \frac{1}{2} & \frac{3}{8} & \frac{3}{8} & \frac{5}{16} & \frac{5}{16} & \frac{35}{128} \\ 0 & 1 & 1 & \frac{3}{2} & \frac{3}{2} & \frac{15}{8} & \frac{15}{8} & \frac{35}{16} \\ 0 & 0 & 1 & \frac{3}{2} & 3 & \frac{15}{4} & \frac{45}{8} & \frac{105}{16} \\ 0 & 0 & 0 & 1 & 2 & 5 & \frac{15}{2} & \frac{105}{8} \\ 0 & 0 & 0 & 0 & 1 & \frac{5}{2} & \frac{15}{2} & \frac{105}{8} \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & \frac{21}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{7}{2} \end{vmatrix}.$$

As an immediate consequence of last relation we can obtain the following recurrence relation [8, Th9] for the

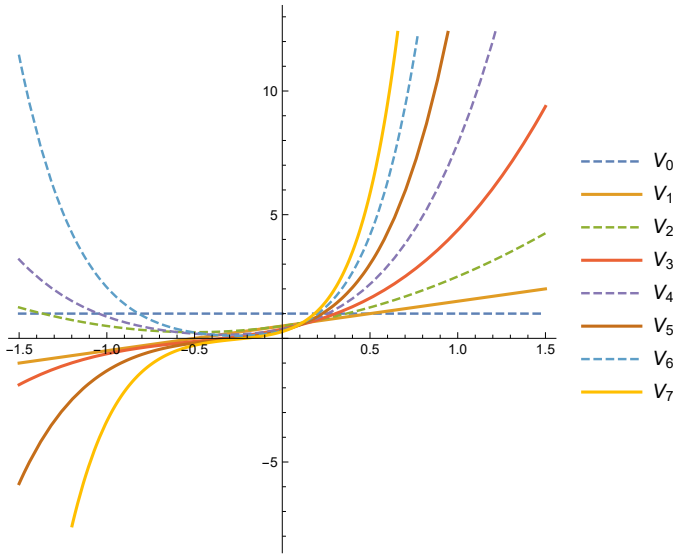



Figure 1: First Appell-Vietoris Polynomials

Appell-Vietoris polynomials

$$\mathcal{V}_n(x) = x^n - \sum_{k=0}^{n-1} \binom{n}{k} d_{n-k} \mathcal{V}_k(x), \ n = 1, 2, \ldots.$$

Finally, we consider the result obtained in [8, Th18], namely

$$\int_0^1 \mathcal{V}_n(x)dx = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} \mathcal{V}_k(0),$$

which can be written as

$$\int_0^1 \mathcal{V}_n(x)dx = \frac{1}{n+1} \left( \sum_{k=0}^{n+1} \binom{n+1}{k} c_k - c_{n+1} \right).$$

Using now Identity $\mathbf{Id}_3$ in Theorem 1, we obtain

$$\int_0^1 \mathcal{V}_n(x)dx = \frac{1}{n+1} \left( 2^{n+2} c_{2n+3} - c_{n+1} \right).$$

## 3.3 Zeros of the Appell-Vietoris polynomials

The literature on the zeros of Appell polynomials is scarse. We can mention [9, Sect. 6.6], where one can find some results on the location of zeros of Appell sequences $(\mathcal{A}_n(x))_{n\geq 0}$ satisfying the condition that

$$\mathcal{A}_n(x), \ \mathcal{A}_{n-1}(x), \ldots, \mathcal{A}_0(x)$$

is a Sturm sequence [15].

We recall that a sequence of polynomials $P_0(x), P_1(x), \ldots, P_n(x)$ is called a Sturm sequence on an interval $(a, b)$ if

1. $P_n(x) \neq 0$ in $(a, b)$;

2. If $P_k(\zeta) = 0$, $k = 1, \ldots, n-1$, then

$$P_{k-1}(\zeta) P_{k+1}(\zeta) < 0.$$

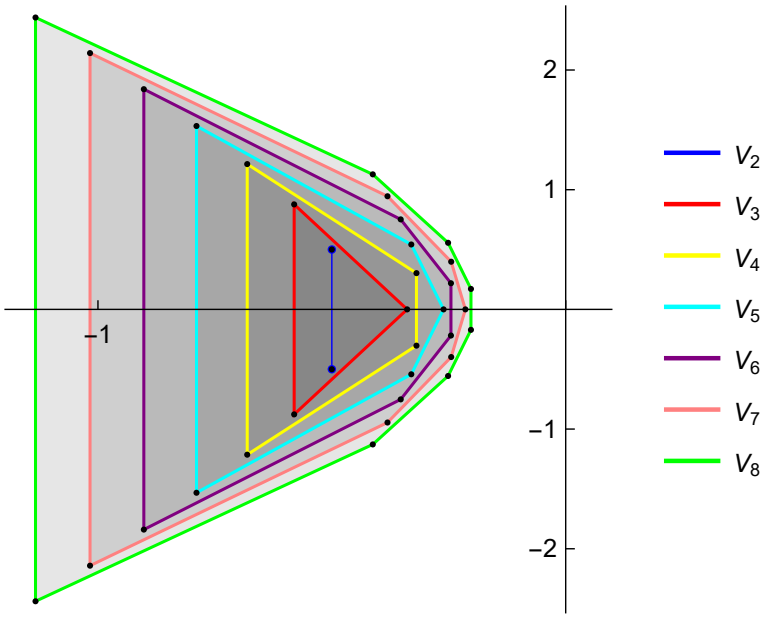



Figure 2: The Gauss-Lucas Theorem for the first Appell-Vietoris polynomials

By observing the first polynomials of the Appell-Vietoris sequence, it is easy to conclude that Condition 2 is not fulfilled.

Naturally, results connecting the zeros of a polynomial and its derivatives can be used, following e.g. [14]. In particular, the Gauss-Lucas Theorem allows to conclude that the zeros of $\mathcal{V}_{n-1}(x)$ are in the convex hull of the zeros of $\mathcal{V}_n(x)$, as illustrated in Fig. 2.

By construction, all the coefficients of the polynomials $\mathcal{V}_n(x)$ (20) are positive, which means, by Descartes' rule of signs, that the real zeros of $\mathcal{V}_n(x)$, if any, are negative. In addition, taking into account that

$$\mathcal{V}_n(0) = c_n > 0$$

and by Theorem 1 (**Id**$_3$)

$$\mathcal{V}_n(-1) = \frac{(-2)^n c_{2n}}{n+1} < 0, \text{ for odd } n,$$

we conclude that, when $n$ is odd, there is at least a real root of $\mathcal{V}_n(x)$ in $(-1, 0)$.

We introduce now the $2n$ degree polynomial

$$\mathcal{W}_{2n}(x) = \mathcal{V}_n(x)\mathcal{V}_n(-x) = \sum_{k=0}^{2n} x^k \sum_{s=0}^{k} (-1)^s \mathcal{T}_s^n \mathcal{T}_{k-s}^n.$$

Observe that if $\zeta$ is a real zero of $\mathcal{W}_{2n}(x)$ then either $\zeta$ or $-\zeta$ is a zero of $\mathcal{V}_n(x)$. Consequently, if $\mathcal{W}_{2n}(x)$ has only nonreal zeros, the same holds for $\mathcal{V}_n(x)$.

The polynomial $\mathcal{W}_{2n}(x)$ can be expressed as

$$\mathcal{W}_{2n}(x) = \sum_{k=0}^{n} a_{k,n} x^{2k} + \sum_{k=0}^{n-1} b_{k,n} x^{2k+1},$$

where

$$a_{k,n} = \sum_{s=0}^{2k} (-1)^s \mathcal{T}_s^n \mathcal{T}_{2k-s}^n \tag{21}$$

Table 4: The values $a_{2k,m}$, $m = 1, \ldots, 5$, $k = 2, \ldots 2m$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | |
| $\frac{15}{4}$ | 2 | 1 | | | | | | |
| $\frac{525}{64}$ | $\frac{35}{2}$ | $\frac{45}{2}$ | 6 | 1 | | | | |
| $\frac{3675}{256}$ | $\frac{2205}{32}$ | $\frac{11655}{64}$ | $\frac{343}{2}$ | $\frac{161}{2}$ | 12 | 1 | | |
| $\frac{363825}{16384}$ | $\frac{24255}{128}$ | $\frac{3465}{4}$ | $\frac{54747}{32}$ | $\frac{112875}{64}$ | $\frac{1725}{2}$ | $\frac{855}{4}$ | 20 | 1 |

and

$$b_{k,n} = \sum_{s=0}^{2k+1} (-1)^s \mathcal{T}_s^n \mathcal{T}_{2k+1-s}^n. \tag{22}$$

Changing $s$ to $2k+1-s$ in (22), it follows that

$$b_{k,n} = -\sum_{s=0}^{2k+1} (-1)^s \mathcal{T}_{2k+1-s}^n \mathcal{T}_s^n = -b_{k,n},$$

which means that $b_{k,n} = 0$, for all values of $k$ and $n$. In other words, we have proved that

$$\mathcal{W}_{2n}(x) = \sum_{k=0}^{n} a_{k,n} x^{2k}. \tag{23}$$

From the properties of the Vietoris sequence and (21), it becomes evident that for even values of $n$, (let's say $n = 2m$), the following relations hold:

$$a_{0,2m} = c_{2m}^2 \quad \text{and} \quad a_{1,2m} = 0.$$

The values of $a_{k,2m}$, for $k = 2, \ldots, 2m$ and $m = 1, \ldots, 5$ are presented in Table 4, arranged in a triangular way. It is noteworthy that this triangle inherits the properties of the triangle $\mathcal{T}_k^n$ that generated it, specifically the non-negativity of its elements and the log concavity of the finite sequence associated with each row, i.e., $a_{2,m}, a_{3,m}, \ldots, a_{2m,m}$.

Since $a_{k,2m} \geq 0$, it follows that both $\mathcal{W}_{2n}(x)$ and $\mathcal{W}_{2n}(-x)$ possess non-negative coefficients (cf. (23)). Consequently, neither $\mathcal{W}_{2n}(x)$ nor $\mathcal{V}_n(x)$ have real zeros, when $n$ is even.

In the case of odd $n$, we know that $\mathcal{V}_n(x)$ has an odd number of real (negative) zeros. By Rolle's theorem, if $\mathcal{V}_n(x)$ has $k > 1$ zeros, then its derivative $n\mathcal{V}_{n-1}(x)$ has at least $k-1$ zeros, which is not possible, since this even-degree polynomial has no real zeros. Therefore we conclude that $\mathcal{V}_n(x)$ has only one negative zero $\zeta_n$, which, as already proved, lies in $(-1, 0)$.

To illustrate the above considerations, we present in Table 5 the zeros of the first Vietoris polynomials (see also Fig. 1).

The numerical results obtained suggest the conjecture that all the zeros of $\mathcal{V}_n(x)$ lie in the open left half-plane. This property, very important in, for example, control theory or in the mathematical theory of differential and difference equations, deserves a more in-depth future study.

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Table 5: The zeros of the first Vietoris polynomials

| $n$ | zeros of $V_n$ |
|---|---|
| 1 | $-0.5$ |
| 2 | $-0.5 \pm 0.5i$ |
| 3 | $-0.58055 \pm 0.87719i$<br>$-0.33891$ |
| 4 | $-0.68090 \pm 1.21396i$<br>$-0.31910 \pm 0.30289i$ |

| $n$ | zeros of $V_n$ |
|---|---|
| 5 | $-0.78934 \pm 1.53203i$<br>$-0.33012 \pm 0.54222i$<br>$-0.26109$ |
| 6 | $-0.90193 \pm 1.83982i$<br>$-0.35274 \pm 0.75196i$<br>$-0.24534 \pm 0.21841i$ |

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